Supplementary Section 7S.9

Infinity

As we saw briefly in section 1.3, modern mathematical logic of the sort studied in *Introduction to Formal Logic with Philosophical Applications* was developed largely in response to some odd and unsettling results in mathematics. One important source of the pressure to refine the concept of logical consequence came from the development of non-Euclidean geometries. But perhaps more important, mathematicians were working with concepts of infinity in new and robust ways which demanded grounding, especially as they led to strange new results and paradoxes.

LOGIC, INFINITY, AND PARADOX

The calculus developed by Leibniz and Newton in the seventeenth and eighteenth centuries was both wildly successful and oddly unnerving. Its central technique involves finding the area under a curve by dividing the curve into infinitely many, infinitely small areas, called infinitesimals by Leibniz and fluxions by Newton, and then adding these infinitely many areas together. On the one hand, the results were precise, perfect, and widely applied in science and mathematics. On the other hand, the idea of adding an infinite number of infinitely small areas seemed preposterous to some mathematicians and philosophers, more so when the results often turned out to be small finite numbers. How can the sum of an infinite number of infinitely small quantities be $\sqrt{7}/2$ or -3? Infinity was supposed to be the realm of God and paradoxes, not productive mathematical methods.

Among the most unsettling results that led Frege and others to seek more secure systems of inference were those of Georg Cantor that showed that there are different sizes of infinity, indeed infinitely many different sizes of infinity. Until the mid-nineteenth century, the infinite was a concept perhaps of more interest to philosophers than to mathematicians. Earlier mathematicians certainly knew about a variety of concepts of mathematical infinity. There were large infinities of addition, like the infinity of the counting numbers, the dense infinity of the rational numbers and the continuous infinity of the real numbers, and the infinity of space. There were small infinities, too, such as the number of points in a finite line segment or the infinitesimals or fluxions used in calculus.

But infinities led to paradoxes. Among the oldest and most influential of the problems are known as Zeno's paradoxes. Little is known of the eponymous Zeno of Elea, who lived in the fifth century B.C.E., beyond that his paradoxes are intended to support the claims of the philosopher Parmenides that reality is one, uniform, and unchanging. While we seem to experience a complex, variegated, and changing world, Parmenides claimed that the real world is stable and constant, unlike the world we perceive. Zeno's paradoxes, invoking the infinite divisibility of space and time, seem to show an error in our beliefs about a changing world.

For example, consider the famous paradox of the racing Achilles and the tortoise. Achilles gives the tortoise a head start, let's say one hundred feet. But now, Achilles can never catch the tortoise. For while he runs the hundred feet initially separating the pair, the tortoise is also in motion, though more slowly than Achilles. When Achilles reaches the tortoise's starting point, the tortoise will have moved, say, ten feet farther. And when Achilles reaches 110 feet, the tortoise will have again moved farther, another foot, say. As Achilles moves one foot farther toward the tortoise, the tortoise is once again a little bit farther along. This goes on infinitely: no matter how many times Achilles reaches a given point formerly occupied by the tortoise, the tortoise will have moved a little farther. Achilles can never catch the tortoise.

Or consider the paradox of the arrow, which assumes that time is composed of atomic instants, ones that cannot be further subdivided. The arc traced by a flying arrow consists of some number of these instants. Consider any one of these instants, and ask whether the arrow is moving at that instant. If the arrow is in motion at the instant, then it must be at one place at the beginning of the instant and at another distinct place at the end of that instant. But then there seem to be parts of an instant, its beginning and its end, contrary to our assumption that time consists of atomic instants. Hence the arrow cannot be in motion at any instant. But the flight of the arrow consists of the sum of its motions at each instant. Since it does not move at any instant, the sum of these instants is zero: the arrow does not move.

Mathematicians and philosophers dealt with the paradoxes largely by constructing some distinctions, between actual and potential infinities, for example, and between categorematic and syncategorematic uses of 'infinity'. The distinction between actual and potential infinity is found in Aristotle's work from the fourth century B.C.E. Aristotle claims that the Achilles paradox, for example, is solved by the observation that Achilles need not traverse an actual infinite series of distances, which would be impossible. Instead, the infinite number of distances is only potentially infinite. We don't actually divide space in the way that Zeno presumes. Thus, the paradox is merely potential and unproblematic. Similarly, if time is not actually infinitely divisible, but only potentially so, the arrow can fly.

In the thirteenth century, the terms 'categorematic' and 'syncategorematic' were introduced to distinguish ways of speaking about the infinite. When one speaks of an infinite quantity as if it actually exists, as when one says that there are infinitely many points on a line, one speaks categorematically and dangerously. But when one says that a line can be extended, potentially, indefinitely, one speaks syncategorematically. Again, uses of this distinction were invoked to avoid accidentally saying something paradoxical or unacceptable about actual infinities.

So mathematicians and philosophers mainly avoided invoking infinities as much as possible, relinquishing their resistance in the case of the successful calculus, but often with guilty consciences, and in the face of severe criticism. For example, the philosopher Bishop George Berkeley wrote a treatise in the eighteenth century, *The Analyst*, in which he accused the proponents of the calculus of basing their work on fundamental errors about the nature of space. It wasn't until the nineteenth century, when Dedekind, Weierstrass, and others arithmetized analysis by showing how to define limits more precisely, that the calculus was seen to be put on firm footing.

In the mid-nineteenth century, though, the mathematician Georg Cantor constructed a startling and influential proof that there are different sizes of infinity. This proof changed the way philosophers and mathematicians thought about and worked with infinity, introducing us to what the late nineteenth-century mathematician David Hilbert called Cantor's paradise of infinitary mathematics.

To get a feel for the different sizes of infinity, we will consider a now-classic concept that is sometimes called the infinite hotel.

THE INFINITE HOTEL

You are the desk clerk at an infinite hotel that has, let us suppose, infinitely many rooms. The hotel is fully booked when a new guest arrives. In a finite hotel, you would have to turn away the potential new guest. But in an infinite hotel, you can add the new guest. To do so, shift every current guest from room n to room $n + 1$: the guest in room 2 moves to room 3; the guest who was in room 3 moves to room 4; the guest from room 4 moves to room 5; and so on. Now room 1 is available for the arriving guest.

If a further finite group of guests arrives, you can perform the same procedure to free up any finite number of rooms. Just add any finite number of guests, m, by shifting all current guests from their current room n to room $n + m$ and putting the new guests in the first m rooms. If seven guests arrive, for example, move all the current guests to rooms with numbers exactly seven greater than their current rooms. If a billion guests arrive, just move them to rooms with numbers a billion greater than the ones they are in currently. Then slot the new guests into the newly vacant rooms with numbers at the beginning of the natural number sequence.

Next, a bus with an infinite number of guests arrives. If you try to shift all guests from room n to room (n + the number of guests on the bus), you have to move the guest in room 2, say, to room 2 + infinity. But since there is no number 'infinity' (or so one might think) you do not know where to put the current guests.

You can still accommodate an infinite number of new guests, but you have to use a new procedure. Shift every current guest from room n to room $2n$. The guest in room 2 moves to room 4, the guest who was in room 3 moves to room 6, and so on. Now, all the even-numbered rooms are filled, and the odd-numbered rooms are vacant. We can put the new guests in the odd-numbered rooms: room 1, room 3, room 5, and so on.

Next, an infinite number of infinite busloads of guests arrives. You can still accommodate them, but again you need a different procedure. Shift all current guests from room n to room 2^n. So the person in room 2 stays in room 2^1 (i.e., room 2); the person who was in room 2 moves to room 2^2 (i.e., room 4); the person who was in room 3 moves to room 2^3 (i.e., room 8); the person who was in room 4 moves to room 2^4 (i.e., room 16); and so on. All of the present guests can be accommodated in the infinite number of rooms that are powers of two, leaving lots of empty rooms. We can place the people on the first bus in room numbers 3^n (for n people on the bus), the people in the second bus in rooms 5^n, the people in the third bus to rooms 7^n, and so on for each (prime number)^n. Since there are an infinite number of prime numbers, there will be an infinite number of infinite such sequences. And there will still be lots of empty rooms left over!

A natural question to arise is whether there are any sets of guests that the infinite hotel could not accommodate. This question is precisely a question about the fine structure of the numbers and whether there are different sizes of infinity.

TWO CONCEPTS OF SIZE

Numbers have at least two different functions: measuring the size of a collection of things; and ordering, or ranking, a series. When we use numbers to measure size, we use the property of the numbers called cardinality. When we use them to measure rank (first, second, third . . .), we use the property called ordinality. Mathematicians sometimes consider the numbers in their different uses as different objects altogether. Thus we have cardinal numbers and ordinal numbers.

One way to characterize cardinal numbers is to invoke one-one correspondence. Consider a basket of apples and a classroom of hungry kindergartners. We can determine whether there are the same number of apples and kindergarteners by giving each student exactly one apple. If there are no extra apples or children, there are (or were, since the kids were hungry) the same number of each. The view that we can define numbers in terms of one-one correspondence has become known as Hume's principle, though the use of one-one correspondence to measure size precedes Hume.

Another way to think about numbers, perhaps more closely related to their ordinal properties, is in terms of wholes and parts: $a > b$ if, and only if, there is some positive number c such that $b + c = a$.

With finite numbers, the characterization in terms of one-one correspondence converges with the characterization in terms of parts and wholes. The size of a group is the same as the correspondence between the objects in the group and some initial segment of the natural numbers. If we have five hungry students, we can line them up (ordinally) and give them each a number from one to five (cardinally).

But these two concepts diverge with infinite collections. The size of the integers seems to be bigger than the size of the even numbers since the size of a whole seems to be greater than the size of its proper part and the even numbers are a proper part of the integers. But, the even numbers (E) and the integers (N) can be put into one-one correspondence with each other.

```
E: 2, 4, 6, 8 . . .
   ↕  ↕  ↕  ↕
N: 1, 2, 3, 4 . . .
```

Let's give names to these different concepts of size. Two sets have the same $size_h$ (for Hume) if they can be put in one-one correspondence with each other. Two sets have the same $size_w$ (for the whole is greater than the sum of its parts) if it is not possible to put either in one-one correspondence with a proper part of itself. So, N and E have the same size_h but different size_ws.

You might think, and before Cantor's work in the mid-nineteenth century it was widely believed, that there is just one size of infinity, that all infinities have the same size. That claim turns out to be false. Moreover, size_h has come to be recognized as the central notion of cardinality. Parallel conclusions can be drawn about infinite ordinal numbers. There are many, indeed infinitely many, different infinite numbers. These are the conclusions of Cantor's influential diagonal argument, one of the most important intellectual discoveries of all time.

CANTOR'S DIAGONAL ARGUMENT

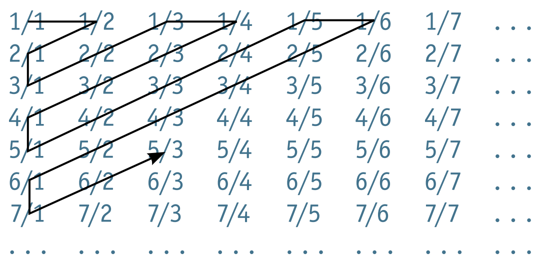
When we make a list, we put objects into one-one correspondence with the natural numbers: item 1, item 2, item 3, and so on. Any infinite list will thus be the same size_h as the natural numbers. For example, we can show how to list the even numbers, as we showed in the previous section; the set of even numbers is the same size_h as the set of integers.

For another example, we can list the prime numbers. The set of primes, P, like the set of even numbers, is a proper subset of the set of natural numbers. But again, it is an infinite set the same size_h as the natural numbers, which we can show by putting the primes in a list.

```
P: 2, 3, 5, 7, 11, 13 . . .
   ↕  ↕  ↕  ↕  ↕   ↕
N: 1, 2, 3, 4,  5,  6, . . .
```

By listing the even numbers, or the odd numbers, or the multiples of seven, or the prime numbers, we are showing that such sets of numbers have the same infinite cardinality as the set of natural numbers, despite being proper subsets of the natural numbers. It is characteristic of an infinite set that it can be put into one-one correspondence with a proper subset of itself.

So both the set of prime numbers and the set of even numbers are the same size$_h$ as the natural numbers while being a smaller size$_w$. Other sets have a size$_w$ larger than the natural numbers while having the same size$_h$. Consider the rational numbers, all ordered pairs of natural numbers. We often call the rational numbers fractions, taking the first of the ordered pair as the numerator and the second as the denominator. There seem to be more rationals than natural numbers. Between every two natural numbers there are many rational numbers, though the reverse is not true. So the rationals have a larger size$_w$. But using a neat trick, we can make a list of the rationals too, showing that they can be put into one-one correspondence with the natural numbers and thus that the natural numbers and the rational numbers have the same size$_h$. Just follow the path indicated by the arrows in the diagram below to construct a complete (if sometimes redundant) list.



Note that while this neat technique traces a path through the diagram which is sometimes diagonal, it is *not* what we call the diagonal argument.

Such constructions may tempt us to think that any set of numbers can be listed and thus that all sets of numbers have the same size$_h$. But if there were some kinds of sets whose members could not be put into a list, then that set would be strictly larger than the set of natural numbers, in both size$_h$ and size$_w$. There would be different sizes of infinity, however we measure size.

Cantor shows that we indeed cannot make certain lists. In terms of the infinite hotel, he shows that there are sets of guests that could not be accommodated. In general, Cantor shows how to construct sets of larger and larger size$_h$s. In particular, his diagonal argument proves that we cannot list the real numbers.

There are different versions of the diagonal argument, and it can be applied to both numbers and, perhaps more generally, to sets. Let's take a look at the argument as it applies to the real numbers. The real numbers may be represented as their decimal expansions, many of which are non-repeating and non-terminating. The structure of the argument is a *reductio ad absurdum*, or indirect proof. We start by supposing, contrary to our desired conclusion, that we can make a list of all of the real numbers. For simplicity's sake, let's imagine that we can list all of the real numbers between zero and one; it turns out that we can't list even these reals. Each such real, in its decimal representation, will consist of a 0, a decimal point, and a sequence (perhaps terminating or repeating, and perhaps not) of natural numbers. We can ignore the zero and the decimal point and just look at the sequence of digits past the decimal point.

So we can write, we are supposing for *reductio*, a complete list of such sequences. Let's represent that hypothetical list, L, abstractly, using a concatenation of variables.

$$
\begin{array}{ll}
L & a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ \ldots \\
 & b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ \ldots \\
 & c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7\ \ldots \\
 & d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7\ \ldots \\
 & \ldots
\end{array}
$$

So, for example, '$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7 \ldots$' could represent '3756920 . . .', which would stand for the real number whose decimal expansion starts 0.3756920 . . .

By hypothesis, L contains the decimal extensions of all real numbers between 0 and 1. Cantor's diagonal technique allows us to find a number between 0 and 1 that does not, in principle, appear in L, contradicting our assumption that L is a complete list.

Consider the number C, defined by concatenating one term from each of the numbers in the list L. We select the number by looking the diagonal of L, taking the first term from the first number in the list, the second term from the second number, and so on.

$$C = a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7\ \ldots$$

C could be in L. It has the same first number as the first number in the list, the same second number as the second number in the list, and so on.

But, given C, we can construct a new number that cannot be on the list, showing the list to be incomplete. Just change each digit in C to create a new number C*. For instance, to construct C*, we can add one to each digit of C other than nine, and replace all nines in C with zeroes.

Now, C* is certainly not in L. C* is different from the first number in L in its first digit, different from the second number in L in its second digit, and so on, for all numbers on the list.

In a quixotic attempt to complete the list, we could add C* to L, to make a new list, L*. But the same procedure allows us to form a new number, say C**, that's not in L*. However complete we make our list, we can always find a number that is not in it.

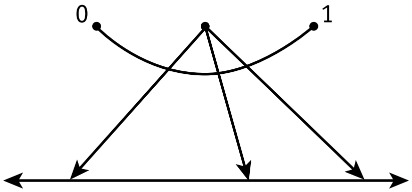
Thus, all possible lists of real numbers are necessarily incomplete. We are in principle prevented from establishing a one-one correspondence between the natural numbers and the real numbers. There are strictly more real numbers than natural numbers.

The preceding proof is called a diagonal argument, due to its method of producing C* along the diagonal of the list.

INTO THE TRANSFINITE AND CANTOR'S THEOREM

With Cantor, let's call the size of the natural numbers $\aleph_0$. Since the real numbers between zero and one have a strictly larger size than $\aleph_0$, we can say that the set of reals between zero and one has a size greater than $\aleph_0$. Just as the set of natural numbers contains many proper subsets with the size $\aleph_0$, the set of real numbers has many subsets of its greater size. To see this, first remember that we use the real numbers as the representations of all of the points on a line. We can show that there are the same number of real numbers overall as there are real numbers between zero and one by providing a mapping between the real numbers (points on a line) between zero and one and all the real numbers (points).

Here is a geometric demonstration.



For each point on the curved line between zero and one, we can find a point on the infinite line, and vice versa. Since there is a one-one mapping between the two lines, there are the same number of points in each line.

If you prefer an analytic proof, take $f(x) = \tan(2x-1)\pi/2$ from $x=0$ to $x=1$. The tangent curve ranges between negative infinity and positive infinity over a domain between zero and one. Thus we can correlate each point on a small segment of the x-axis with a unique real number on the whole y-axis, and vice versa. There is a one-one correspondence between the real numbers between 0 and 1 and all of the real numbers.

We have seen now that infinity is at least more complicated than was thought prior to the nineteenth century. There are at least two different sizes of infinity, even in terms of size$_h$. It turns out that there are in fact infinitely many different sizes of infinity, since a generalized version of the diagonal argument can be run on any set of any size. For any set, there is another set of greater size.

To get a feel for the properties of infinite numbers, let's take a look at some properties of numbers and whether they hold just for finite numbers or for infinite numbers as well. I won't prove that these properties hold here.

The familiar properties at 7S.9.1. hold for all cardinal numbers, whether finite or transfinite.

7S.9.1 For all cardinal numbers *a*, *b*, and *c*:

$a + b = b + a$
$ab = ba$
$a + (b + c) = (a + b) + c$
$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
$a \cdot (b + c) = ab + ac$
$a^{(b + c)} = a^b \cdot a^c$
$(ab)^c = a^c \cdot b^c$
$(a^b)^c = a^{bc}$

Not all properties of finite numbers hold for infinite numbers. We already saw one property that holds only for infinite sets, the property of having a proper subset which is the same size_h as itself. The properties at 7S.9.2 hold for infinite numbers, but not for finite numbers.

7S.9.2 For infinite cardinals a:

$$a + 1 = a$$
$$2a = a$$
$$a \cdot a = a$$

We can show the three claims at 7S.9.2 by producing bijective (one-one) mappings between sets of each size. That's what we did in the discussion of the infinite hotel.

Consider one final important property, 7S.9.3, which holds both of finite and transfinite numbers.

7S.9.3 $2^a > a$

Whether a is finite or infinite, 2^a will always be a number with a larger cardinality. 7S.9.3 has an analog in set theory: the power set (or set of all subsets) of a set is always strictly larger than the given set. The set-theoretic claim is called Cantor's theorem, and its proof is a set-theoretic version of the diagonal argument, which I'll leave for an appendix to this section.

Given Cantor's theorem, which shows that there are infinitely many different sizes of infinity, we can start naming the infinite numbers of differing cardinalities, proceeding beyond the sequence of natural numbers. We can define a sequence of infinite cardinalities:

$$\aleph_0,\ \aleph_1,\ \aleph_2,\ \aleph_3,\ \aleph_4 \ldots$$

While there are an infinite number of infinite cardinalities on this list, set theorists, by various ingenious methods including the addition of axioms that neither follow from nor contradict the standard axioms, generate even larger cardinals than these. Cardinal counting gets pretty wild. There are ethereal cardinals, subtle cardinals, almost ineffable cardinals, totally ineffable cardinals, remarkable cardinals, superstrong cardinals, and superhuge cardinals, among many others. All of these cardinal numbers are transfinite and larger than any of the sequence of alephs.

Returning to just the alephs, the natural numbers have the size $\aleph_0$. We saw that the rational numbers have the same cardinality. There are more real numbers than natural numbers, as we saw in the diagonal argument. But how many reals are there? Are the reals the next largest size of infinity, $\aleph_1$, or are there other sizes of infinity between the size of the natural numbers and the size of the real numbers? Cantor's continuum hypothesis is the claim that the reals are of size $\aleph_1$, but the size of the real numbers is one of the most interesting open questions in mathematics.

THE CONTINUUM HYPOTHESIS

Certain questions in the history of mathematics have proven difficult to answer. Fermat's theorem, that there are no n for which there are a, b, and c, such that $a^n + b^n = c^n$, was conjectured in 1637, written in the margin of Fermat's copy of Diophantus's *Arithmetica*. It was proven in 1994. Goldbach's conjecture, that every even number greater than four can be written as the sum of two odd primes, remains unproven, though most mathematicians believe that it is also true.

Euclid's parallel postulate, which we saw in section 1.3 of *IFLPA*, is more interesting. It can fail, as it does in non-Euclidean spaces. But it can also hold, as it does in Euclidean space. Thus, we have decided that the question whether the parallel postulate is true is, strictly speaking, ill-formed. There is no one true answer. There are different kinds of spaces, each defined by a different version of the parallel postulate.

Cantor's continuum hypothesis is interesting in part because we do not know if it is like Fermat's theorem or Goldbach's conjecture, in having a solution, or whether it is ill-formed, like the parallel postulate. Cantor provided a method for generating larger and larger transfinite numbers. He shows that the cardinal number of the reals is equal to $2^{\aleph_0}$. He also shows that $2^{\aleph_0}$ is greater than $\aleph_0$. Cantor's theorem does not show, however, that it is the next greater transfinite number. Let's take $\aleph_1$ to be the name we give to the next transfinite cardinal after $\aleph_0$. The continuum hypothesis is that $\aleph_1 = 2^{\aleph_0}$. The generalized continuum hypothesis is that $\aleph_{n+1} = 2^{\aleph_n}$.

To understand how the continuum hypothesis might be false, remember that certain operations on finite numbers that generate larger numbers, like exponentiation, skip numbers. When we multiply a finite cardinal by two or seventeen, or add six, or raise to the π power, we generate cardinals that are not merely one larger. Only the successor function yields the next natural number. So it seems possible that 2^n, for transfinite n, is more like ordinary exponentiation in skipping some transfinite numbers, rather than like succession, which gives the next largest number. In other words, we do not know that $2^{\aleph_0}$ is $\aleph_1$. Indeed, we do not even know that transfinite cardinal numbers can be ordered linearly.

At the 1900 Paris Congress, David Hilbert cited the continuum hypothesis as one of the ten most important unsolved problems in mathematics. Cantor believed the continuum hypothesis, but he could not prove it. Two results in the twentieth century further entrenched the problem. In 1940, Kurt Gödel showed that the continuum hypothesis is consistent with the standard axioms of set theory. But in 1963, Paul Cohen showed that its negation is also consistent with set theory. Thus, the continuum hypothesis is independent of the standard axioms. We can consistently consider the continuum to be of all different sizes: $\aleph_1$, $\aleph_2$, $\aleph_3$, and so on. Even the additions of many large cardinal axioms to the standard axioms of set theory fail to settle the question.

But we can settle the question of the size of the continuum by adopting some stronger axioms for set theory. Some mathematicians believe that the continuum hypothesis, even the generalized version, is so intuitively true that we should just adopt it, or an equivalent, as part of set theory. As we will see, Gödel favored this approach. Alternatively, we could take the question to be ill-formed, like the question of whether the parallel postulate is true. Perhaps there are different set theories, with different sizes of the continuum. Mathematicians are divided on whether the continuum hypothesis is true, though opinion has generally turned against it.

ORDINALS AND COUNTING

As we saw above, we use cardinal numbers to measure size, while we use ordinal numbers for ranking; first, second, third, and so on. Cantor defined cardinal numbers in terms of ordinal numbers, making the ordinals more fundamental. Frege's attempt to define the numbers followed Cantor's work; Frege sought independent definitions of the ordinals and cardinals. Remember, the development of modern logic, like the logic in the first five chapters of *Introduction to Formal Logic with Philosophical Applications*, was in the service of Frege's project in the foundations of arithmetic. To finish this section, let's look briefly at the ordinal numbers, and at how we can define arithmetic by using the more general set theory.

Cantor developed set theory in order to generate his theory of transfinites. Frege assumed a similar set theory in his work. Despite some differences, Cantor and Frege both used inconsistent set theories, which we now call naive set theory for its assumption that any property determines a set. The inconsistency was discovered by Bertrand Russell; it is called Russell's paradox. Russell's paradox shows that some properties taken to define sets lead to contradictions.

To understand Russell's paradox, one needs a little understanding of set theory. Sets are collections that have members, the items in the set. They can include other sets as members. Let's say that there are seventy million pet dogs in the United States and ninety million pet cats. So there's a set of pet dogs, D, with seventy million members, and a set of pet cats, C, with ninety million members. There's also a set of pet dogs or cats, B, with 160 million members. And there's a set, let's call it P, with just two members, both sets: the set of dogs and the set of cats. 7S.9.4 lists all four of these sets.

7S.9.4 D: {all the pet dogs in the United States}
C: {all the pet cats in the United States}
B: {anything which is either a pet dog or a pet cat in the United States}
P: {{the pet dogs}, {the pet cats}}

The dogs and cats are not members of the set P. They are members of the two sets that are members of this set.

Since sets can have other sets as members, as P does, they might even have themselves as members. Consider a possible set of all sets. Since it is itself a set, it should be a member of itself. It turns out that this possible set is too large actually to be a set, a fact that is a consequence of Russell's paradox.

Russell, presenting the paradox to Frege, inquired about the property of not including oneself. Consider the set of all and only sets that do not include themselves. If it includes itself, then it shouldn't include itself, since it is the set only of sets that do not include themselves. But if it doesn't include itself, then it should include itself since it is the set of all sets that do include themselves. The set of all sets that do not include themselves both includes itself and does not include itself, a contradiction. Naive set theory is inconsistent. Some properties cannot consistently determine sets.

To avoid Russell's paradox, set theory is ordinary presented axiomatically, rather than naively, these days. Instead of assuming that every property determines a set, we start with some simple sets, perhaps just the empty set, and rules for constructing sets, like the axioms of Zermelo-Fraenkel set theory.

Ordinal numbers, set-theoretically, are just special kinds of sets, ones that are well ordered, the definition of which is at 7S.9.5.

7S.9.5 A set A is well ordered by a relation $<$, if for all x, y, and z in A

1. $\sim x < x$
2. $(x < y \cdot y < z) \supset x < z$
3. Either $x < y$, $x = y$, or $y < x$
4. Every nonempty subset of A has a smallest element.

For convenience, we standardly pick a particular ordinal to represent each particular number. We choose one example of a well-ordering for each number and use it as the definition of that number.

To move through the ordinals from smaller to larger, we most often look for the successor of a number, the set that stands for the next ordinal number. Ordinals that can be constructed in this way are called successor ordinals. In transfinite set theory, there are also sets that are called limit elements. We get to them not by finding a successor of a set, but by collecting all the sets we have counted so far into one further set. This operation of collecting several sets into one is called union. If we combine all the sets that correspond to the finite ordinals into a single set, we get another well-ordered set. This new set will be another ordinal: there will be a well-ordering on it, and it will have a minimal element. This limit ordinal will be larger than all of the ordinals in it.

So, there are two kinds of ordinals: successor ordinals and limit ordinals. Limit ordinals are the way in which we jump from considering successors to the next infinite ordinal number. It is like getting to the end of an infinite sequence and jumping to the next level of infinity.

7S.9.6 displays a list of ordinal numbers in order of their sizes.

7S.9.6 $1, 2, 3, \ldots \omega$
$\omega+1, \omega+2, \omega+3 \ldots 2\omega$
$2\omega+1, 2\omega+2, 2\omega+3 \ldots 3\omega$
$4\omega, 5\omega, 6\omega \ldots \omega^2$
$\omega^2, \omega^3, \omega^4 \ldots \omega^\omega$
$\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega, \ldots \varepsilon^0$

The list 7S.9.6 is of ordinals, so by '1', I mean the first ordinal, rather than the cardinal '1'. ω is the first transfinite ordinal, corresponding to the set of natural numbers, the cardinal number $\aleph_0$. The limit ordinals are the ones found after the ellipses on each line, the completions of an infinite series.

Summary

Cantor's theory of transfinites transformed the way we think of infinity. His diagonal argument shows that there are different levels of infinity. We form ordinals to represent the ranks of these different levels of infinity by taking certain series to completion. Completing an infinite series violates the restriction on actual infinity and syncategorematic infinities that blocked Zeno's paradoxes (and others). That such a completion is mathematically consistent and fecund entails that new responses to Zeno are necessary.

The mathematics of infinity has developed robustly in the last century and a half. While some philosophers and mathematicians initially resisted the surprising results, set theorists today work productively on higher transfinites, seeking proper and more full axiomatizations of set theory and even asking whether there are multiple set-theoretic universes. Axiomatic set theory is a vibrant area of contemporary research, at the intersection of logic, mathematics, and philosophy.

TELL ME MORE

- What are the axioms of Zermelo-Fraenkel set theory? See 6S.13: Second-Order Logic and Set Theory.

For Further Research and Writing

1. What is the relationship between modern mathematical logic and theories of infinity? How do Frege's claims in *Begriffsschrift* about his motivations for developing modern logic connect with the theories of infinity?
2. What are Zeno's paradoxes? How does the distinction between actual and potential infinities help to solve those paradoxes? How do the modern solutions to the paradoxes differ from Aristotle's solution? See Marcus and McEvoy as well as Dowden for further discussions.
3. Distinguish size_h from size_w. Is one characterization of size more intuitive than another? Why do mathematicians use size_h to define cardinality? Tiles's *The Philosophy of Set Theory* could be helpful.
4. What is Cantor's diagonal argument? Discuss versions in number theory and in set theory. Presentations in Tiles, Dauben, and Yarnelle are all accessible.

5. What is Cantor's continuum hypothesis? What questions about the structure of the transfinite numbers does it leave open? See the Gödel article and Tiles's discussion of the question.
6. How did Cantor's diagonal argument change the way we think about infinity? Moore's survey will be especially useful here.
7. How does naive set theory differ from axiomatic set theory? Is there a correct axiomatization of set theory? What considerations favor one axiomatization over another? Maddy's dense and fecund paper is an excellent source, as is Kneale and Kneale's book.

Suggested Readings

Dauben, Joseph Warren. *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Princeton, NJ: Princeton University Press, 1979. A biography of Cantor, who discovered the different levels of infinity, and his troubled life.

Dowden, Bradley. "Zeno's Paradoxes." In *The Internet Encyclopedia of Philosophy*, edited by James Fieser and Bradley Dowden. http://www.iep.utm.edu/zeno-par/. Dowden presents the modern solutions to Zeno's paradoxes.

Frege, Gottlob. *Begriffsschrift*. In *From Frege to Gödel*, edited by Jean van Heijenort, 1–82. Cambridge, MA: Harvard University Press, 1982.

Galilei, Galileo. *Dialogues Concerning Two New Sciences*. Translated by Henry Crew and Alfonso de Salvio. New York: Macmillan, (1638) 1914. Galileo's observations about infinity are mainly in the first day of the dialogue.

Gödel, Kurt. 1964. "What Is Cantor's Continuum Problem? (1964)" In Solomon Feferman et al., eds., *Kurt Gödel: Collected Works, Vol. II*, 254–270. New York: Oxford University Press, 1995.

Kneale, William, and Martha Kneale. *The Development of Logic*. Oxford, UK: Clarendon Press, 1962. This classic book contains substantial sections on Cantor and Frege, their set theories, and their treatments of numbers.

Maddy, Penelope. "Believing the Axioms." *Journal of Symbolic Logic* 53, no. 2 (1988): 481–511 and 53, no. 3 (1988): 736–764.

Marcus, Russell, and Mark McEvoy. *An Historical Introduction to the Philosophy of Mathematics*. London: Bloomsbury, 2016. The chapter on Cantor's work is similar to much of what is provided here, but also contains Cantor's original presentation of the diagonal argument. The chapter on Frege's logicism contains his introduction to *Begriffsschrift*. The introductory chapter contains a discussion of axiomatic theories which might also be useful. The chapter on the presocratics has a discussion of Zeno's paradoxes and Aristotle's distinction between actual and potential infinities. The chapter on Hilbert's work is also relevant.

Moore, A. W. *The Infinite*, 2nd ed. London: Routledge, 1990. A wide-ranging, accessible survey on infinity in philosophy and mathematics.

Suri, Gaurav, and Hartosh Singh Bal. *A Certain Ambiguity*. Princeton, NJ, and Oxford, UK: Princeton University Press, 2007. A novel about infinity, mathematics, and religion, with excellent and engaging mathematical asides.

Tiles, Mary. *The Philosophy of Set Theory: An Historical Introduction to Cantor's Paradise*. Mineola, NY: Dover, 2004. An accessible, careful treatment of Cantor's work that puts his transfinite numbers in some excellent context.

Yarnelle, John. *An Introduction to Transfinite Mathematics*. Boston: D. C. Heath, 1964. A clear and accessible introduction to transfinite arithmetic.

APPENDIX

I mentioned that Cantor's theorem is the set-theoretic analogue of the arithmetic claim that $2^a > a$. In set-theoretic terms, this claim is that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$. '$\mathbb{C}(A)$' refers to the cardinality of a set A; $\mathbb{C}$ is the measure of the size of a set. For finite sets, $\mathbb{C}(A)$ is just the number of elements of A. '$\wp(A)$' refers to the power set of A, the set of all subsets of a set A. 7S.9.7 shows two finite sets and their power sets.

7S.9.7 $S_1=\{a, b\}$ $\wp(S_1)=\{\{a\}, \{b\}, \{a, b\}, \varnothing\}$
$S_2=\{2, 4, 6\}$ $\wp(S_2)=\{\{2\}, \{4\}, \{6\}, \{2, 4\}, \{2, 6\}, \{4, 6\}, \{2, 4, 6\}, \varnothing\}$

In general the power set of a set with n elements will have 2^n elements, which is why the number-theoretic claim that $2^n > a\ n$ is the arithmetic correlate of the set-theoretic claim that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$. For infinite n, sets with n members are the same size as sets with $n+1$ members, or with $2n$ members, or with n^2 members, as we saw at 7S.9.2. With infinite numbers, it is not always clear that what we think of as a larger set is in fact larger. We might believe that sets with n members are the same size as sets with 2^n members. This conclusion would be erroneous. $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.

The claim that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$ used to be called Cantor's paradox; it is now called Cantor's theorem. The proof of the theorem is a set-theoretic version of the diagonalization argument. Understanding it requires some familiarity with set theory. Most basically, a set is a collection of objects, a plurality considered as a unit. We can define sets either by listing their elements, or by stating a rule for inclusion in the set. At 7S.9.8, A is defined in the first way; B is defined in the second way.

7S.9.8 A = {Alvin, Simon, Theodore}
B = {x | x is one of the three most popular singing chipmunks}

An element, $\in$, of a set is just one of its members. At 7S.9.9, we see two true claims about elements of the sets defined at 7S.9.8.

7S.9.9 Alvin $\in$ A
Theodore $\in$ B

A subset S of a set A is a set which includes only members of A. If S omits at least one member of A, it is called a proper subset. The set C, defined at 7S.9.10, is a subset of A.

7S.9.10 C = {Alvin, Simon}

We can express the subset relation of C to A as ‘$C \subseteq A$’. C is also a proper subset of A, which means that it is a subset of A while not being identical to A, and which we can write as $C \subset A$.

To prove Cantor’s theorem, we need two more set-theoretic definitions, at 7S.9.11.

7S.9.11 A function is called *one-one* if it every element of the domain maps to a different element of the range: $f(a)=f(b) \supset a=b$

A *function maps a set A onto another set B* if the range of the function is the entire set B, in other words, if no elements of B are left out of the mapping.

To prove Cantor’s theorem, 7S.9.12, we want to show that the cardinality of the power set of a set is strictly larger than the cardinality of the set itself (i.e. $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$). It suffices to show that there is no function which maps A one-one and onto its power set.

7S.9.12 Assume that there is a function $f: A \rightarrow \wp(A)$

Consider the set $B = \{x \mid x \in A \bullet x \notin f(x)\}$
B is a subset of A, since it consists only of members of A.
So, B is an element of $\wp(A)$, by definition of the power set.
That means that B itself is in the range of f.
Since, by assumption, f is one-one and onto, there must be an element of A, b, such that f(b) is B itself.

Is $b \in B$?
If it is, then there is a contradiction, since B is defined only to include sets that are not members of their images.
If it is not, then there is a contradiction, since B should include all elements that are not members of their images.
Either way, we have a contradiction.

So, our assumption fails.
There is no such function $f: A \rightarrow \wp(A)$.
$\wp(A)$ is strictly larger than A.
$\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.

QED