7 Complex numbers

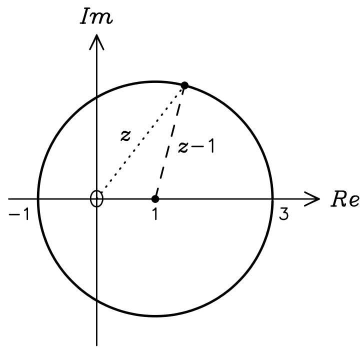
(1) Sketch the locus of points $z = x + iy$ in the Argand plane that satisfy

(a) $|z-1| = 2$, (b) $\left|\frac{z-4}{z-1}\right| = 1$ and (c) $\arg\left(\frac{z-4}{z-1}\right) = \frac{\pi}{2}$

(a) $z - 1 = (x-1) + iy$

$\therefore\ |z-1|^2 = (x-1)^2 + y^2 = 2^2$

i.e. A circle of radius 2 and centre $z = 1$, or $(1, 0)$



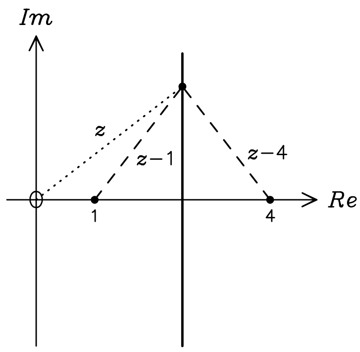
(b) $\left|\frac{z-4}{z-1}\right| = \frac{|z-4|}{|z-1|} = 1$

$\therefore\ |z-4|^2 = |z-1|^2$

$\therefore\ (x-4)^2 + y^2 = (x-1)^2 + y^2$

$\therefore\ 16 - 8x = 1 - 2x$

i.e. The straight line $x = \mathcal{R}e\{z\} = \frac{5}{2}$



(c) Let $w = \dfrac{z-4}{z-1} = \dfrac{(z-4)(z-1)^*}{|z-1|^2}$

$$= \frac{\big[(x-4)+\mathrm{i}y\big]\big[(x-1)-\mathrm{i}y\big]}{(x-1)^2+y^2}$$

$$\therefore\ w = \frac{(x^2+y^2-5x+4)+\mathrm{i}3y}{(x-1)^2+y^2}$$

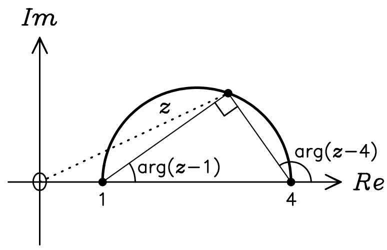
$$\arg\left(\frac{z-4}{z-1}\right) = \arg(w) = \frac{\pi}{2} \quad \Rightarrow \quad \mathcal{R}e\{w\}=0 \quad \text{and} \quad \mathcal{I}m\{w\}>0$$

$$\therefore\ x^2+y^2-5x+4=0 \quad \text{and} \quad y>0$$

$$\text{i.e.}\ \left(x-\tfrac{5}{2}\right)^2+y^2=\left(\tfrac{3}{2}\right)^2 \quad \text{and} \quad y>0$$

$\arg\left(\dfrac{z-4}{z-1}\right) = \arg(z-4) - \arg(z-1)$

A semi-circle with $\mathcal{I}m\{z\}>0$ of radius $\frac{3}{2}$ and centre $z=\frac{5}{2}$



If the value of the given argument had been $3\pi/2$ (radians), instead of $\pi/2$, the locus of z would have been the lower half of the same circle.