The Chemistry Maths Book

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Solutions

Chapter 18 Matrices and linear transformations

Section 18.1

1. Construct transformation matrices that represent the following rotations about the *z*-axis: **(i)** anticlockwise through $45°$, **(ii)** anticlockwise through $90°$, **(iii)** clockwise through $90°$.

By equation (18.10) an anticlockwise rotation through angle θ is represented by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(i) $\theta = \pi/4, \quad \sin\theta = \cos\theta = 1/\sqrt{2} \quad \rightarrow \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(ii) $\theta = \pi/2, \quad \sin\theta = 1, \cos\theta = 0 \quad \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(iii) $\theta = -\pi/2, \quad \sin\theta = -1, \cos\theta = 0 \quad \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2. Construct a transformation matrix that represents the interchange of *x* and *y* coordinates of a point.

By equations (18.3) and (18.5),

$$\left.\begin{aligned} x' &= 0 + y \\ y' &= x + 0 \end{aligned}\right\} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Section 18.2

For the following matrices,

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & -4 \\ 2 & -3 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} \quad \mathbf{Q} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{a} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \quad \mathbf{b} = (2 \quad 5 \quad -2)$$

find, *if possible*:

3. $\det \mathbf{A}$, $\operatorname{tr} \mathbf{A}$

Matrix **A** is not square, and has no determinant or trace.

4. $\det \mathbf{D} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 3 \times 2 \times (-1) = -6$

$\operatorname{tr} \mathbf{D} = 3 + 2 - 1 = 4$

5. $\det \mathbf{P} = \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} = 4$

$\operatorname{tr} \mathbf{P} = 1 + 4 = 5$

6. $\det \mathbf{a}$, $\operatorname{tr} \mathbf{a}$

Matrix **a** is not square, and has no determinant or trace.

7. $\mathbf{A}^{\mathsf{T}} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \\ 3 & 4 \end{pmatrix}$

8. $\mathbf{C}^{\mathsf{T}} = \begin{pmatrix} -5 & 4 & 2 \\ 3 & -1 & -1 \end{pmatrix}$

9. $\mathbf{D}^{\mathsf{T}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{D}$

10. $\mathbf{P}^{\mathsf{T}} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$

11. $\mathbf{a}^{\mathsf{T}} = (0 \quad -3 \quad 1)$

12. $\mathbf{b}^{\mathsf{T}} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$

Section 18.3

For the above matrices find, *if possible*:

13. $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -4 \\ 2 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1+0 & -2+1 & 3-4 \\ 0+2 & 3-3 & 4+0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & 4 \end{pmatrix}$

14. $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -4 \\ 2 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1-0 & -2-1 & 3+4 \\ 0-2 & 3+3 & 4-0 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 7 \\ -2 & 6 & 4 \end{pmatrix}$

15. $\mathbf{B}-\mathbf{A}=\begin{pmatrix}0 & 1 & -4\\ 2 & -3 & 0\end{pmatrix}-\begin{pmatrix}1 & -2 & 3\\ 0 & 3 & 4\end{pmatrix}=\begin{pmatrix}-1 & 3 & -7\\ 2 & -6 & -4\end{pmatrix}$

16. $\mathbf{C}+\mathbf{D}$

The matrices have different dimensions, and addition is not defined.

17. $\mathbf{a}+\mathbf{b}^{\mathsf{T}}=\begin{pmatrix}0\\ -3\\ 1\end{pmatrix}+\begin{pmatrix}2\\ 5\\ -2\end{pmatrix}=\begin{pmatrix}2\\ 2\\ -1\end{pmatrix}$

18. $\mathbf{a}^{\mathsf{T}}+\mathbf{b}=(0 \quad -3 \quad 1)+(2 \quad 5 \quad -2)=(2 \quad 2 \quad -1)$

19. $3\mathbf{P}=3\begin{pmatrix}1 & -2\\ 0 & 4\end{pmatrix}=\begin{pmatrix}3\times 1 & 3\times(-2)\\ 3\times 0 & 3\times 4\end{pmatrix}=\begin{pmatrix}3 & -6\\ 0 & 12\end{pmatrix}$

20. $2\mathbf{A}+3\mathbf{B}=2\begin{pmatrix}1 & -2 & 3\\ 0 & 3 & 4\end{pmatrix}+3\begin{pmatrix}0 & 1 & -4\\ 2 & -3 & 0\end{pmatrix}=\begin{pmatrix}2+0 & -4+3 & 6-12\\ 0+6 & 6-9 & 8+0\end{pmatrix}$

$$=\begin{pmatrix}2 & -1 & -6\\ 6 & -3 & 8\end{pmatrix}$$

21. $\mathbf{AB}$

The number of columns of **A** is not equal to the number of rows of **B**, and, by equation (18.25), the matrix product is not defined.

22. $\mathbf{BC}=\begin{pmatrix}0 & 1 & -4\\ 2 & -3 & 0\end{pmatrix}\begin{pmatrix}-5 & 3\\ 4 & -1\\ 2 & -1\end{pmatrix}=\begin{pmatrix}0\times(-5)+1\times 4+(-4)\times 2 & 0\times 3+1\times(-1)+(-4)\times(-1)\\ 2\times(-5)+(-3)\times 4+0\times 2 & 2\times 3+(-3)\times(-1)+0\times(-1)\end{pmatrix}$

$$=\begin{pmatrix}-4 & 3\\ -22 & 9\end{pmatrix}$$

dimenions: $(2\times 3)\times(3\times 2) \;\rightarrow\; (2\times 2)$

23. $\mathbf{CB}=\begin{pmatrix}-5 & 3\\ 4 & -1\\ 2 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & -4\\ 2 & -3 & 0\end{pmatrix}=\begin{pmatrix}0+6 & -5-9 & 20+0\\ 0-2 & 4+3 & -16+0\\ 0-2 & 2+3 & -8+0\end{pmatrix}$

$$=\begin{pmatrix}6 & -14 & 20\\ -2 & 7 & -16\\ -2 & 5 & -8\end{pmatrix}$$

dimenions: $(3\times 2)\times(2\times 3) \;\rightarrow\; (3\times 3)$

24. $\mathbf{CP} = \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -5+0 & 10+12 \\ 4+0 & -8-4 \\ 2+0 & -4-4 \end{pmatrix}$

$= \begin{pmatrix} -5 & 22 \\ 4 & -12 \\ 2 & -8 \end{pmatrix}$

dimenions: $(3\times 2)\times(2\times 2) \;\to\; (3\times 2)$

25. PC

Dimensions: $(2\times 2)\times(3\times 2)$ The number of columns of **P** is not equal to the number of rows of **C**, and the matrix product is not defined.

26. $\mathbf{D}^2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3\times 3 & 0 & 0 \\ 0 & 2\times 2 & 0 \\ 0 & 0 & -1\times(-1) \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

27. $\mathbf{PQ} = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3+0 & 0-2 \\ 0+0 & 0+4 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$

28. $\mathbf{QP} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3+0 & -6+0 \\ 0+0 & 0+4 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 0 & 4 \end{pmatrix}$

29. $\mathbf{Ba} = \begin{pmatrix} 0 & 1 & -4 \\ 2 & -3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-3-4 \\ 0+9+0 \end{pmatrix} = \begin{pmatrix} -7 \\ 9 \end{pmatrix}$

30. $\mathbf{ab} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}(2 \quad 5 \quad -2) = \begin{pmatrix} 0 & 0 & 0 \\ -6 & -15 & 6 \\ 2 & 5 & -2 \end{pmatrix}$

31. $\mathbf{ba} = (2 \quad 5 \quad -2)\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \big(2\times 0 + 5\times(-3) + (-2)\times 1\big) = (-17) = -17$

32. $\mathbf{a}^{\mathsf{T}}\mathbf{b}^{\mathsf{T}} = (\mathbf{ba})^{\mathsf{T}} = -17$ (from Exercise 31)

33. $\mathbf{b}^{\mathsf{T}}\mathbf{a}^{\mathsf{T}} = (\mathbf{ab})^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & 0 \\ -6 & -15 & 6 \\ 2 & 5 & -2 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 0 & -6 & 2 \\ 0 & -15 & 5 \\ 0 & 6 & -2 \end{pmatrix}$ (from Exercise 30)

34. Ca

Dimenions: $(3\times 2)\times(3\times 1)$ The number of columns of **C** is not equal to the number of rows of **a**, and the matrix product is not defined.

35. $\mathbf{a}^{\mathrm{T}}\mathbf{C} = \begin{pmatrix}0 & -3 & 1\end{pmatrix}\begin{pmatrix}-5 & 3\\ 4 & -1\\ 2 & -1\end{pmatrix} = \begin{pmatrix}-10 & 2\end{pmatrix}$

dimensions: $(1\times 3)\times(3\times 2) = (1\times 2)$

36. If $\mathbf{A} = \begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ -2 & 1 & -1\end{pmatrix}$, show that $\mathbf{A}^3 - \mathbf{A}^2 - 3\mathbf{A} + \mathbf{I} = \mathbf{0}$.

We have $\mathbf{A} = \begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ -2 & 1 & -1\end{pmatrix} \;\rightarrow\; \mathbf{A}^2 = \begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ -2 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ -2 & 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 3 & 2\\ 0 & 5 & 2\\ 2 & -2 & 1\end{pmatrix}$

$$\rightarrow\; \mathbf{A}^3 = \begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ -2 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & 3 & 2\\ 0 & 5 & 2\\ 2 & -2 & 1\end{pmatrix} = \begin{pmatrix}3 & 6 & 5\\ 6 & 7 & 8\\ -4 & 1 & -3\end{pmatrix}$$

Therefore $\mathbf{A}^3 - \mathbf{A}^2 - 3\mathbf{A} + \mathbf{I} = \begin{pmatrix}3 & 6 & 5\\ 6 & 7 & 8\\ -4 & 1 & -3\end{pmatrix} - \begin{pmatrix}1 & 3 & 2\\ 0 & 5 & 2\\ 2 & -2 & 1\end{pmatrix} - 3\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ -2 & 1 & -1\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$$= \begin{pmatrix}(3-1-3+1) & (6-3-3+0) & (5-2-3+0)\\ (6-0-6+0) & (7-5-3+1) & (8-2-6+0)\\ (-4-2+6+0) & (1+2-3+0) & (3-1-3+1)\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \mathbf{0}$$

37. If $\mathbf{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}$, find **A** such that $\mathbf{AB} = \begin{pmatrix}3 & 2\\ 1 & 4\end{pmatrix}$

We have $\mathbf{AB} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix} = \begin{pmatrix}2a+3b & a+2b\\ 2c+3d & c+2d\end{pmatrix}$

$$= \begin{pmatrix}3 & 2\\ 1 & 4\end{pmatrix} \text{ if } \begin{cases}2a+3b=3,\ a+2b=2 & \rightarrow\ a=0,\ b=1\\ 2c+3d=1,\ c+2d=4 & \rightarrow\ d=7,\ c=-10\end{cases}$$

Therefore $\mathbf{A} = \begin{pmatrix}0 & -1\\ -10 & 7\end{pmatrix}$

Given the matrices **A** and **B**, find **AB** and **BA**:

38. $\mathbf{A}=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-1 & 3\\ 3 & 2\end{pmatrix}$ $\rightarrow$ $\begin{cases}\mathbf{AB}=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 3\\ 3 & 2\end{pmatrix}=\begin{pmatrix}-5 & 4\\ 4 & -1\end{pmatrix}\\ \mathbf{BA}=\begin{pmatrix}-1 & 3\\ 3 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}-5 & 4\\ 4 & -1\end{pmatrix}\end{cases}$

39. $\mathbf{A}=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ $\rightarrow$ $\begin{cases}\mathbf{AB}=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\\ \mathbf{BA}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}\end{cases}$

Find the commutator of the following pairs of matrices:

40. $\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$, $\begin{pmatrix}-1 & 3\\ 3 & 2\end{pmatrix}$

By Exercise 38, $\mathbf{AB}-\mathbf{BA}=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 3\\ 3 & 2\end{pmatrix}-\begin{pmatrix}-1 & 3\\ 3 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$

$$=\begin{pmatrix}-5 & 4\\ 4 & -1\end{pmatrix}-\begin{pmatrix}-5 & 4\\ 4 & -1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

and the matrices commute.

41. $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

By Exercise 39, $\mathbf{AB}-\mathbf{BA}=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$

$$=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}-\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$$

and the matrices do not commute.

42. The spin matrices for a nucleus with spin quantum number 1 are

$$\mathbf{I}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \mathbf{I}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad \mathbf{I}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

(i) Find the commutators $\left[\mathbf{I}_x, \mathbf{I}_y\right]$, $\left[\mathbf{I}_y, \mathbf{I}_z\right]$, $\left[\mathbf{I}_z, \mathbf{I}_x\right]$. **(ii)** Find $\mathbf{I}_x^2 + \mathbf{I}_y^2 + \mathbf{I}_z^2$.

(i)

$$[\mathbf{I}_x, \mathbf{I}_y] = \mathbf{I}_x\mathbf{I}_y - \mathbf{I}_y\mathbf{I}_x$$

$$= \frac{1}{2}\hbar^2\left[\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix} - \begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right]$$

$$= \frac{1}{2}\hbar^2\left[\begin{pmatrix}i & 0 & -i\\ 0 & 0 & 0\\ i & 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0 & -i\\ 0 & 0 & 0\\ i & 0 & i\end{pmatrix}\right] = \frac{1}{2}\hbar^2\begin{pmatrix}2i & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -2i\end{pmatrix}$$

$$= i\hbar^2\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} = i\hbar\mathbf{I}_z$$

and similarly for the other pairs. Therefore

$$[\mathbf{I}_x, \mathbf{I}_y] = i\hbar\mathbf{I}_z, \qquad [\mathbf{I}_y, \mathbf{I}_z] = i\hbar\mathbf{I}_x, \qquad [\mathbf{I}_z, \mathbf{I}_x] = i\hbar\mathbf{I}_y$$

(ii)

$$\mathbf{I}_x^{\,2} = \frac{1}{2}\hbar^2\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \frac{1}{2}\hbar^2\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{pmatrix}$$

$$\mathbf{I}_y^{\,2} = \frac{1}{2}\hbar^2\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix} = \frac{1}{2}\hbar^2\begin{pmatrix}1 & 0 & -1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{pmatrix}$$

$$\mathbf{I}_z^{\,2} = \hbar^2\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} = \hbar^2\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Therefore $\quad \mathbf{I}_x^{\,2} + \mathbf{I}_y^{\,2} + \mathbf{I}_z^{\,2} = \hbar^2\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} = 2\,\hbar^2\mathbf{1}$

43. Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$, find matrices **B** and **C** for which $\mathbf{BA} = \mathbf{AC} = \mathbf{A}$.

$\mathbf{BA} = \mathbf{A}$ if **B** is the unit matrix of order 2:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus $$\mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} = \mathbf{A}$$

$\mathbf{AC} = \mathbf{A}$ if **C** is the unit matrix of order 4:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $$\mathbf{AC} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} = \mathbf{A}$$

Find the general matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which:

44. $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

We have $$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } \begin{cases} a+2c=0 & \to \quad a=-2c \\ b+2d=0 & \to \quad b=-2d \end{cases}$$

Therefore $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2c & -2d \\ c & d \end{pmatrix}$$

for all values of *c* and *d*.

45. $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

From Exercise 44,

$$\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}-2c & -2d\\ c & d\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Then $\begin{pmatrix}-2c & -2d\\ c & d\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}=\begin{pmatrix}-2c-4d & -4c-8d\\ c+2d & 2c+4d\end{pmatrix}$

$$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \text{ if } c=-2d$$

Therefore $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}4d & -2d\\ -2d & d\end{pmatrix}=d\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}$

for all values of *d*.

Check: $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

46. Given $\mathbf{A}=\begin{pmatrix}-5 & 3\\ 4 & -1\\ 2 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -2\\ 0 & 4\end{pmatrix}$, show that $(\mathbf{AB})^{\mathsf T}=\mathbf{B}^{\mathsf T}\mathbf{A}^{\mathsf T}$

We have $\mathbf{AB}=\begin{pmatrix}-5 & 3\\ 4 & -1\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & 4\end{pmatrix}=\begin{pmatrix}-5 & 22\\ 4 & -12\\ 2 & -8\end{pmatrix} \rightarrow (\mathbf{AB})^{\mathsf T}=\begin{pmatrix}-5 & 4 & 2\\ 22 & -12 & -8\end{pmatrix}$

and $\mathbf{B}^{\mathsf T}\mathbf{A}^{\mathsf T}=\begin{pmatrix}1 & 0\\ -2 & 4\end{pmatrix}\begin{pmatrix}-5 & 4 & 2\\ 3 & -1 & -1\end{pmatrix}=\begin{pmatrix}-5 & 4 & 2\\ 22 & -12 & -8\end{pmatrix}$

$$=(\mathbf{AB})^{\mathsf T}$$

Section 18.4

Find the inverse matrix, if possible:

47. $\mathbf{A}=\begin{pmatrix}2 & -3\\ 4 & 1\end{pmatrix}$

As in Example 18.16,

$$\mathbf{A}^{-1}=\frac{1}{2\times1-(-3)\times4}\begin{pmatrix}1 & 3\\ -4 & 2\end{pmatrix}=\frac{1}{14}\begin{pmatrix}1 & 3\\ -4 & 2\end{pmatrix}$$

48. $\mathbf{A}=\begin{pmatrix}1&2&3\\-2&1&2\\3&-1&-1\end{pmatrix}$

By equation (18.45) the inverse is the adjoint divided by the determinant: $\mathbf{A}^{-1}=\widehat{\mathbf{A}}\big/\det\mathbf{A}$.

We have $\det\mathbf{A}=\begin{vmatrix}1&2&3\\-2&1&2\\3&-1&-1\end{vmatrix}=\begin{vmatrix}1&2\\-1&-1\end{vmatrix}-2\begin{vmatrix}-2&2\\3&-1\end{vmatrix}+3\begin{vmatrix}-2&1\\3&-1\end{vmatrix}=1+8-3$

$=6$

The adjoint $\widehat{\mathbf{A}}$ is the transpose of the matrix **C** of the cofactors of the elements of the determinant:

$$\mathbf{C}=\begin{pmatrix}1&4&-1\\-1&-10&7\\1&-8&5\end{pmatrix}\to\widehat{\mathbf{A}}=\begin{pmatrix}1&-1&1\\4&-10&-8\\-1&7&5\end{pmatrix}$$

Then $\mathbf{A}^{-1}=\dfrac{1}{6}\begin{pmatrix}1&-1&1\\4&-10&-8\\-1&7&5\end{pmatrix}$

49. $\mathbf{A}=\begin{pmatrix}2&-1&1\\-1&1&-2\\-3&1&0\end{pmatrix}$

We have $\det\mathbf{A}=2\begin{vmatrix}1&-2\\1&0\end{vmatrix}+\begin{vmatrix}-1&-2\\-3&0\end{vmatrix}+\begin{vmatrix}-1&1\\-3&1\end{vmatrix}=4-6+2=0$

The matrix is singular, and has no inverse.

50. $\mathbf{A}=\begin{pmatrix}\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\\0&1&0\\\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\end{pmatrix}$

We have $\det\mathbf{A}=\dfrac{1}{\sqrt2}\begin{vmatrix}1&0\\0&\frac{1}{\sqrt2}\end{vmatrix}-0-\dfrac{1}{\sqrt2}\begin{vmatrix}0&1\\\frac{1}{\sqrt2}&0\end{vmatrix}=\dfrac12-0+\dfrac12$

$=1$

The inverse is therefore equal to the adjoint. If **C** is the matrix of cofactors (as in Exercise 48),

$$\mathbf{C}=\begin{pmatrix}\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\\0&1&0\\\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\end{pmatrix}\to\widehat{\mathbf{A}}=\mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&1&0\\-\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\end{pmatrix}$$

51. $\mathbf{A}=\begin{pmatrix}3&4&0&0\\1&2&0&0\\0&0&3&1\\0&0&4&2\end{pmatrix}$

The matrix is in “block-diagonal” form

$$\mathbf{A}=\begin{pmatrix}3&4&0&0\\1&2&0&0\\0&0&3&1\\0&0&4&2\end{pmatrix}=\begin{pmatrix}\mathbf{B}&\mathbf{0}\\\mathbf{0}&\mathbf{D}\end{pmatrix}$$

where **B, D** and **0** are the (2×2) matrices

$$\mathbf{B}=\begin{pmatrix}3&4\\1&2\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}3&1\\4&2\end{pmatrix},\quad \mathbf{0}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Then $\mathbf{A}^{-1}=\begin{pmatrix}\mathbf{B}^{-1}&\mathbf{0}\\\mathbf{0}&\mathbf{D}^{-1}\end{pmatrix}$

because $\mathbf{AA}^{-1}=\begin{pmatrix}\mathbf{B}&\mathbf{0}\\\mathbf{0}&\mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{B}^{-1}&\mathbf{0}\\\mathbf{0}&\mathbf{D}^{-1}\end{pmatrix}=\begin{pmatrix}\mathbf{BB}^{-1}&\mathbf{0}\\\mathbf{0}&\mathbf{DD}^{-1}\end{pmatrix}=\begin{pmatrix}\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{I}\end{pmatrix}$

Now $\mathbf{B}= \;\rightarrow\; \mathbf{B}^{-1}=\frac{1}{2}\begin{pmatrix}2&-4\\-1&3\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}3&1\\4&2\end{pmatrix}\;\rightarrow\;\mathbf{D}^{-1}=\frac{1}{2}\begin{pmatrix}2&-1\\-4&3\end{pmatrix}$

Therefore $\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}2&-4&0&0\\-1&3&0&0\\0&0&2&-1\\0&0&-4&3\end{pmatrix}$

52. For each matrix (**A**) of Exercises 47 – 51, verify (if $\mathbf{A}^{-1}$ exists) that $\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$.

47. $\mathbf{AA}^{-1}=\frac{1}{14}\begin{pmatrix}2&-3\\4&1\end{pmatrix}\begin{pmatrix}1&3\\-4&2\end{pmatrix}=\frac{1}{14}\begin{pmatrix}14&0\\0&14\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$

$\mathbf{A}^{-1}\mathbf{A}=\frac{1}{14}\begin{pmatrix}1&3\\-4&2\end{pmatrix}\begin{pmatrix}2&-3\\4&1\end{pmatrix}=\frac{1}{14}\begin{pmatrix}14&0\\0&14\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\mathbf{I}$

48. $\mathbf{AA}^{-1}=\frac{1}{6}\begin{pmatrix}1&2&3\\-2&1&2\\3&-1&-1\end{pmatrix}\begin{pmatrix}1&-1&1\\4&-10&-8\\-1&7&5\end{pmatrix}=\frac{1}{6}\begin{pmatrix}(1+8-3)&(-1-20+21)&(1-16+15)\\(-2+4-2)&(2-10+14)&(-2-8+10\\(3-4+1)&(-3+10-7)&(3+8-5)\end{pmatrix}$

$$=\frac{1}{6}\begin{pmatrix}6&0&0\\0&6&0\\0&0&6\end{pmatrix}=\mathbf{I}$$

48. (cont)

$$\mathbf{A}^{-1}\mathbf{A}=\frac{1}{6}\begin{pmatrix}1&-1&1\\4&-10&-8\\-1&7&5\end{pmatrix}\begin{pmatrix}1&2&3\\-2&1&2\\3&-1&-1\end{pmatrix}=\frac{1}{6}\begin{pmatrix}(1+2+3)&(2-1-1)&(3-2-1)\\(4+20-24)&(8-10+8)&(12-20+8\\(-1-14+15)&(-2+7-5)&(-3+14-5)\end{pmatrix}$$

$$=\frac{1}{6}\begin{pmatrix}6&0&0\\0&6&0\\0&0&6\end{pmatrix}=\mathbf{I}$$

49. **A** is singular

50. $$\mathbf{A}\mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\\0&1&0\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A}=\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\\0&1&0\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbf{I}$$

51. $$\mathbf{A}\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix}3&4&0&0\\1&2&0&0\\0&0&3&1\\0&0&4&2\end{pmatrix}\begin{pmatrix}2&-4&0&0\\-1&3&0&0\\0&0&2&-1\\0&0&-4&3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&2\\0&0&2&2\end{pmatrix}=\mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A}=\frac{1}{2}\begin{pmatrix}2&-4&0&0\\-1&3&0&0\\0&0&2&-1\\0&0&-4&3\end{pmatrix}\begin{pmatrix}3&4&0&0\\1&2&0&0\\0&0&3&1\\0&0&4&2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&2\\0&0&2&2\end{pmatrix}=\mathbf{I}$$

53. If $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}5&6\\7&8\end{pmatrix}$, show that $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$

We have $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\to\mathbf{A}^{-1}=-\frac{1}{2}\begin{pmatrix}4&-2\\-3&1\end{pmatrix}$

$\mathbf{B}=\begin{pmatrix}5&6\\7&8\end{pmatrix}\to\mathbf{B}^{-1}=-\frac{1}{2}\begin{pmatrix}8&-6\\-7&5\end{pmatrix}$

Then $\mathbf{AB}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}19&22\\43&50\end{pmatrix}\to(\mathbf{AB})^{-1}=\frac{1}{4}\begin{pmatrix}50&-22\\-43&19\end{pmatrix}$

and $\mathbf{B}^{-1}\mathbf{A}^{-1}=-\frac{1}{2}\begin{pmatrix}8&-6\\-7&5\end{pmatrix}\times\left(-\frac{1}{2}\right)\begin{pmatrix}4&-2\\-3&1\end{pmatrix}=\frac{1}{4}\begin{pmatrix}50&-22\\-43&19\end{pmatrix}$

$=(\mathbf{AB})^{-1}$

Section 18.5

54. The linear transformation $\mathbf{r}' = \mathbf{A}\mathbf{r}$, where

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \qquad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents an anticlockwise rotation through angle θ about the z-axis. **(i)** Write down the corresponding linear equations. **(ii)** Find **A** for a clockwise rotation through $\pi/4$ about the z-axis. **(iii)** Show that

$$\mathbf{A}^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and explain its geometric meaning. **(iv)** Explain the geometric meaning of the equation $\mathbf{A}^3 = \mathbf{I}$.

(i)

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}$$

(ii) For the clockwise rotation, $\theta = -\pi/4$, $\sin\theta = -1/\sqrt{2}$, $\cos\theta = 1/\sqrt{2}$.

Then $$\mathbf{A} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii)

$$\mathbf{A}^2 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta & 0 \\ 2\sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{A}^2$ is anticlockwise rotation through angle 2θ about the z-axis.

(iv) If $\mathbf{A}^3 = \mathbf{I}$ then rotation is through $2\pi = 360°$ (or, more generally, through a multiple of 2π)

and $\mathbf{A} = 2\pi/3 = 120°$ (or a multiple thereof).

55. For transformations in three dimensions, write down the matrices that represent **(i)** rotation about the *x*-axis, **(ii)** rotation about the *y*-axis, **(iii)** reflection in the *xy*-plane, **(iv)** reflection in the *yz*-plane, **(v)** reflection in the *zx*-plane, **(vi)** inversion through the origin.

(i)
$$\begin{aligned} x' &= x \\ y' &= y\cos\theta - z\sin\theta \\ z' &= y\sin\theta + z\cos\theta \end{aligned} \quad \to \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

(ii)
$$\begin{aligned} x' &= x\cos\theta + z\sin\theta \\ y &= y \\ z' &= -x\sin\theta + z\cos\theta \end{aligned} \quad \to \quad \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

(iii)
$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= -z \end{aligned} \quad \to \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(iv)
$$\begin{aligned} x' &= -x \\ y' &= y \\ z' &= z \end{aligned} \quad \to \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(v)
$$\begin{aligned} x' &= x \\ y' &= -y \\ z' &= z \end{aligned} \quad \to \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(vi)
$$\begin{aligned} x' &= -x \\ y' &= -y \\ z' &= -z \end{aligned} \quad \to \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The coordinates of four points in the *xy*-plane are given by the columns of the matrix

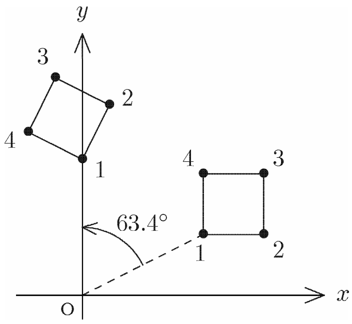
$$\mathbf{X} = \begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

(see Example 18.19). Find $\mathbf{X}' = \mathbf{AX}$ for each of the following matrices $\mathbf{A}$, and draw appropriate diagrams to illustrate the transformations:

56. $\mathbf{A} = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$$\mathbf{X}' = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

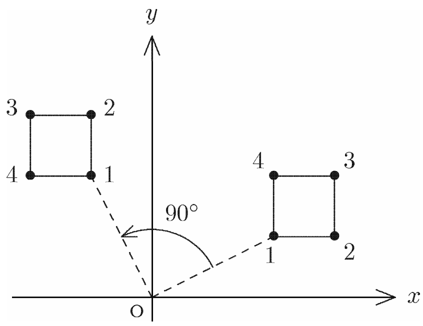
$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 1 & -1 & -2 \\ 5 & 7 & 8 & 6 \end{pmatrix}$$



57. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

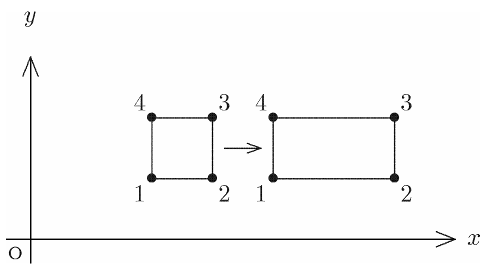
$$= \begin{pmatrix} -1 & -1 & -2 & -2 \\ 2 & 3 & 3 & 2 \end{pmatrix}$$



58. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{X}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

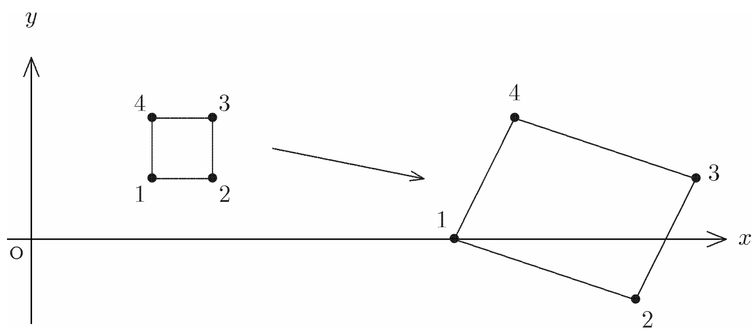
$$= \begin{pmatrix} 4 & 6 & 6 & 6 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$



59. $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

$$\mathbf{X}' = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 10 & 11 & 8 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$



60. **(i)** Find the single matrix **A** that represents the sequence of consecutive transformations **(a)** anticlockwise rotation through θ about the *x*-axis, followed by **(b)** reflection in the *xy*-plane, followed by **(c)** anticlockwise rotation through ϕ about the *z*-axis. **(ii)** Find **A** for $\theta = \pi/3$ and $\phi = -\pi/6$. **(iii)** Find $\mathbf{r}' = \mathbf{Ar}$ for this **A** and $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

(i)
$$\mathbf{A} = \underset{(\mathbf{c})}{\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{(\mathbf{b})}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} \underset{(\mathbf{a})}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & -\sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\cos\theta\sin\phi & \sin\theta\sin\phi \\ \sin\phi & \cos\theta\cos\phi & -\sin\theta\cos\phi \\ 0 & -\sin\theta & -\cos\theta \end{pmatrix}$$

(ii) We have $\theta = \pi/3 \;\to\; \sin\theta = \sqrt{3}/2, \;\; \cos\theta = 1/2$

$$\phi = -\pi/6 \;\to\; \sin\phi = -1/2, \;\; \cos\phi = \sqrt{3}/2$$

Then
$$\mathbf{A} = \begin{pmatrix} \sqrt{3}/2 & 1/4 & -\sqrt{3}/4 \\ -1/2 & \sqrt{3}/4 & -3/4 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

(iii)
$$\mathbf{r}' = \mathbf{Ar} = \begin{pmatrix} \sqrt{3}/2 & 1/4 & -\sqrt{3}/4 \\ -1/2 & \sqrt{3}/4 & -3/4 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} + \sqrt{3}/4 \\ -1 + 3/4 \\ 1/2 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 5\sqrt{3} \\ -1 \\ 2 \end{pmatrix}$$

Section 18.6

61. For each of the matrices **(i)**, **(iii)**, and **(vi)** in Exercise 55, (a) show that the matrix is orthogonal, (b) find its inverse.

(i) Let $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}$$

where $$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}$$

(a) For normalization: $$\mathbf{a}^{\mathsf{T}}\mathbf{a} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$\mathbf{b}^{\mathsf{T}}\mathbf{b} = \begin{pmatrix} 0 & \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\mathbf{c}^{\mathsf{T}}\mathbf{c} = \begin{pmatrix} 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} = \sin^2\theta + \cos^2\theta = 1$$

For orthogonality: $$\mathbf{a}^{\mathsf{T}}\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} = 0$$

$$\mathbf{a}^{\mathsf{T}}\mathbf{c} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} = 0$$

$$\mathbf{b}^{\mathsf{T}}\mathbf{c} = \begin{pmatrix} 0 & \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} = -\cos\theta\sin\theta + \cos\theta\sin\theta = 0$$

Determinant: $$\det\mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

(b) The matrix is orthogonal. Therefore

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathsf{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad (\text{for } \theta \to -\theta)$$

(iii) Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}$

(a) $\mathbf{a}^{\mathsf{T}}\mathbf{a} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1, \quad \mathbf{a}^{\mathsf{T}}\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

and similarly for the other pairs of column vectors. Also

$$\det \mathbf{A} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1$$

(b) The matrix is orthogonal. Therefore

$\mathbf{A}^{-1} = \mathbf{A}^{\mathsf{T}} = \mathbf{A}$ (for reflection in the *xy*-plane)

(vi) $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

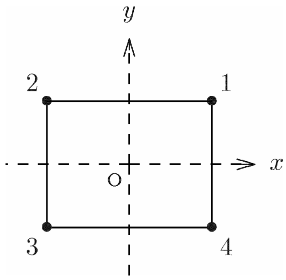
(a) as for **(iii)**

(b) $\mathbf{A}^{-1} = \mathbf{A}^{\mathsf{T}} = \mathbf{A}$ (for inversion)

Section 18.7

62. The symmetry properties of the plane figure formed by the four points at the corners of a rectangle (not a square), Figure 18.9, can be described in terms of four symmetry operations, the identity operation and three rotations.

Figure 18.9



(i) Describe these symmetry operations. **(ii)** Construct the group multiplication table.

The symmetry elements of the rectangle can be chosen as the three 2-fold axes of rotation: Oz, O*x*, O*y* (an alternative choice is the axis Oz and the bisecting planes xz and yz).

(i) The symmetry operations are

E the identity operation that leaves every point unmoved

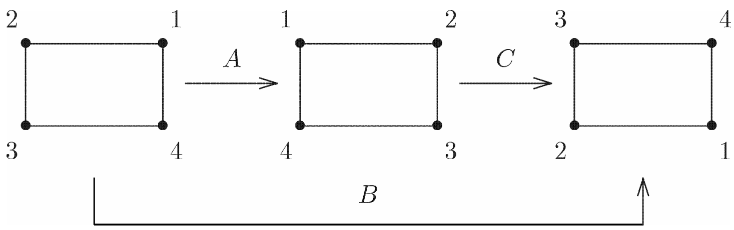
A rotation through 180° about the O*z*-axis

B rotation through 180° about the O*x*-axis

C rotation through 180° about the O*y*-axis

(ii) We have, for example,

AC = B

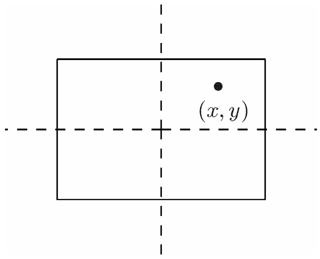


The group multiplication is

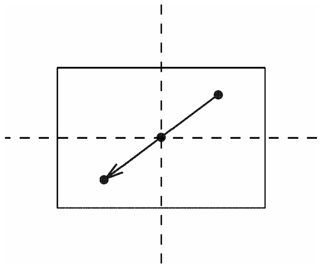
	E	*A*	*B*	*C* (applied first)
E	*E*	*A*	*B*	*C*
A	*A*	*E*	*C*	*B*
B	*B*	*C*	*E*	*A*
C	*C*	*B*	*A*	*E*

63. Construct a two-dimensional matrix representation of the group in Exercise 62 by applying each symmetry operation in turn to the coordinates (x, y) of a point in the plane of the figure.

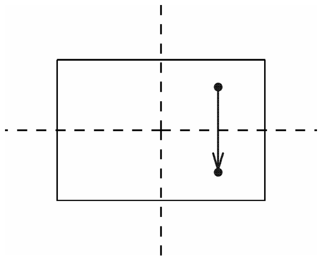
$$E\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \to \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} \quad \to \quad \mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} \quad \to \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \quad \to \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$