The Chemistry Maths Book

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Solutions

Chapter 17 Determinants

Section 17.1

1. Use determinants to solve the pair of equations $\begin{aligned} 4x + \ \ y &= 11 \\ 3x + 2y &= 12 \end{aligned}$

$$D = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 2 \times 4 - 1 \times 3 = 5$$

$$D_1 = \begin{vmatrix} 11 & 1 \\ 12 & 2 \end{vmatrix} = 11 \times 2 - 1 \times 12 = 10 \quad \rightarrow \quad x = D_1/D = 10/5 = 2$$

$$D_2 = \begin{vmatrix} 4 & 11 \\ 3 & 12 \end{vmatrix} = 48 - 33 = 15 \qquad \rightarrow \quad y = D_2/D = 15/5 = 3$$

Evaluate:

2. $\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2 \times 3 = 6$

3. $\begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} = 0 - 1 \times (-2) = 2$

4. $\begin{vmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{vmatrix} = \cos^2 n\theta + \sin^2 n\theta = 1$

Section 17.2

5. Use determinants to solve the equations $\begin{aligned} x + \ \ y + \ z &= 6 \\ x + 2y + 3z &= 14 \\ x + 4y + 9z &= 36 \end{aligned}$

The determinant of the coefficients is, by equation (17.10),

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 6 - 6 + 2 = 2$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 36 & 4 & 9 \end{vmatrix} = 6\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - 14\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} + 36\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 36 - 70 + 36 = 2$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 1 & 36 & 9 \end{vmatrix} = \begin{vmatrix} 14 & 3 \\ 36 & 9 \end{vmatrix} - \begin{vmatrix} 6 & 1 \\ 36 & 9 \end{vmatrix} + \begin{vmatrix} 6 & 1 \\ 14 & 3 \end{vmatrix} = 18 - 18 + 4 = 4$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 1 & 4 & 36 \end{vmatrix} = \begin{vmatrix} 2 & 14 \\ 4 & 36 \end{vmatrix} - \begin{vmatrix} 1 & 6 \\ 4 & 36 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ 2 & 14 \end{vmatrix} = 16 - 12 + 2 = 6$$

Therefore $\quad x = D_1/D = 1, \ \ y = D_2/D = 2, \ \ z = D_3/D = 3$.

Evaluate the determinants by expansion along **(i)** the first row, **(ii)** the second column:

6. $$\begin{vmatrix} 2 & 3 & 5 \\ 0 & 1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \begin{cases} \textbf{(i)} \quad 2\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} + 5\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} = -14 + 18 - 15 = -11 \\ \\ \textbf{(ii)} \quad -3\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} = 18 - 13 - 16 = -11 \end{cases}$$

7. $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = \begin{cases} \textbf{(i)} \quad \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 + 2 + 2 = 6 \\ \\ \textbf{(ii)} \quad -\begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 2 + 3 + 1 = 6 \end{cases}$$

8. $$\begin{vmatrix} 1 & 3 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{vmatrix} = \begin{cases} \textbf{(i)} \quad \begin{vmatrix} -1 & 2 \\ 0 & 4 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = -4 + 0 + 0 = -4 \\ \\ \textbf{(ii)} \quad -3\begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0 - 4 + 0 = -4 \end{cases}$$

9. $$\begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = \begin{cases} \textbf{(i)} \quad 0 - 3\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 2 & 6 \end{vmatrix} = 0 + 6 + 24 = 30 \\ \\ \textbf{(ii)} \quad -3\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 0 - 6\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 6 + 0 + 24 = 30 \end{cases}$$

10. **(i)** Find the cofactors of all the elements of $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

(ii) Confirm that the same value of the determinant is obtained by expansion along every row and every column

(i)

$a_{11} = 1 \;\rightarrow\; C_{11} = +\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1$

$a_{12} = 2 \;\rightarrow\; C_{12} = -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -3$

$a_{13} = 3 \;\rightarrow\; C_{13} = +\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = -2$

$a_{21} = 2 \;\rightarrow\; C_{21} = -\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = -5$

$a_{22} = 0 \;\rightarrow\; C_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2$

$a_{23} = -1 \;\rightarrow\; C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = 3$

$a_{31} = 1 \;\rightarrow\; C_{31} = +\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = -2$

$a_{32} = -1 \;\rightarrow\; C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 7$

$a_{33} = 1 \;\rightarrow\; C_{33} = +\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$

(ii) $D = C_{11} + 2C_{12} + 3C_{13} = -1 - 6 - 6 = -13$ $\quad$ $D = C_{11} + 2C_{21} + C_{31} = -1 - 10 - 2 = -13$

$D = 2C_{21} + 0 - C_{23} = -10 + 0 - 3 = -13$ $\quad$ $D = 2C_{12} + 0 - C_{32} = -6 + 0 - 7 = -13$

$D = C_{31} - C_{32} + C_{33} = -2 - 7 - 4 = -13$ $\quad$ $D = 3C_{13} - C_{23} + C_{33} = -6 - 3 - 4 = -13$

Section 17.3

Evaluate:

11. $$\begin{vmatrix} 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & -1 & 2 & 3 \\ 2 & 2 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 4 \\ -1 & 2 & 3 \\ 2 & 1 & 0 \end{vmatrix} - 0 + \begin{vmatrix} 3 & 1 & 4 \\ 1 & -1 & 3 \\ 2 & 2 & 0 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= 2\left[\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + 0 + 4\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}\right]$$

$$+ \left[3\begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}\right]$$

$$-3\left[3\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 0\right]$$

$$= 2\left[-3 + 0 - 20\right] + \left[-18 + 6 + 16\right] - 3\left[-15 + 3 + 0\right]$$

$$= -46 + 4 + 36 = -6$$

12. $$\begin{vmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= 3\times 2\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6\times 2 - 4\times 2 = 4$$

13. $$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 2 & -3 \\ 2 & 1 & -1 & 2 \\ 5 & 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 2 & -3 \\ 1 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & -3 \\ 2 & -1 & 2 \\ 5 & 1 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$

$$-3\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix}$$

$$= (-2 + 14 - 12) + (3 - 24 + 21) = 0$$

14. $\begin{vmatrix} -2 & 6 & 17 & -5 \\ 0 & 3 & 22 & -17 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & -6 \end{vmatrix} = (-2)\times 3\times 4\times(-6) = 144$

Section 17.4

Use Cramer's rule to solve the systems of equations:

15.
$$\begin{aligned} 3x - 2y - 2z &= 0 \\ x + y - z &= 0 \\ 2x + 2y + z &= 0 \end{aligned}$$

The determinant of the coefficients is

$$D = \begin{vmatrix} 3 & -2 & -2 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 9 + 6 + 0$$

$$= 15 \neq 0$$

By Cramer's rule for homogeneous equation, with all $b_k = 0$ in equation (17.26),

$$x = y = z = 0$$

16.
$$\begin{aligned} w + 2x + 3y + z &= 5 \\ 2w + x + y + z &= 3 \\ w + 2x + y &= 4 \\ x + y + 2z &= 0 \end{aligned}$$

The determinant of the coefficients is (expansion along the first column)

$$D = \begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} + 0$$

$$= \left[\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right] - 2\left[2\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$$+ \left[2\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$$= \left[2 - 4 + 1\right] - 2\left[4 - 12 + 1\right] + \left[2 - 3 + 0\right] = -1 + 14 - 1 = 12$$

Similarly:

by expansion along the first column,

$$D_1 = \begin{vmatrix} 5 & 2 & 3 & 1 \\ 3 & 1 & 1 & 1 \\ 4 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{vmatrix} = -5+21-4+0=12 \quad \to \quad w = D/D_1 = 1$$

by expansion along the fourth row,

$$D_2 = \begin{vmatrix} 1 & 5 & 3 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 0+0-6+18=12 \quad \to \quad x = D/D_2 = 1$$

$$D_3 = \begin{vmatrix} 1 & 2 & 5 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 4 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix} = 0+6+0+6=12 \quad \to \quad y = D/D_3 = 1$$

$$D_4 = \begin{vmatrix} 1 & 2 & 3 & 5 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 0 \end{vmatrix} = 0-9-3+0=-12 \quad \to \quad z = D/D_4 = -1$$

Therefore $w = x = y = 1, \ z = -1$

17. **(i)** Show that the following equations have no solution unless $k = 3$, **(ii)** solve for this value of *k*.

$$\left.\begin{array}{ll} (1) & 2x - y + z = 2 \\ (2) & 3x + y - 2z = 1 \\ (3) & x - 3y + 4z = k \end{array}\right\} \to D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} \xrightarrow{\text{by first row}} = -4+14-10 = 0$$

The determinant of the coefficients is zero, so that the equations are either linearly dependent or inconsistent.

(i) Subtraction of equation (2) from twice (1) gives

$$x - 3y + 4z = 3$$

and this is inconsistent with equation (3) unless $k = 3$.

(ii) The equations are now consistent, but linearly dependent. Solving for *x* and *y* in terms of *z*,

$$(1)+(2) \quad \to \quad 5x - z = 3 \quad \to \quad x = (z+3)/5$$

$$(2)-(1) \quad \to \quad x + 2y - 3z = -1 \quad \to \quad y = (7z-4)/5$$

for all values of *z*.

18. (i) Find *k* for which the following equations have a nontrivial solution, **(ii)** solve for this value of *k*.

(i)

$$\left.\begin{array}{ll}(1) & kx+5y+3z=0\\ (2) & 5x+\ \ y-\ \ z=0\\ (3) & kx+2y+\ \ z=0\end{array}\right\} \rightarrow D=\begin{vmatrix} k & 5 & 3\\ 5 & 1 & -1\\ k & 2 & 1\end{vmatrix} \xrightarrow{\text{by first column}} =3k+5-8k$$

$$=0 \text{ when } k=1$$

The homogeneous equations are consistent when $k=1$.

(ii) We solve for *x* and *y* in terms of *z*:

$$(2)+(3) \ \rightarrow\ 6x+3y=0 \ \rightarrow\ y=-2x$$
$$(1)-(3) \ \rightarrow\ 3y+2z=0 \ \rightarrow\ y=-2z/3$$

Therefore, $x=z/3,\ y=-2z/3$ for all values of *z*.

Find **(i)** the values of λ for which the following systems of equations have nontrivial solutions, **(ii)** the solutions for these values of λ.

19.

$$2x+\ \ y=\lambda x$$
$$x+2y=\lambda y$$

(i) The equations can be written as the secular equations

$$\begin{array}{ccccc}(2-\lambda)x & + & y & = & 0\\ x & + & (2-\lambda) & = & 0\end{array}$$

and have nonzero solution when the secular determinant is zero:

$$D=\begin{vmatrix} 2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix}=(2-\lambda)^2-1=0 \ \rightarrow\ \lambda=1,3$$

(ii) We solve the secular equations for *y* in terms of *x*:

$$\lambda=1: \quad (2-\lambda)x+y \ \rightarrow\ x+y=0 \ \rightarrow\ y=-x$$
$$\lambda=3: \quad (2-\lambda)x+y \ \rightarrow\ -x+y=0 \ \rightarrow\ y=x$$

for all values of *x*.

20.
$$\begin{aligned} 3x + y &= \lambda x \\ x + 3y + z &= \lambda y \\ y + 3z &= \lambda z \end{aligned}$$

(i) The secular equations

$$\begin{array}{llllll} (1) & (3-\lambda)x & + & y & & = 0 \\ (2) & x & + & (3-\lambda)y & + \; z & = 0 \\ (3) & & & y & + \; (3-\lambda)z & = 0 \end{array}$$

have nonzero solution when the secular determinant is zero:

$$D = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (3-\lambda)\left[(3-\lambda)^2 - 2\right] = 0 \text{ when } \lambda = 3, \lambda = 3 \pm \sqrt{2}$$

(ii) We solve the secular equations for x and y in terms of z:

$$\lambda = 3: \begin{array}{l} (1) \to \; y = 0 \\ (2) \to \; x + z = 0 \end{array} \Bigg\} \to \begin{array}{l} x = -z \\ y = 0 \end{array}$$

$$\lambda = 3 + \sqrt{2}: \begin{array}{l} (1) \to \; -\sqrt{2}x + y = 0 \\ (3) \to \; y - \sqrt{2}z = 0 \end{array} \Bigg\} \to \begin{array}{l} x = z \\ y = \sqrt{2}z \end{array}$$

and, similarly,

$$\lambda = 3 - \sqrt{2}: \quad x = z, \; y = -\sqrt{2}z \,.$$

21.
$$\begin{aligned} x + 2y - 3z &= \lambda x \\ 2x + 4y - 6z &= \lambda y \\ -x - 2y + 3z &= \lambda z \end{aligned}$$

(i) The secular equations

$$\begin{array}{llllllll} (1) & (1-\lambda)x & + & 2y & - & 3z & = 0 \\ (2) & 2x & + & (4-\lambda)y & - & 6z & = 0 \\ (3) & -x & - & 2y & + & (3-\lambda)z & = 0 \end{array}$$

have nonzero solution when the secular determinant is zero:

$$D = \begin{vmatrix} 1-\lambda & 2 & -3 \\ 2 & 4-\lambda & -6 \\ -1 & -2 & 3-\lambda \end{vmatrix} = \lambda^2(8-\lambda) = 0 \text{ when } \lambda = 0 \text{(double)}, \; \lambda = 8$$

(ii)
$$\lambda = 0: \begin{array}{l} (1) \to \; x + 2y - 3z = 0 \\ (2) \to \; 2x + 4y - 6z = 0 \\ (3) \to \; -x - 2y + 3z = 0 \end{array} \Bigg\} \to x = 3z - 2y \text{ for all values of } y \text{ and } z$$

$$\lambda = 8: \begin{array}{l} (1) \to \; -7x + 2y - 3z = 0 \\ (2) \to \; 2x - 4y - 6z = 0 \\ (3) \to \; -x - 2y - 5z = 0 \end{array} \Bigg\} \to \begin{array}{l} x = -z \\ y = -2z \end{array} \Bigg\} \text{ for all values of } z$$

Section 17.5

Use the properties of determinants to show that:

22. $D = \begin{vmatrix} 3 & 6 & -3 \\ 2 & 1 & 5 \\ 1 & 2 & -1 \end{vmatrix} = 0$

row $1 = 3\times$row 3 $\rightarrow$ $D = 0$ by property 6

23. $\begin{vmatrix} 2 & 2 & -1 \\ 3 & 2 & 2 \\ -1 & 0 & -3 \end{vmatrix} = 0$

row $3 =$ row $1 -$ row 2 $\rightarrow$ $D = 0$ by property 8

24. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \dfrac{1}{2}\begin{vmatrix} a-b & a+b \\ c-d & c+d \end{vmatrix}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[\text{column 2 from column 1}]{\text{subtract}} \begin{vmatrix} a-b & b \\ c-d & d \end{vmatrix} \qquad \text{(property 7)}$$

$$\downarrow$$

$$\begin{vmatrix} a-b & \frac{1}{2}(a+b) \\ c-d & \frac{1}{2}(c+d) \end{vmatrix} \xleftarrow[\text{to column 2}]{\text{add } \frac{1}{2}\times\text{column 1}} \qquad \text{(property 7)}$$

$$\downarrow$$

$$= \frac{1}{2}\begin{vmatrix} a-b & a+b \\ c-d & c+d \end{vmatrix} \qquad \text{(property 2)}$$

25. Differentiate the following determinant with respect to x: $\begin{vmatrix} 1 & 2x & 3x^2 \\ 4x^3 & 5x^4 & 6x^5 \\ 7x^6 & 8x^7 & 9x^8 \end{vmatrix}$

$$\frac{d}{dx}\begin{vmatrix} 1 & 2x & 3x^2 \\ 4x^3 & 5x^4 & 6x^5 \\ 7x^6 & 8x^7 & 9x^8 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 6x \\ 4x^3 & 5x^4 & 6x^5 \\ 7x^6 & 8x^7 & 9x^8 \end{vmatrix} + \begin{vmatrix} 1 & 2x & 3x^2 \\ 12x^2 & 20x^3 & 30x^4 \\ 7x^6 & 8x^7 & 9x^8 \end{vmatrix} + \begin{vmatrix} 1 & 2x & 3x^2 \\ 4x^3 & 5x^4 & 6x^5 \\ 42x^5 & 56x^6 & 72x^7 \end{vmatrix}$$

Section 17.6

Evaluate the following determinants by reduction to triangular form:

26. $\begin{vmatrix} 3 & 2 & -2 \\ 6 & 1 & 5 \\ -9 & 3 & 4 \end{vmatrix} \xrightarrow[\text{add } 3\times\text{row 1 to row 3}]{\text{subtract } 2\times\text{row 1 from row 2}} \begin{vmatrix} 3 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & 9 & -2 \end{vmatrix}$

$\xrightarrow{\text{add } 3\times\text{row 2 to row 3}} \begin{vmatrix} 3 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & 0 & 25 \end{vmatrix} \rightarrow 3\times(-3)\times 25 = -225$

27. $\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \xrightarrow[\text{and from row 4}]{\text{subtract row 1 from row 3}} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix}$

$\xrightarrow[\text{from row 4}]{\text{subtract row 2}} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{vmatrix} \xrightarrow[\text{from row 4}]{\text{subtract row 3}} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1$

28. $\begin{vmatrix} 2 & 4 & 6 & 3 \\ 8 & 15 & 14 & 13 \\ 7 & 9 & 13 & 9 \\ 1 & 2 & 5 & 5 \end{vmatrix} \xrightarrow[\frac{7}{2}\times\text{row 1 from row 3},\ \frac{1}{2}\times\text{row 1 from row 4}]{\text{subtract } 4\times\text{row 1 from row 2}} \begin{vmatrix} 2 & 4 & 6 & 3 \\ 0 & -1 & -10 & 1 \\ 0 & -5 & -8 & -\frac{3}{2} \\ 0 & 0 & 2 & \frac{7}{2} \end{vmatrix}$

$\xrightarrow[\text{from row 3}]{\text{subtract } 5\times\text{row 2}} \begin{vmatrix} 2 & 4 & 6 & 3 \\ 0 & -1 & -10 & 1 \\ 0 & 0 & 42 & -\frac{13}{2} \\ 0 & 0 & 2 & \frac{7}{2} \end{vmatrix} \xrightarrow[\text{from row 4}]{\text{subtract } \frac{1}{21}\times\text{row 3}} \begin{vmatrix} 2 & 4 & 6 & 3 \\ 0 & -1 & -10 & 1 \\ 0 & 0 & 42 & -\frac{13}{2} \\ 0 & 0 & 0 & \frac{80}{21} \end{vmatrix}$

$= 2\times(-1)\times 42\times\frac{80}{21} = -320$

Section 17.7

29. Expand the determinant

$$D = \begin{vmatrix} \psi_1(x_1) & \psi_1(x_2) & \psi_1(x_3) & \psi_1(x_4) \\ \psi_2(x_1) & \psi_2(x_2) & \psi_2(x_3) & \psi_2(x_4) \\ \psi_3(x_1) & \psi_3(x_2) & \psi_3(x_3) & \psi_3(x_4) \\ \psi_4(x_1) & \psi_4(x_2) & \psi_4(x_3) & \psi_4(x_4) \end{vmatrix}$$

There are $4! = 24$ permutations of 4 objects x_1, x_2, x_3, x_4 and therefore 24 products

$$\psi_1(x_i)\,\psi_2(x_j)\,\psi_3(x_k)\,\psi_4(x_l)$$

with + sign if the permutation is obtained from x_1, x_2, x_3, x_4 by an even number of transpositions, and with $-$ sign for an odd number of transpositions. For example

$$\psi_1(x_1)\,\psi_2(x_2)\,\psi_3(x_3)\,\psi_4(x_4) \xrightarrow{1\leftrightarrow 2} -\psi_1(x_2)\,\psi_2(x_1)\,\psi_3(x_3)\,\psi_4(x_4)$$

$$\psi_1(x_1)\,\psi_2(x_2)\,\psi_3(x_3)\,\psi_4(x_4) \xrightarrow[3\leftrightarrow 4]{1\leftrightarrow 2} +\psi_1(x_2)\,\psi_2(x_1)\,\psi_3(x_4)\,\psi_4(x_3)$$

Then

$$\begin{aligned}
&\psi_1(x_1)\,\psi_2(x_2)\,\psi_3(x_3)\,\psi_4(x_4) \\
&-\psi_1(x_1)\,\psi_2(x_2)\,\psi_3(x_4)\,\psi_4(x_3) \\
&+\psi_1(x_1)\,\psi_2(x_3)\,\psi_3(x_4)\,\psi_4(x_2) \\
&-\psi_1(x_1)\,\psi_2(x_3)\,\psi_3(x_2)\,\psi_4(x_4) \\
&+\psi_1(x_1)\,\psi_2(x_4)\,\psi_3(x_2)\,\psi_4(x_3) \\
&-\psi_1(x_1)\,\psi_2(x_4)\,\psi_3(x_3)\,\psi_4(x_2) \\
&+\psi_1(x_2)\,\psi_2(x_1)\,\psi_3(x_4)\,\psi_4(x_3) \\
&-\psi_1(x_2)\,\psi_2(x_1)\,\psi_3(x_3)\,\psi_4(x_4) \\
&+\psi_1(x_2)\,\psi_2(x_3)\,\psi_3(x_1)\,\psi_4(x_4) \\
&-\psi_1(x_2)\,\psi_2(x_3)\,\psi_3(x_4)\,\psi_4(x_1) \\
&+\psi_1(x_2)\,\psi_2(x_4)\,\psi_3(x_3)\,\psi_4(x_1) \\
&-\psi_1(x_2)\,\psi_2(x_4)\,\psi_3(x_1)\,\psi_4(x_3) \\
&+\psi_1(x_3)\,\psi_2(x_1)\,\psi_3(x_2)\,\psi_4(x_4) \\
&-\psi_1(x_3)\,\psi_2(x_1)\,\psi_3(x_4)\,\psi_4(x_2) \\
&+\psi_1(x_3)\,\psi_2(x_2)\,\psi_3(x_4)\,\psi_4(x_1) \\
&-\psi_1(x_3)\,\psi_2(x_2)\,\psi_3(x_1)\,\psi_4(x_4) \\
&+\psi_1(x_3)\,\psi_2(x_4)\,\psi_3(x_1)\,\psi_4(x_2) \\
&-\psi_1(x_3)\,\psi_2(x_4)\,\psi_3(x_2)\,\psi_4(x_1) \\
&+\psi_1(x_4)\,\psi_2(x_1)\,\psi_3(x_3)\,\psi_4(x_2) \\
&-\psi_1(x_4)\,\psi_2(x_1)\,\psi_3(x_2)\,\psi_4(x_3) \\
&+\psi_1(x_4)\,\psi_2(x_2)\,\psi_3(x_1)\,\psi_4(x_3) \\
&-\psi_1(x_4)\,\psi_2(x_2)\,\psi_3(x_3)\,\psi_4(x_1) \\
&+\psi_1(x_4)\,\psi_2(x_3)\,\psi_3(x_2)\,\psi_4(x_1) \\
&-\psi_1(x_4)\,\psi_2(x_3)\,\psi_3(x_1)\,\psi_4(x_2)
\end{aligned}$$