15 Fourier series and transforms

(1) Show that the Fourier transform of a real function, $f(x) = f(x)^*$, is conjugate symmetric: $F(-k) = F(k)^*$. What else follows about $F(k)$ if, additionally, (a) $f(-x) = f(x)$; and (b) $f(-x) = -f(x)$?

Defining the Fourier transform as

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad \text{— (1)}$$

where $i^2 = -1$,

$$F(-k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx \qquad \text{— (2)}$$

$$= \int_{-\infty}^{\infty} \left[f(x)^*\, e^{-ikx} \right]^* dx = \left[\int_{-\infty}^{\infty} f(x)^*\, e^{-ikx}\, dx \right]^*$$

$dx = dx^*$

Hence, $F(-k) = F(k)^*$ if $f(x) = f(x)^*$.

(a) Substituting $x = -y$ in eqn (2)

$dx = -dy$

$$F(-k) = \int_{-\infty}^{\infty} f(-y)\, e^{-iky}\, dy \qquad \text{— (3)}$$

If $f(-y) = f(y)$, therefore, $F(-k) = F(k)$. Hence, the Fourier transform of a real and symmetric function is also real and even: $F(k) = F(-k) = F(k)^*$.

(b) From eqn (3), $F(-k) = -F(k)$ if $f(-y) = -f(y)$. Hence, the Fourier transform of a real and antisymmetric function is purely imaginary and odd: $-F(k) = F(-k) = F(k)^*$.

(2) Prove that the Fourier transform of the convolution of two functions

$$\mathsf{f}(x) = \mathsf{g}(x) \otimes \mathsf{h}(x) = \int_{-\infty}^{\infty} \mathsf{g}(u)\, \mathsf{h}(x-u)\, \mathsf{d}u$$

is proportional to the product of their Fourier transforms, $\mathsf{G}(\mathsf{k}) \times \mathsf{H}(\mathsf{k})$.

$$\mathsf{F}(\mathsf{k}) = \int_{-\infty}^{\infty} \mathsf{f}(x)\, \mathsf{e}^{-\mathsf{i}\mathsf{k}x}\, \mathsf{d}x = \int_{x=-\infty}^{x=\infty} \big[\mathsf{g}(x) \otimes \mathsf{h}(x)\big]\, \mathsf{e}^{-\mathsf{i}\mathsf{k}x}\, \mathsf{d}x$$

$$= \int_{x=-\infty}^{x=\infty} \int_{u=-\infty}^{u=\infty} \mathsf{g}(u)\, \mathsf{h}(x-u)\, \mathsf{e}^{-\mathsf{i}\mathsf{k}x}\, \mathsf{d}u\, \mathsf{d}x$$

$$= \int_{u=-\infty}^{u=\infty} \int_{x=-\infty}^{x=\infty} \mathsf{g}(u)\, \mathsf{h}(x-u)\, \mathsf{e}^{-\mathsf{i}\mathsf{k}x}\, \mathsf{d}x\, \mathsf{d}u$$

But

$y = x - u$
$\mathsf{d}y = \mathsf{d}x$

$$\int_{x=-\infty}^{x=\infty} \mathsf{h}(x-u)\, \mathsf{e}^{-\mathsf{i}\mathsf{k}x}\, \mathsf{d}x = \int_{y=-\infty}^{y=\infty} \mathsf{h}(y)\, \mathsf{e}^{-\mathsf{i}\mathsf{k}(u+y)}\, \mathsf{d}y = \mathsf{e}^{-\mathsf{i}\mathsf{k}u} \underbrace{\int_{-\infty}^{\infty} \mathsf{h}(y)\, \mathsf{e}^{-\mathsf{i}\mathsf{k}y}\, \mathsf{d}y}_{\mathsf{H}(\mathsf{k})}$$

$$\therefore \quad \mathsf{F}(\mathsf{k}) = \mathsf{H}(\mathsf{k}) \int_{u=-\infty}^{u=\infty} \mathsf{g}(u)\, \mathsf{e}^{-\mathsf{i}\mathsf{k}u}\, \mathsf{d}u = \underline{\mathsf{G}(\mathsf{k}) \times \mathsf{H}(\mathsf{k})}$$

The proportionality mentioned in the question is actually an equality for the definition of the Fourier transform used here; other conventions involve factors of 2π. The reciprocity between a Fourier transform and its inverse means that

$$\int_{-\infty}^{\infty} \big[\mathsf{g}(x) \times \mathsf{h}(x)\big]\, \mathsf{e}^{-\mathsf{i}\mathsf{k}x}\, \mathsf{d}x = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathsf{G}(u)\, \mathsf{H}(k-u)\, \mathsf{d}u \propto \mathsf{G}(\mathsf{k}) \otimes \mathsf{H}(\mathsf{k})$$