Supplementary Section 6S.11

Axiomatic Systems

Our inferential system is a natural deduction system. Frege's original system, like all systems developed in the early days of modern logic, is different. Frege used what we now call a Hilbert-style system, after the great German mathematician David Hilbert, or an axiomatic system, or just a Hilbert system.

Hilbert systems are more difficult to manage than natural deduction systems. Natural deduction systems, and others, were developed, beginning in the 1930s, to simplify the unwieldy inferences of Hilbert systems and to mirror the way in which mathematicians actually reason. But Hilbert systems have certain advantages. They tend to have very few rules of inference and austere vocabulary. More importantly, Hilbert systems tend to be strictly logical. In systems of natural deduction, some lines in derivations, usually the first ones, are assumptions. They are often contingent formulas and not logical truths. Other propositions that follow from those assumptions are also contingent. We can turn any valid inference of **PL** into a logical truth (or an axiom schema of the metalanguage) by the method in section 3.8. But most inferences do not contain logical truths as lines in the derivation. In contrast, in Hilbert systems every line in a deduction is a provable theorem.

Let's take a moment to consider the differences in rules of inference. Our system **PL** has twenty-five rules of inference and equivalence. The advantage of having so many rules is that our inferences are relatively easy and perspicuous. In contrast, Hilbert systems often contain only two rules and three or four axioms. The axioms may be written as logical truths of the object language or metalinguistic schemas for producing logical truths. In the former case, the system will contain a rule that says that any wff which shares the form of the axiom is also a theorem. In the latter case, which is perhaps more common, the system will have a rule, called substitution, which says that any proposition that results from a substitution of wffs for the metalinguistic variables in the axiom schemas is a theorem. The other rules vary with the vocabulary of the language used. Since such systems often use negation and the material conditional, the other rules often include modus ponens, perhaps as the only other rule.

SYSTEM H

6S.11.1 is a Hilbert-style axiom system for propositional logic, which I'll call **H**, after both Hilbert and Geoffrey Hunter (see Hunter, *Metalogic*) with three axiom schemas, two substitution rules, and one rule of inference.

6S.11.1	H1	$\alpha \supset (\beta \supset \alpha)$
	H2	$[\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)]$
	H3	$(\sim\beta \supset \sim\alpha) \supset (\alpha \supset \beta)$

Axiom substitution (AS): Any wff that results from consistently substituting wffs for the terms in any of the schemas H1–H3 is an axiom.

Theorem substitution (TS): Any wff that results from consistently substituting wffs for each of the wffs in a proven theorem is also a theorem.

Modus ponens (MP): If α and β are any formulas of the language, β may be inferred from α and $\alpha \supset \beta$.

System **H** is equivalent to our familiar natural deduction system. Both Frege's axiomatization and the standard systems of natural deduction are complete, which means that every theorem of **PL** is provable. So the same logical truths that we can prove in **PL** are provable in this system **H**.

Let's look at some derivations in **H** before returning to some more-general comments about Hilbert-style axiom systems for logic.

6S.11.2 is a proof in **H** of the tautology 'P ⊃ P'.

6S.11.2	1. $P \supset [(P \supset P) \supset P]$	H1, AS
	2. $\{P \supset [(P \supset P) \supset P]\} \supset \{[P \supset (P \supset P)] \supset (P \supset P)\}$	H2, AS
	3. $[P \supset (P \supset P)] \supset (P \supset P)$	1, 2, MP
	4. $P \supset (P \supset P)$	H1, AS
	5. $P \supset P$	3, 4, MP

QED

Note that we could apply the same technique to derive any formula of the form $\alpha \supset \alpha$. For example, by substituting '(A ⊃ ~B) ⊃ ~A' for 'P' in the above proof, we can construct a proof of '((A ⊃ ~B) ⊃ ~A) ⊃ ((A ⊃ ~B) ⊃ ~A)'.

Thus, once a theorem has been established, we can invoke theorem substitution (TS) to use it in proofs of further theorems. This result makes working with Hilbert systems more tolerable. Given the proof of 6S.11.2, for example, we can prove the slightly more complicated theorem, '~P ⊃ (P ⊃ P)', at 6S.11.3.

6S.11.3	1. $P \supset P$	6S.11.2, TS
	2. $(P \supset P) \supset (\sim P \supset (P \supset P))$	H1, AS
	3. $\sim P \supset (P \supset P)$	1, 2, MP

QED

6S.11.4 is another proof in **H**, of '$(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$'.

6S.11.4

1.	$[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$	H2, AS
2.	$\{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}\}$	H1, AS
3.	$(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}$	1, 2, MP
4.	$\{(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}\} \supset \{\{(Q \supset R) \supset [P \supset (Q \supset R)]\} \supset \{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\}\}$	H2, AS
5.	$\{(Q \supset R) \supset [P \supset (Q \supset R)]\} \supset \{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\}$	4, 3, MP
6.	$(Q \supset R) \supset [P \supset (Q \supset R)]$	H1, AS
7.	$(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]$	5, 6, MP
8.	$\{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}$	H2, AS
9.	$[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]$	8, 7, MP
10.	$\{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\} \supset \{(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}\}$	H1, AS
11.	$(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}$	10, 9, MP
12.	$\{(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}\} \supset \{\{(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]\} \supset \{(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]\}\}$	H2, AS
13.	$\{(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]\} \supset \{(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]\}$	12, 11, MP
14.	$(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]$	H1, AS
15.	$(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$	13, 14, MP

QED

EXERCISES 6S.11

Prove each of the following four propositions in *H*. You may use TS with the theorems at 6S.11.1, 6S.11.3, and 6S.11.4. You may also use results of the earlier exercises in later exercises. Exercise 3 is especially challenging.

1. $P \supset [Q \supset (P \supset Q)]$
2. $\sim P \supset (P \supset Q)$
3. $[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]$
4. $[P \supset (P \supset Q)] \supset (P \supset Q)$

OTHER AXIOMATIC SYSTEMS

Our system **H** has three axiom schemas and uses a language with just negation and the material conditional. Other systems use different vocabulary. I'll show you two other systems, though we won't work with them. 6S.11.5 is a system we can call **R**, after J. Barkley Rosser, which uses conjunction, negation, and the conditional, and has the rules of axiom substitution and modus ponens.

6S.11.5	R1	$\alpha \supset (\alpha \cdot \alpha)$
	R2	$(\alpha \cdot \beta) \supset \alpha$
	R3	$(\alpha \supset \beta) \supset [{\sim}(\beta \cdot \gamma) \supset {\sim}(\gamma \cdot \alpha)]$

Frege's system for propositional logic, the first fifty-one propositions of *Begriffsschrift*, used six axioms, propositions 1, 2, 8, 28, 31, and 41 of his work. (The *Begriffsschrift* included axioms for quantificational logic, as well as propositional logic.) Let's call Frege's propositional system **B**. The axioms of **B** are at 6S.11.6.

6S.11.6	B1	$\alpha \supset (\beta \supset \alpha)$
	B2	$[\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)]$
	B3	$[\alpha \supset (\beta \supset \gamma)] \supset [\beta \supset (\alpha \supset \gamma)]$
	B4	$(\alpha \supset \beta) \supset ({\sim}\beta \supset {\sim}\alpha)$
	B5	${\sim}{\sim}\alpha \supset \alpha$
	B6	$\alpha \supset {\sim}{\sim}\alpha$

Metalogical proofs, proofs about the systems of logic we discuss, tend to be easier the more simple our object language is: the smaller the vocabulary of a language and the fewer the axioms, the easier it can be to prove things about the system. To this end, we try to formulate the most austere languages we can. One way to achieve austerity is to make sure that our axioms are independent, that no axiom can be proved except by substitution into one axiom schema; it can't be gotten from the other schemas. The axioms of Frege's system **B** are not independent, though the axioms of **R** and **H** are.

Perhaps at the height of austerity for propositional logic is the system **N**, after Jean Nicod, which uses just a single operator, the Sheffer stroke, and a single axiom. (For more on system **N**, other systems, and good references, see Mendelson, section 1.6.)

$$[\alpha \mid (\beta \mid \gamma)] \mid \{[\delta \mid (\delta \mid \delta)] \mid \{(\varepsilon \mid \beta) \mid [(\alpha \mid \varepsilon) \mid (\alpha \mid \varepsilon)]\}\}$$

PREDICATE LOGICS

The differences between working with axiom systems like **H**, **R**, **B**, or **N** and working with natural deduction systems like the ones in the first five chapters of *Introduction to Formal Logic with Philosophical Applications* is remarkable. The amount of logical ingenuity required for even a simple inference in an axiom system is much greater. Still, once you have learned to work with axiom systems for propositional logic, axiom systems for predicate logic are fairly reasonable.

Frege's language for predicate logic includes just a universal quantifier, and his inference system includes just rules of universal instantiation and universal generalization.

This is neither odd nor unique; we can introduce the existential quantifier as a shorthand for '$\sim(\forall\alpha)\sim$' and add the instantiation and generalization rules to systems without them if we wish. Derivations using those rules are similar to the ones we have seen for natural deductions.

Approaches other than Frege's have also been developed, though we will not go any further with them here. See the suggested readings, below, for further references.

AXIOMATICS BEYOND LOGIC

While modern logic began with the development of axiomatic systems, and other systems like natural deduction were developed later, axiomatizing logic is not particularly important. We can easily determine the logical truths of a language by providing a semantics. While metalogical proofs are easier in axiomatic systems than in natural deduction systems, especially where the number of rules grows, and it is always useful to be able to see the minimal assumptions of a theory, systems of propositional and predicate logic are so well defined that the axiom systems tend to be more of a curiosity than a robust source of insights.

The importance of axiom systems in other fields may be different. Mathematicians often see their work as, at heart, merely working out the logical consequences of various, perhaps arbitrary axiomatizations. One's choice of axioms utterly determines the mathematical theorems one proves. While there are other views about the axioms and the nature of mathematics, mathematicians tend to rely heavily, at least in the background, on particular axiomatizations.

Nonlogical theories are usually assumed to be placed within a background logical theory so that they really have two different kinds of axioms: logical and nonlogical (or proper). The proper axioms (of, say, geometry or quantum mechanics) are added to the logical axioms so that there are both rules for inference and contentful claims about the subject matter that do not appear in the logical axioms.

The most prominent and established axiomatic theory is that of Euclid's *Elements*, from ~300 B.C.E. While Euclid did not have contemporary logical machinery, and was much more casual about the nature of logical inference, he still provided a set of mainly logical claims, which he called common notions, as the background structure for geometry.

Euclid's Common Notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

To these common notions, Euclid added five proper geometric axioms.

Euclid's Geometric Axioms

1. To draw a line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which the angles are less than two right angles.

This same structure, background logical axioms and contentful proper axioms, governs axiomatic presentations of mathematical and physical theories today. The main difference is that we are able to be more precise about the nature of inference. Frege's logic and the development of proof theory in the twentieth century were responses to worries about the nature of inference. Frege wanted a gap-free logic, and so specified his rules of inference carefully and syntactically.

To construct formal physical theories, we generally add proper mathematical axioms as well as properly physical ones. The most basic, foundational mathematical theories are set theories (though some recent developments in the foundations of mathematics include even further abstraction to what are called category theories). There are lots of different formulations of set theories, but one of the most prominent is called **ZF**, for Zermelo-Fraenkel set theory. For more on set theory, you might consult the texts from Enderton or Mendelson that are listed in the readings below.

All higher-level mathematical theories, including arithmetic and geometry, can be formulated in the language of set theory. So the most austere axiomatization of mathematics is typically presented in set theory. Still, we can axiomatize higher-level mathematical theories themselves. The standard set of axioms for arithmetic, often credited to Giuseppe Peano, are presented in section 5.6. The Peano postulates may be considered as independent of set theory or translated into the language of set theory. To do the latter, we provide set-theoretic definitions of number. There are various options for defining numbers in set theory. Whether any of those options constitutes a reduction of arithmetic to set theory, and how and whether to choose a particular definition of numbers in terms of sets are all interesting philosophical questions worth exploring.

One of David Hilbert's achievements was his contemporary (1899) axiomatization of geometry, which uses twenty axioms in five different categories; see the suggested readings for references to it. Even more recently, in 1932, George Birkhoff presented a shorter set of axioms, which, unlike Hilbert's, assumes arithmetical notions, like measures of distance.

Birkhoff's Postulates for Geometry

Postulate I: Postulate of Line Measure. The points A, B, . . . of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B - x_A| = d(A,B)$ for all points A and B.

Postulate II: Point-Line Postulate. One and only one straight line l contains two given distinct points P and Q.

Postulate III: Postulate of Angle Measure. The half-lines l, m, . . . through any point O can be put into 1:1 correspondence with the real numbers a(mod 2π) so that if A≠0 and B≠0 are points on l and m, respectively, the difference $a_m - a_l$ (mod 2π) is ∡AOB. Further, if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.

Postulate IV: Postulate of Similarity. If in two triangles ABC and A′B′C′, and for some constant k>0, d(A′, B′)=kd(A, B), d(A′, C′)=kd(A, C) and ∡B′A′C′=±∡BAC, then d(B′, C′)=kd(B,C), ∡C′B′A′=±∡CBA, and ∡A′C′B′=±∡ACB.

We can axiomatize further mathematical theories, for topology, say, or for probability. Beyond mathematical theories, we can axiomatize physical theories, like Newtonian gravitation or quantum mechanics. All such formal theories presume, at least in principle, the language and inferential systems of symbolic logic. They may not presume the classical logics of *Introduction to Formal Logic with Philosophical Applications*, **PL** and **F**. Some people argue that the best logic for mathematics, for example, is second-order. Others argue that we need a three-valued logic to make sense of quantum physics. Such debates provide healthy research projects for contemporary logicians.

AXIOMS AND THEOREMS

Let's end this section on axioms with some brief reflections on a philosophical puzzle about the nature of axiomatizations. One, perhaps old-fashioned, way to think about the relation between axioms of a theory and other theorems is that the axioms are supposed to be obviously known; the theorems are inferences based on axiomatic foundations. It is easy to impose this view of the relation on Euclidean geometry and other axiomatizations. But that simple story does not seem to capture the relation precisely. Our choices of the axioms seem to be based in part on the theorems we can infer from them. Bertrand Russell describes the process neatly, for mathematics, though similar comments could be made for any axiomatic system.

> When pure mathematics is organized as a deductive system—i.e. as the set of all those propositions that can be deduced from an assigned set of premises—it becomes obvious that, if we are to believe in the truth of pure mathematics, it cannot be solely because we believe in the truth of the set of premises. Some of the premises are much less obvious that some of their consequences and are believed chiefly because of their consequences. This will be found to be always the case when a science is arranged as a deductive system. It is not the logically simplest propositions of the system that are the most obvious, or that provide the chief part of our reasons for believing in the system. (Russell 1924: 325)

We hold some logical or mathematical or physical claims to be true: logical claims like 'if p then p'; arithmetic facts like '$2 + 2 = 4$'; simple claims about the world like, 'apples fall from trees down to the ground'. We seek systematizations of those particular beliefs both to see if they are consistent and to make connections with other claims, aiming to achieve an elegant systematization.

In constructing an axiomatic theory, we work in two directions. We derive theorems from axioms and we decide on axioms on the basis of the theorems we can prove from them. The latter process of refining choices of axioms is called, in mathematics, reverse mathematics; it is a lively research program.

Nelson Goodman observes the same two-directional process in formal, deductive logic.

> How do we justify a *de*duction? Plainly by showing that it conforms to the general rules of deductive inference. An argument that so conforms is justified or valid, even if its conclusion happens to be false. . . . Principles of deductive inference are justified by their conformity with accepted deductive practice. Their validity depends upon accordance with the particular deductive inferences we actually make and sanction. If a rule yields unacceptable inferences, we drop it as invalid. Justification of general rules thus derives from judgments rejecting or accepting particular deductive inferences. (Goodman, "The New Riddle of Induction," 63–64)

One problem with this view, from Russell and Goodman, of the relation between the axioms and the theorems is that it appears to make our reasoning in axiomatic theories circular. We justify our particular claims or beliefs in terms of general principles from which they follow. We justify our general principles in terms of the specific claims they yield.

> This looks flagrantly circular. . . . But this circle is a virtuous circle. The point is that rules and particular inferences alike are justified by being brought into agreement with each other. *A rule is amended if it yields an inference we are unwilling to accept; an inference is rejected if it violates a rule we are unwilling to amend*. The process of justification is the delicate one of making mutual adjustments between rules and accepted inferences; and in the agreement achieved lies the only justification needed for either. (ibid, 64)

The virtuous circle that Goodman mentions here has come to be known as reflective equilibrium, and it has been invoked in logic, mathematics, linguistics, ethics, and political philosophy. Goodman even argues that accepting reflective equilibrium as the proper way of thinking about the relation between axioms and theorems leads to a dissolution of the classic problem of induction.

Summary

Modern mathematical logic was developed axiomatically, and axiomatic theories can be both useful and important in logic and other fields. It is often useful to see the minimal assumptions of a theory, though mathematical and physical theories are often axiomatizable in various conflicting ways.

Our natural deduction proof systems are generally easier to use than axiomatic theories. The main advantages of axiomatic logical theories are metalogical, and metalogic is a typical component of a next logic course. So it's good to be able to recognize the different kinds of theories and how they are related.

TELL ME MORE

- How do we know that System **H** is equivalent to **PL**? See 6.4: Metalogic.
- What is the Sheffer stroke? See 6S.8: Adequacy.
- What are the axioms for Zermelo-Fraenkel set theory? How do we write a second-order version of the mathematical induction scheme? See 6S.13: Second-Order Logic and Set Theory.

For Further Research and Writing

1. Students interested in the way in which metalogical proofs are simplified by working with axiomatic systems should consult Hunter's *Metalogic*. It is not an easy task to work through the book, especially without firm guidance, but sections 22–29 cover some important topics in the metatheory of propositional logic and may be accessible to students with backgrounds in mathematics or computer science, especially those familiar with the methods of mathematical induction.
2. For questions about the relationship between set theory and other mathematical theories, there's no better place to start than Benacerraf's classic "What Numbers Could Not Be" and the vast literature in its wake. Benacerraf presents two conflicting sets of definitions of numbers in terms of sets and asks whether any sense can be made of picking one over the other.
3. Even less technical questions about axioms are raised by Lewis Carroll's "What the Tortoise Said to Achilles." That paper, and related ones, can be found in a neat little collection from Cahn, Talisse, and Aikin.

Suggested Readings

Benacerraf, Paul. "What Numbers Could Not Be." *Philosophical Review* 74, no. 1 (January 1965): 47–73. Benacerraf compares set-theoretic definitions of numbers from Zermelo and von Neumann, arguing that neither is the right definition. The consequences for our understanding of the relationship between set theory and other branches of mathematics are puzzling.

Cahn, Steven M., Robert B. Talisse, and Scott F. Aikin. *Thinking About Logic: Classic Essays*. Boulder, CO: Westview Press, 2011. A collection of essays in the philosophy of logic, including Lewis Carroll's "What the Tortoise Said to Achilles" and two response papers.

Carroll, Lewis. "What the Tortoise Said to Achilles." In Cahn, Talisse, and Aikin. A classic and engaging dialogue, the point of which is debatable, but may concern the relation between axioms and rules of inference.

Enderton, Herbert. *Elements of Set Theory*. Boston: Academic Press, 1977. A classic, accessible, if technical, introduction to set theory.

Frege, Gottlob. *Begriffsschrift*. In *From Frege to Gödel*, edited by Jean van Heijenort, 1–82. Cambridge, MA: Harvard University Press, 1982. The original axiomatic logic system, covering propositional logic, predicate logic, and even some higher-order quantification. See the references to work by Richard Mendelsohn and George Boolos in section 6.10, translating the main propositions of the early portions of the *Begriffsschrift* into contemporary notation.

Goodman, Nelson. "The New Riddle of Induction." In *Fact Fiction and Forecast*, 59–83. Harvard University Press, 1983. Goodman challenges a traditional view of axiom systems on which the axioms are unassailably obvious truths and other theorems are known because they are derived from the axioms.

Hilbert, David. *Foundations of Geometry*, 2nd ed. Translated by Leo Unger. La Salle, IL: Open Court, (1899) 1971. Hilbert's axiomatization of Euclidean geometry.

Hunter, Geoffrey. *Metalogic*. Berkeley: University of California Press, 1971. Our system **H** is just Hunter's system PS. Geoffrey Hunter's axiomatic system for predicate logic, QS, uses two additional rules, beyond axiomatic versions of universal instantiation and universal generalization.

Kleene, Stephen. *Mathematical Logic*. Mineola, NY: Dover, 1967. Kleene uses ten axiom schemas for propositional logic; see pp. 34 and 387.

Kneale, W., and M. Kneale. *The Development of Logic*. Oxford, UK: Clarendon Press, 1962. Section IX.2 discusses axioms and rules, in Frege's work and beyond.

Mendelson, Elliott. *Introduction to Mathematical Logic*, 4th ed. section 1.4 presents an axiom system for propositional logic much like the one in this chapter. Section 1.6 contains a variety of alternative axiomatizations and citations to their original uses, including one, called L_3, that uses axioms instead of axiom schemata; and two (including **N**, from this section) that use only a single axiom. Section 2.3 has Mendelson's axiomatization of first-order predicate logic. Chapter 4 is an extended treatment of set theory.

Rosser, J. Barkley. *Logic for Mathematicians*. New York: McGraw-Hill, 1953. The source of system B, above.

Russell, Bertrand. "Logical Atomism." In *Logic and Knowledge*, edited by R. C. Marsh, 126–50. London: Allen & Unwin, (1924) 1956.

Quine, W. V. "Completeness of the Propositional Calculus." In *Selected Logic Papers*, 159–163. Cambridge, MA: Harvard University Press, 1966. A neat alternative proof of the completeness of a system of propositional logic, with some interesting theorems and references to their proofs.

SOLUTIONS TO EXERCISES 6S.11

1. $P \supset [Q \supset (P \supset Q)]$

1. $[Q \supset (P \supset Q)] \supset \{P \supset [Q \supset (P \supset Q)]\}$	H1, AS
2. $Q \supset (P \supset Q)$	H1, AS
3. $P \supset [Q \supset (P \supset Q)]$	1, 2, MP

QED

2. $\sim P \supset (P \supset Q)$

1. $[\sim P \supset (\sim Q \supset \sim P)] \supset \{[(\sim Q \supset \sim P) \supset (P \supset Q)] \supset [\sim P \supset (P \supset Q)]\}$	6S.11.4, TS
2. $\sim P \supset (\sim Q \supset \sim P)$	H1, AS
3. $[(\sim Q \supset \sim P) \supset (P \supset Q)] \supset [\sim P \supset (P \supset Q)]$	1, 2, MP
4. $(\sim Q \supset \sim P) \supset (P \supset Q)$	H3, AS
5. $\sim P \supset (P \supset Q)$	3, 4, MP

QED

3. $[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]$

1. $Q \supset (P \supset Q)$	H1, AS
2. $[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$	H2, AS
3. $[(P \supset Q) \supset (P \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}$	H1, AS
4. $\{[(P \supset Q) \supset (P \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\} \supset \{[P \supset (Q \supset R)] \supset \{[(P \supset Q) \supset (P \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\}\}$	H1, AS
5. $[P \supset (Q \supset R)] \supset \{[(P \supset Q) \supset (P \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\}$	4, 3 MP
6. $\{[P \supset (Q \supset R)] \supset \{[(P \supset Q) \supset (P \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\}\} \supset \{\{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[P \supset (Q \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\}\}$	H2, AS
7. $\{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[P \supset (Q \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\}$	6, 5, MP
8. $[P \supset (Q \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}$	7, 2, MP
9. $\{Q \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}$	H2, AS
10. $\{\{Q \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}\} \supset \{[P \supset (Q \supset R)] \supset \{\{Q \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}\}\}$	H1, AS
11. $[P \supset (Q \supset R)] \supset \{\{Q \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}\}$	10, 9, MP

12. $\{[P \supset (Q \supset R)] \supset \{\{Q \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}\}\} \supset \{\{[P \supset (Q \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\} \supset \{[P \supset (Q \supset R)] \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}\}\}$ H2, AS
13. $\{[P \supset (Q \supset R)] \supset \{Q \supset [(P \supset Q) \supset (P \supset R)]\}\} \supset \{[P \supset (Q \supset R)] \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}\}$ 12, 11, MP
14. $[P \supset (Q \supset R)] \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}$ 13, 8, MP
15. $\{[P \supset (Q \supset R)] \supset \{[Q \supset (P \supset Q)] \supset [Q \supset (P \supset R)]\}\} \supset \{\{[P \supset (Q \supset R)] \supset [Q \supset (P \supset Q)]\} \supset \{[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]\}\}$ H2, AS
16. $\{[P \supset (Q \supset R)] \supset [Q \supset (P \supset Q)]\} \supset \{[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]\}$ 15, 14, MP
17. $[Q \supset (P \supset Q)] \supset \{[P \supset (Q \supset R)] \supset [Q \supset (P \supset Q)]\}$ H1, AS
18. $[P \supset (Q \supset R)] \supset [Q \supset (P \supset Q)]$ 17, 1, MP
19. $[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]$ 16, 18, MP

QED

4. $[P \supset (P \supset Q)] \supset (P \supset Q)$

1. $[(P \supset Q) \supset (P \supset Q)] \supset \{P \supset [(P \supset Q) \supset Q]\}$ Exercise 3, TS
2. $(P \supset Q) \supset (P \supset Q)$ 6S.11.1, TS
3. $P \supset [(P \supset Q) \supset Q]$ 1, 2, MP
4. $\{P \supset [(P \supset Q) \supset Q]\} \supset \{[P \supset (P \supset Q)] \supset (P \supset Q)\}$ H2, AS
5. $[P \supset (P \supset Q)] \supset (P \supset Q)$ 4, 3, MP

QED