1 Basic algebra and arithmetic

(1) The pH scale of acidity is a logarithmic measure of the (molar) concentration of hydrogen ions present in an aqueous solution: $\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$. How many times more concentrated are the H^+ ions in a solution of pH 1 compared to a solution of pH 6?

$$\mathrm{pH} = -\log_{10}\left[\mathrm{H}^+\right] \iff \left[\mathrm{H}^+\right] = 10^{-\mathrm{pH}}$$

Denoting the concentration of H^+ ions in a solution with pH x as $[\mathrm{H}^+]_x$

$$\frac{[\mathrm{H}^+]_1}{[\mathrm{H}^+]_6} = \frac{10^{-1}}{10^{-6}} = 10^{-1}\,10^{6} = 10^{-1+6} = \underline{10^{5}}$$

$10^5 = 100000$

(2) By letting $A = a^{\mathrm{M}}$ and $B = a^{\mathrm{N}}$, and using the definition of a logarithm, show that

$$\log(AB) = \log(A) + \log(B) \quad \text{and} \quad \log(A/B) = \log(A) - \log(B)$$

Similarly, show that

$$\log(A^{\beta}) = \beta\log(A) \quad \text{and} \quad \log_b(A) = \log_a(A) \times \log_b(a)$$

$$A = a^{\mathrm{M}} \iff \mathrm{M} = \log_a(A)$$

$$B = a^{\mathrm{N}} \iff \mathrm{N} = \log_a(B)$$

But $AB = a^{\mathrm{M}}\,a^{\mathrm{N}} = a^{\mathrm{M+N}}$

$$\therefore \quad \log_a(AB) = \log_a(a^{\mathrm{M+N}}) = \mathrm{M} + \mathrm{N} = \log_a(A) + \log_a(B)$$

This result holds for logarithms to any base, because we did not specify the value of a above.

i.e. $\underline{\log(AB) = \log(A) + \log(B)}$

$$\frac{1}{B} = \frac{1}{a^{\mathrm{N}}} = a^{-\mathrm{N}} \iff \log_a\left(\frac{1}{B}\right) = \log_a(a^{-\mathrm{N}}) = -\mathrm{N} = -\log_a(B)$$

Hence, using the previous result for the logarithm of the product of A and $1/B$, we obtain

$$\underline{\log(A/B) = \log(A) - \log(B)}$$

$$A^{\beta} = (a^{\mathrm{M}})^{\beta} = a^{\mathrm{M}\beta} \iff \log_a(A^{\beta}) = \log_a(a^{\mathrm{M}\beta}) = \mathrm{M}\beta = \beta\log_a(A)$$

$$\text{i.e.} \quad \underline{\log(A^{\beta}) = \beta\log(A)}$$

$$\log_b(A) = \log_b(a^{\mathrm{M}}) = \mathrm{M}\,\log_b(a) \implies \underline{\log_b(A) = \log_a(A) \times \log_b(a)}$$

(3) Derive the formula for the two solutions of the quadratic equation $ax^2 + bx + c = 0$.

$$\text{If } a \neq 0, \qquad x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}\left(\frac{4a}{4a}\right) = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

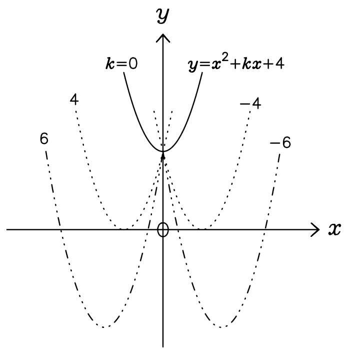
$$\text{i.e.} \quad \underline{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

While this formula is only valid for $a \neq 0$, the case of $a = 0$ is even simpler as we then just have $bx + c = 0$. This linear equation has the solution $x = -c/b$.

(4) For what values of k does $x^2 + kx + 4 = 0$ have real roots?

Using the general formula for the roots of a quadratic, which was derived in the previous question

$$x = \frac{-k \pm \sqrt{k^2 - 16}}{2}$$



For the values of x to be real, the quantity to be square rooted ($b^2 - 4ac$) must not be negative. Hence, we require that $k^2 \geqslant 16$.

$$\therefore \quad \underline{|k| \geqslant 4}$$

In other words, k has to be less than -4 or greater than $+4$.

(5) Derive the formulae for the sums of an arithmetic progression and a geometric progression.

$$\text{Let} \quad a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l = S_N \quad — (1)$$

$$\text{where} \quad l = a + (N-1)d$$

$$\therefore \quad l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a = S_N \quad — (2)$$

$$(1) + (2) \Rightarrow \quad (a+l) + (a+l) + \cdots + (a+l) + (a+l) = 2S_N$$

$$\therefore \quad N(a+l) = N\big[2a + (N-1)d\big] = 2S_N$$

$$\therefore \quad S_N = \underline{\text{Sum of AP} = \frac{N}{2}\big[2a + (N-1)d\big]}$$

$$\text{Let} \quad a + ar + ar^2 + \cdots + ar^{N-2} + ar^{N-1} = S_N \quad — (3)$$

$$r \times (3) \Rightarrow \quad ar + ar^2 + \cdots + ar^{N-2} + ar^{N-1} + ar^N = rS_N \quad — (4)$$

$$(3) - (4) \Rightarrow \quad a(1-r^N) = (1-r)S_N$$

$$\therefore \quad S_N = \underline{\text{Sum of GP} = \frac{a(1-r^N)}{1-r}}$$

(6) By expressing a recurring decimal number as the sum of an infinite GP, show that $0.121212\cdots = 4/33$. What is $0.3181818\cdots$ as a fraction?

$$\begin{aligned} 0.12121212\cdots &= 0.12 + 0.0012 + 0.000012 + \cdots \\ &= \text{sum of infinite GP with } a = 0.12 \text{ and } r = 0.01 \\ &= \frac{0.12}{1-0.01} \\ &= \frac{12}{99} \end{aligned}$$

$$\therefore\quad \underline{0.12121212\cdots = 4/33}$$

$$\begin{aligned} 0.318181818\cdots &= 0.3 + 0.018 + 0.00018 + 0.0000018 + \cdots \\ &= \frac{3}{10} + \frac{0.018}{1-0.01} \\ &= \frac{3}{10} + \frac{18}{990} \\ &= \frac{33}{110} + \frac{2}{110} = \frac{35}{110} \end{aligned}$$

$$\therefore\quad \underline{0.318181818\cdots = 7/22}$$