The Chemistry Maths Book

Erich Steiner

University of Exeter

Second Edition 2008

Solutions

Chapter 16 Vectors

Section 16.2

1. The position vectors of the points A, B, C and D are ***a, b, c***, and ***d***. Express the following quantities in terms of ***a, b, c***, and ***d***: **(i)** $\overrightarrow{AB}$ and $\overrightarrow{BA}$, **(ii)** the position of the centroid of the points, **(iii)** the position of the midpoint of $\overrightarrow{BC}$, **(iv)** the position of an arbitrary point on the line $\overrightarrow{BC}$ (the equation of the line).

(i) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$, $\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b}$

(ii) the mean position of the four points is $\frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$

(iii) the mean position of ***b*** and ***c is*** $\frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$

(iv) $\boldsymbol{b} + \lambda\overrightarrow{BC} = \boldsymbol{b} + \lambda(\boldsymbol{c} - \boldsymbol{b})$, all λ

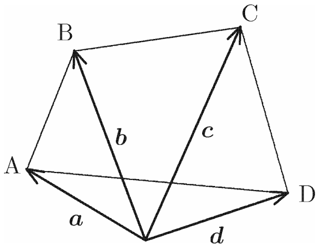


Figure 1

Section 16.3

2. Two sides of the triangle ABC are $\overrightarrow{AB} = (2, 1, -1)$ and $\overrightarrow{AC} = (3, 2, 0)$. Find $\overrightarrow{BC}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = (3, 2, 0) - (2, 1, -1) = (1, 1, 1)$

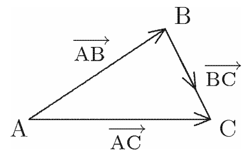


Figure 2

Find **(i)** the vector $\boldsymbol{a} = (a_1, a_2, a_3)$ with the given initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$, **(ii)** the length of $\boldsymbol{a}$, **(iii)** the unit vector parallel to $\boldsymbol{a}$.

3. $P(1, -2, 0)$, $Q(4, 2, 0)$

(i) $\boldsymbol{a} = \overrightarrow{PQ} = (4, 2, 0) - (1, -2, 0) = (3, 4, 0)$

(ii) $|\boldsymbol{a}| = \sqrt{3^2 + 4^2 + 0} = 5$

(iii) $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{1}{5}(3, 4, 0) = (3/5, 4/5, 0)$

4. $P(-3, 2, 1)$, $Q(-1, -3, 2)$

(i) $\boldsymbol{a} = \overrightarrow{PQ} = (-1, -3, 2) - (-3, 2, 1) = (2, -5, 1)$

(ii) $|\boldsymbol{a}| = \sqrt{4 + 25 + 1} = \sqrt{30}$

(iii) $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{1}{\sqrt{30}}(2, -5, 1) = (2/\sqrt{30}, -5/\sqrt{30}, 1/\sqrt{30})$

5. P(0, 0, 0) , Q(2, 3, 1)

(i) $\boldsymbol{a} = \overrightarrow{\text{PQ}} = (2, 3, 1)$

(ii) $|\boldsymbol{a}| = \sqrt{4+9+1} = \sqrt{14}$

(iii) $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|} = \dfrac{1}{\sqrt{14}}(2, 3, 1) = (2/\sqrt{14}\,, 3/\sqrt{14}\,, 1/\sqrt{14})$

For $\boldsymbol{a} = (1, 2, 3)$, $\boldsymbol{b} = (-2, 3, -4)$, $\boldsymbol{c} = (0, 4, -1)$, find

6. $\boldsymbol{a}+\boldsymbol{b},\ \boldsymbol{b}+\boldsymbol{a}$

$\boldsymbol{a}+\boldsymbol{b} = (1, 2, 3)+(-2, 3, -4) = (-1, 5, -1) = \boldsymbol{b}+\boldsymbol{a}$

7. $3\boldsymbol{a},\ -\boldsymbol{a},\ \boldsymbol{a}/3$

$3\boldsymbol{a} = 3(1, 2, 3) = (3, 6, 9), \quad -\boldsymbol{a} = -(1, 2, 3) = (-1, -2, -3), \quad \boldsymbol{a}/3 = (1/3, 2/3, 1)$

8. $3\boldsymbol{a}+2\boldsymbol{b}-3\boldsymbol{c}$

$3\boldsymbol{a}+2\boldsymbol{b}-3\boldsymbol{c} = (3, 6, 9)+(-4, 6, -8)-(0, 12, -3) = (-1, 0, 4)$

9. $3\boldsymbol{a}-3\boldsymbol{c},\ 3(\boldsymbol{a}-\boldsymbol{c})$

$3(\boldsymbol{a}-\boldsymbol{c}) = 3\boldsymbol{a}-3\boldsymbol{c} = (3, 6, 9)-(0, 12, -3) = (3, -6, 12)$

10. $|\boldsymbol{a}+\boldsymbol{b}|,\ |\boldsymbol{a}|+|\boldsymbol{b}|$

By Exercise 6, $\quad \boldsymbol{a}+\boldsymbol{b} = (-1, 5, -1) \ \to\ |\boldsymbol{a}+\boldsymbol{b}| = \sqrt{1+25+1} = \sqrt{27}$

and $\quad |\boldsymbol{a}| = \sqrt{1+4+9} = \sqrt{14}, \quad |\boldsymbol{b}| = \sqrt{4+9+16} = \sqrt{29}$

$\to\ |\boldsymbol{a}|+|\boldsymbol{b}| = \sqrt{14}+\sqrt{29}$

11. Three masses, $m_1 = 2,\ m_2 = 3,\ m_3 = 1$, have position vectors $\boldsymbol{r}_1 = (3, -2, 1)$, $\boldsymbol{r}_2 = (2, -1, 0)$, $\boldsymbol{r}_3 = (0, 1, -2)$, respectively. Find **(i)** the position vector of the centre of mass, **(ii)** the position vectors of the masses with respect to the centre of mass.

(i) By equation (16.12), the centre of mass is

$$\boldsymbol{R} = \frac{1}{(2+3+1)}\Big[2(3, -2, 1)+3(2, -1, 0)+(0, 1, -2)\Big] = \frac{1}{6}(12, -6, 0) = (2, -1, 0)$$

(ii)

$$\boldsymbol{r}_1 - \boldsymbol{R} = (3, -2, 1)-(2, -1, 0) = (1, -1, 1)$$
$$\boldsymbol{r}_2 - \boldsymbol{R} = (2, -1, 0)-(2, -1, 0) = (0, 0, 0)$$
$$\boldsymbol{r}_3 - \boldsymbol{R} = (0, 1, -2)-(2, -1, 0) = (-2, 2, -2)$$

12. Three charges, $q_1 = 3,\ q_2 = -2,\ q_3 = 1$, have position vectors $\boldsymbol{r}_1 = (2, 2, 1)$, $\boldsymbol{r}_2 = (2, -2, 3)$, $\boldsymbol{r}_3 = (0, -4, -3)$, respectively. Find **(i)** the dipole moment of the system of charges with respect to the origin, **(ii)** the position of the point with respect to which the dipole moment is zero.

(i) $\boldsymbol{\mu}(\boldsymbol{0}) = 3\boldsymbol{r}_1 - 2\boldsymbol{r}_2 + \boldsymbol{r}_3 = 3(2, 2, 1) - 2(2, -2, 3) + (0, -4, -3) = (2, 6, -6)$

(ii) By equation (16.16), the dipole moment with respect to $\boldsymbol{R}$ is

$\boldsymbol{\mu}(\boldsymbol{R}) = \boldsymbol{\mu}(\boldsymbol{0}) - Q\boldsymbol{R}$ where $Q = 3 - 2 + 1 = 2$ is the total charge. Then

$\boldsymbol{\mu}(\boldsymbol{R}) = \boldsymbol{0}$ when $\boldsymbol{\mu}(\boldsymbol{0}) = Q\boldsymbol{R}$

$$\rightarrow \quad \boldsymbol{R} = \frac{1}{2}(2, 6, -6) = (1, 3, -3)$$

13. Forces are said to be in equilibrium if the total force is zero. Find $\boldsymbol{f}$ such that $\boldsymbol{f}$, $\boldsymbol{f}_1 = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{f}_2 = 2\boldsymbol{j} - \boldsymbol{k}$ are in equilibrium.

$$\boldsymbol{f} + \boldsymbol{f}_1 + \boldsymbol{f}_2 = \boldsymbol{0} \text{ when } \boldsymbol{f} = -(\boldsymbol{f}_1 + \boldsymbol{f}_2) = -\left[(2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) + (2\boldsymbol{j} - \boldsymbol{k})\right]$$
$$= -2\boldsymbol{i} + \boldsymbol{j}$$

14. For the vectors $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{j} - 2\boldsymbol{k}, \boldsymbol{c} = -\boldsymbol{k}$, find **(i)** $\boldsymbol{x} = 2\boldsymbol{a} + \boldsymbol{b} + 4\boldsymbol{c}$, **(ii)** a vector perpendicular to $\boldsymbol{c}$ and $\boldsymbol{x}$, **(iii)** a vector perpendicular to $\boldsymbol{b}$ and $\boldsymbol{c}$.

(i) $\boldsymbol{x} = 2\boldsymbol{a} + \boldsymbol{b} + 4\boldsymbol{c} = 2(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) + (2\boldsymbol{j} - 2\boldsymbol{k}) + 4(-\boldsymbol{k}) \quad \rightarrow \quad \boldsymbol{x} = 4\boldsymbol{i}$

(ii) Vector $\boldsymbol{c}$ lies along z-direction, vector $\boldsymbol{x}$ along the x-direction. Therefore any vector along the y-direction is perpendicular $\boldsymbol{c}$ and $\boldsymbol{x}$: $\rightarrow \quad \lambda\boldsymbol{j}$, where λ is arbitrary.

(iii) Vector $\boldsymbol{b}$ lies in the yz-plane, vector $\boldsymbol{c}$ along the z-direction. Therefore a vector along the x-direction is perpendicular to $\boldsymbol{b}$ and $\boldsymbol{c}$: $\rightarrow \quad \lambda\boldsymbol{i}$.

Section 16.4

Differentiate with respect to t.

15. $2t\boldsymbol{i} + 3t^2\boldsymbol{j}$

$$\frac{\partial}{\partial t}(2t\boldsymbol{i} + 3t^2\boldsymbol{j}) = \left(\frac{\partial}{\partial t}2t\right)\boldsymbol{i} + \left(\frac{\partial}{\partial t}3t^2\right)\boldsymbol{j} = 2\boldsymbol{i} + 6t\boldsymbol{j}$$

16. $(\cos 2t, 3\sin t, 2t)$

$$\frac{\partial}{\partial t}(\cos 2t, 3\sin t, 2t) = \left(\frac{\partial}{\partial t}(\cos 2t), \frac{\partial}{\partial t}(3\sin t), \frac{\partial}{\partial t}(2t)\right) = (-2\sin 2t, 3\cos t, 2)$$

17. A body of mass *m* moves along the curve $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, where $x = at$ and $y = (at - \frac{1}{2}gt^2)$ at time *t*. **(i)** Find the velocity and acceleration at time *t*. **(ii)** Find the force acting on the body. Describe the motion of the body **(iii)** in the *x*-direction, **(iv)** in the *y*-direction, **(v)** overall.

We have $$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

$$= at\boldsymbol{i} + (at - \tfrac{1}{2}gt^2)\boldsymbol{j}$$

Then

(i) $$\boldsymbol{v} = \frac{\partial \boldsymbol{r}}{\partial t} = a\boldsymbol{i} + (a - gt)\boldsymbol{j}, \quad \boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} = -g\boldsymbol{j}$$

(ii) $$\boldsymbol{F} = -mg\boldsymbol{j}$$

(iii) Motion along the *x*-direction is at constant speed $v_x = a$, to the right (x increasing) if $a > 0$, to

the left if $a < 0$.

(iv) Vertical motion (along the *y*-direction) is under the influence of gravity ($\boldsymbol{F} = -mg\boldsymbol{j}$). The initial speed is $v_y(0) = a$ when $y(0) = 0$. The maximum height, when the vertical velocity is zero, is given by

$$\frac{dy}{dt} = a - gt = 0 \text{ when } t = a/g \quad \to \quad y(\max) = a^2/2g$$

(v) As illustrated in Figure 3 the motion of the body is parabolic with initial velocity $\boldsymbol{v}(0) = a\boldsymbol{i} + a\boldsymbol{j}$. This would be the motion of a projectile fired at angle of $45°$ to the horizontal if there were no air resistance or of other forces acting on the body.

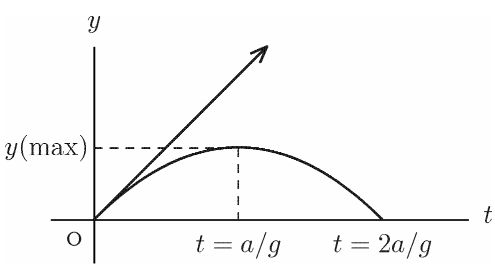


Figure 3

18. A body of mass m moves along the curve $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, where $x = 2\cos 3t$, $y = 2\sin 3t$ and $z = 3t$ at time t. Find **(i)** the velocity and acceleration at time t, **(ii)** the force acting on the body. Describe the motion of the body **(iii)** in the x- and y- directions, **(iv)** in the xy-plane, **(v)** in the z-direction, **(vi)** overall.

We have
$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$
$$= 2\cos 3t\,\boldsymbol{i} + 2\sin 3t\,\boldsymbol{j} + 3t\,\boldsymbol{k}$$

Then

(i)
$$\boldsymbol{v} = \frac{\partial \boldsymbol{r}}{\partial t} = -6\sin 3t\,\boldsymbol{i} + 6\cos 3t\,\boldsymbol{j} + 3\boldsymbol{k}$$
$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} = -18\cos 3t\,\boldsymbol{i} - 18\sin 3t\,\boldsymbol{j} = -9(x\boldsymbol{i} + y\boldsymbol{j})$$

(ii)
$$\boldsymbol{F} = m\boldsymbol{a} = -9m(x\boldsymbol{i} + y\boldsymbol{j})$$

(iii) The component of the force acting in the x-direction is $F_x = -9mx$ and this is the equation of motion of the harmonic oscillator, equation (12.29), with force constant $k = 9m$:

$$m\frac{d^2x}{dt^2} = -kx$$

The body therefore undergoes simple harmonic motion along the x-direction with frequency

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{3}{2\pi}$$

The body undergoes the same motion along the y-direction

(iv) The x- and y- coordinates of the body,

$$x = 2\cos 3t\,, \qquad y = 2\sin 3t$$

are parametric equation of the circle with radius 2 and centre at $(x, y) = (0, 0)$, on the z-axis (see Example 9.13). The motion in the xy-plane is therefore circular around the z-axis. The body perform a complete revolution in time $t = 2\pi/3$. The frequency of circulation is therefore $\nu = 3/2\pi$ (as in (iii) for each direction)

(v) The body moves in the positive z-direction with constant speed $\boldsymbol{v}_z = 3$

(vi) The motion is circular in a plane, parallel to the xy-plane, that moves at constant speed in the z-direction. The resulting motion is around a right-handed helix as described in Example 10.13

Section 16.5

For $\boldsymbol{a} = (1, 3, -2)$, $\boldsymbol{b} = (0, 3, 1)$, $\boldsymbol{c} = (1, -1, -3)$, find:

19. $\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{b} \cdot \boldsymbol{a}$

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (1,3,-2) \cdot (0,3,1) = (1 \times 0) + (3 \times 3) + (-2 \times 1) \\ &= 7 = \boldsymbol{b} \cdot \boldsymbol{a} \end{aligned}$$

20. $(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{c}$, $\boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{c}$

$$(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{c} = \left[(1,3,-2) - (0,3,1)\right] \cdot (1,-1,-3) = (1,0,-3) \cdot (1,-1,-3) = 1 + 0 + 9 = 10$$

$$\boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{c} = (1,3,-2) \cdot (1,-1,-3) - (0,3,1) \cdot (1,-1,-3) = (1 - 3 + 6) - (0 - 3 - 3) = 10$$

21. $(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b}$

$$(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} = \left[(1,3,-2) \cdot (1,-1,-3)\right] \times (0,3,1) = 4 \times (0,3,1) = (0,12,4)$$

22. Show that $\boldsymbol{a} = (1, 2, 3)$, $\boldsymbol{b} = (0, -3, 2)$ and $\boldsymbol{c} = (-13, 2, 3)$ are orthogonal vectors.

$$\boldsymbol{a} \cdot \boldsymbol{b} = (1, 2, 3) \cdot (0,-3,2) = 0 - 6 + 6 = 0$$

$$\boldsymbol{b} \cdot \boldsymbol{c} = (0,-3,2) \cdot (-13, 2, 3) = 0 - 6 + 6 = 0$$

$$\boldsymbol{c} \cdot \boldsymbol{a} = (-13, 2, 3) \cdot (1, 2, 3) = -13 + 4 + 9 = 0$$

23. Find the value of λ for which $\boldsymbol{a} = (\lambda, 3, 1)$ and $\boldsymbol{b} = (2, 1, -1)$ are orthogonal.

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} = (\lambda, 3, 1) \cdot (2, 1, -1) &= 2\lambda + 3 - 1 \\ &= 0 \text{ when } \lambda = -1 \end{aligned}$$

For $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{j}$, $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, find

24. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{i} \cdot \boldsymbol{j} = 0$

25. $\boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{j} \cdot (2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) = 0 - 3 + 0 = -3$

26. $\boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{i} \cdot (2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) = 2 + 0 + 0 = 2$

27. Find the angles between the direction of $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \sqrt{2}\boldsymbol{k}$ and the *x*-, *y*-, and *z*- directions.

Let the angles be α, β and γ, respectively. We need $|\boldsymbol{a}| = \sqrt{1+1+2} = 2$.

Then
$$\boldsymbol{a}\cdot\boldsymbol{i} = |\boldsymbol{a}|\cos\alpha \;\to\; 1 = 2\cos\alpha \;\to\; \cos\alpha = \frac{1}{2} \;\to\; \alpha = \frac{\pi}{3}$$
$$\boldsymbol{a}\cdot\boldsymbol{j} = |\boldsymbol{a}|\cos\beta \;\to\; -1 = 2\cos\beta \;\to\; \cos\beta = -\frac{1}{2} \;\to\; \beta = \frac{2\pi}{3}$$
$$\boldsymbol{a}\cdot\boldsymbol{k} = |\boldsymbol{a}|\cos\gamma \;\to\; \sqrt{2} = 2\cos\gamma \;\to\; \cos\gamma = \frac{1}{\sqrt{2}} \;\to\; \gamma = \frac{\pi}{4}$$

28. A body undergoes the displacement $\boldsymbol{d}$ under the influence of a force $\boldsymbol{f} = 3\boldsymbol{i} + 2\boldsymbol{j}$. Calculate the work done **(i)** by the force on the body when $\boldsymbol{d} = 2\boldsymbol{i} - \boldsymbol{j}$, **(ii)** by the body against the force when $\boldsymbol{d} = \boldsymbol{i} - 3\boldsymbol{k}$, **(iii)** by the body against the force when $\boldsymbol{d} = 2\boldsymbol{k}$.

(i) $W = \boldsymbol{f}\cdot\boldsymbol{d} = (3\boldsymbol{i}+2\boldsymbol{j})\cdot(2\boldsymbol{i}-\boldsymbol{j}) = 6-2 = 4$

(ii) $W = -\boldsymbol{f}\cdot\boldsymbol{d} = -(3\boldsymbol{i}+2\boldsymbol{j})\cdot(\boldsymbol{i}-3\boldsymbol{k}) = -3$

(iii) $W = -\boldsymbol{f}\cdot\boldsymbol{d} = -(3\boldsymbol{i}+2\boldsymbol{j})\cdot(2\boldsymbol{k}) = 0$

29. A body undergoes a displacement from $\boldsymbol{r}_1 = (0, 0, 0)$ to $\boldsymbol{r}_2 = (2, 3, 1)$ under the influence of the conservative force $\boldsymbol{F} = x\boldsymbol{i} + 2y\boldsymbol{j} + 3z\boldsymbol{k}$. **(i)** Calculate the work $W(\boldsymbol{r}_1 \to \boldsymbol{r}_2)$ done on the body. **(ii)** Find the potential-energy function $V(\boldsymbol{r})$ of which the components of the force are (–) the partial derivatives. **(iii)** Confirm that $W(\boldsymbol{r}_1 \to \boldsymbol{r}_2) = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2)$.

(i) The work done by a conservative force is independent of the path.The contribution of the three components of force are then additive:

$$W(\boldsymbol{r}_1 \to \boldsymbol{r}_2) = \int_C \left[F_x\,dx + F_y\,dy + F_z\,dz\right]$$
$$= \int_{x_1}^{x_2} F_x\,dx + \int_{y_1}^{y_2} F_y\,dy + \int_{z_1}^{z_2} F_z\,dz$$

Therefore $$W(\boldsymbol{r}_1 \to \boldsymbol{r}_2) = \int_0^2 x\,dx + 2\int_o^3 y\,dy + 3\int_0^1 z\,dz = 2 + 9 + \frac{3}{2} = \frac{25}{2}$$

(ii) $$F_x = x = -\frac{\partial V}{\partial x},\quad F_y = 2y = -\frac{\partial V}{\partial y},\quad F_z = 3z = -\frac{\partial V}{\partial z}$$
$$\to\; V = -\frac{1}{2}x^2 - y^2 - \frac{3}{2}z^2 + C$$

(iii) $$V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2) = \left[C\right] - \left[-2 - 9 - \frac{3}{2} + C\right] = \frac{25}{2}$$

30. Calculate the energy of interaction between the system of charges $q_1 = 2$, $q_2 = -3$ and $q_3 = 1$ at positions $\boldsymbol{r}_1 = (3, -2, 1)$, $\boldsymbol{r}_2 = (0, 1, 2)$ and $\boldsymbol{r}_3 = (0, 2, 1)$, respectively, and the applied electric field $\boldsymbol{E} = -2\boldsymbol{k}$.

The energy of interaction is

$$V = -\boldsymbol{\mu}\cdot\boldsymbol{E} = 2\mu_z$$

where $\mu_z = q_1 z_1 + q_2 z_2 + q_3 z_3 = 2 - 6 + 1 = -3$

Then $V = -6$, independent of choice of origin for the neutral system.

Section 16.6

For $\boldsymbol{a} = (1, 3, -2)$, $\boldsymbol{b} = (0, 3, 1)$, $\boldsymbol{c} = (0, -1, 2)$, find:

31. $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{b}\times\boldsymbol{a}$

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{b} &= (1, 3, -2)\times(0, 3, 1)\\ &= \boldsymbol{i}\left[3\times1-(-2)\times3\right]+\boldsymbol{j}\left[(-2)\times0-1\times1\right]+\boldsymbol{k}\left[1\times3-0\times3\right]\\ &= 9\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}\end{aligned}$$

$$\boldsymbol{b}\times\boldsymbol{a} = -\boldsymbol{a}\times\boldsymbol{b} = -9\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$$

32. $\boldsymbol{b}\times\boldsymbol{c}$, $|\boldsymbol{b}\times\boldsymbol{c}|$

$$\begin{aligned}\boldsymbol{b}\times\boldsymbol{c} &= (0, 3, 1)\times(0, -1, 2)\\ &= \boldsymbol{i}\left[3\times2-1\times(-1)\right]+\boldsymbol{j}\left[1\times0-2\times0\right]+\boldsymbol{k}\left[0\times(-1)-0\times3\right]\\ &= 7\boldsymbol{i}\end{aligned}$$

$$|\boldsymbol{b}\times\boldsymbol{c}| = 7$$

33. $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}$, $\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$

$$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} &= (1, 6, -1)\times(0, -1, 2)\\ &= \boldsymbol{i}\left[6\times2-(-1)\times(-1)\right]+\boldsymbol{j}\left[(-1)\times0-1\times2\right]+\boldsymbol{k}\left[1\times(-1)-6\times0\right]\\ &= 11\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}\end{aligned}$$

We have

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{c} &= (1, 3, -2)\times(0, -1, 2)\\ &= \boldsymbol{i}\left[3\times2-(-2)\times(-1)\right]+\boldsymbol{j}\left[(-2)\times0-1\times2\right]+\boldsymbol{k}\left[1\times(-1)-3\times0\right]\\ &= 4\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}\end{aligned}$$

and, from Exercise 32

$$\boldsymbol{b}\times\boldsymbol{c} = 7\boldsymbol{i}$$

Therefore $\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c} = 11\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$

34. $\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}$

$$\boldsymbol{c}\times\boldsymbol{a}=-\boldsymbol{a}\times\boldsymbol{c}\quad\rightarrow\quad\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}=\boldsymbol{0}$$

35. $(\boldsymbol{a}\times\boldsymbol{c})\cdot\boldsymbol{b}$

From Exercise 33, $\boldsymbol{a}\times\boldsymbol{c}=4\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$

Therefore $\quad(\boldsymbol{a}\times\boldsymbol{c})\cdot\boldsymbol{b}=(4\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k})\cdot(3\boldsymbol{j}+\boldsymbol{k})=0-6-1=-7$

36. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$, $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$

From Exercise 32, $\boldsymbol{b}\times\boldsymbol{c}=7\boldsymbol{i}$

Therefore $\quad\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k})\times(7\boldsymbol{i})=-14\boldsymbol{j}-21\boldsymbol{k}$

From Exercise 31, $\boldsymbol{a}\times\boldsymbol{b}=9\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$

Therefore $\quad(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(9\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k})\times(-\boldsymbol{j}+2\boldsymbol{k})=\boldsymbol{i}-18\boldsymbol{j}-9\boldsymbol{k}$

37. Show that $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$.

By Equation (16.52)

$$\boldsymbol{a}\times\boldsymbol{b}=(a_y b_z-a_z b_y)\boldsymbol{i}+(a_z b_x-a_x b_z)\boldsymbol{j}+(a_x b_y-a_y b_x)\boldsymbol{k}$$

Therefore $\quad(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a}=(a_y b_z-a_z b_y)a_x+(a_z b_x-a_x b_z)a_y+(a_x b_y-a_y b_x)a_z$

$$=\cancel{a_y b_z a_x}-\cancel{a_z b_y a_x}+\cancel{a_z b_x a_y}-\cancel{a_x b_z a_y}+\cancel{a_x b_y a_z}-\cancel{a_y b_x a_z}=0$$

and similarly for $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}$

38. The quantity $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is called a **triple scalar product**. Show that

(i) $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$ **(ii)** $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}$ (determinant)

We have $\quad\boldsymbol{b}\times\boldsymbol{c}=(b_y c_z-b_z c_y)\boldsymbol{i}+(b_z c_x-b_x c_z)\boldsymbol{j}+(b_x c_y-b_y c_x)\boldsymbol{k}$

(i) Therefore $\quad\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=a_x(b_y c_z-b_z c_y)+a_y(b_z c_x-b_x c_z)+a_z(b_x c_y-b_y c_x)$

$$=\left[a_x b_y c_z+c_x a_y b_z+b_x c_y a_z\right]-\left[a_x c_y b_z+b_x a_y c_z+c_x b_y a_z\right]$$

The triple scalar product is invariant under a cyclic permutation of the symbols *a*, *b*, *c*. Thus, if $(a,b,c)\rightarrow(c,a,b)$ then

$$\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})=(a_y b_z-a_z b_y)c_x+(a_z b_x-a_x b_z)c_y+(a_x b_y-a_y b_x)c_z$$

$$=\left[a_x b_y c_z+c_x a_y b_z+b_x c_y a_z\right]-\left[a_x c_y b_z+b_x a_y c_z+c_x b_y a_z\right]$$

and similarly for $\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$

(ii) By equations (16.52) and (16.53), and the discussion of determinants in Chapter 7, the vector product $\boldsymbol{b}\times\boldsymbol{c}$ can be written in the form of a determinant,

$$\boldsymbol{b}\times\boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \boldsymbol{i}(b_y c_z - b_z c_y) + \boldsymbol{j}(b_z c_x - b_x c_z) + \boldsymbol{k}(b_x c_y - b_y c_x)$$

The triple scalar product $\boldsymbol{a}\bullet(\boldsymbol{b}\times\boldsymbol{c})$ is obtained by replacing the vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ by the corresponding components a_x, a_y, a_z:

$$\boldsymbol{a}\bullet(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z) + a_z(b_x c_y - b_y c_x)$$

39. The quantity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ is called a **triple vector product**.

(i) By expanding in terms of components, show that $\mathbf{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\bullet\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\bullet\boldsymbol{b})\boldsymbol{c}$

(ii) Confirm this formula for the vectors $\boldsymbol{a} = (1, 3, -2)$, $\boldsymbol{b} = (0, 3, 1)$, $\boldsymbol{c} = (0, -1, 2)$.

(i) Let
$$\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{i}(b_y c_z - b_z c_y) + \boldsymbol{j}(b_z c_x - b_x c_z) + \boldsymbol{k}(b_x c_y - b_y c_x)$$
$$= \boldsymbol{d} = \boldsymbol{i}d_x + \boldsymbol{j}d_y + \boldsymbol{k}d_z$$

Then
$$\begin{aligned}
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= \boldsymbol{i}(a_y d_z - a_z d_y) + \boldsymbol{j}(a_z d_x - a_x d_z) + \boldsymbol{k}(a_x d_y - a_y d_x) \\
&= \boldsymbol{i}\left[a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z)\right] \\
&\quad + \boldsymbol{j}\left[a_z(b_y c_z - b_z c_y) - a_x(b_x c_y - b_y c_x)\right] \\
&\quad + \boldsymbol{k}\left[a_x(b_z c_x - b_x c_z) - a_y(b_y c_z - b_z c_y)\right] \\
&= \boldsymbol{i}(a_x c_x + a_y c_y + a_z c_z)b_x - \boldsymbol{i}(a_x b_x + a_y b_y + a_z b_z)c_x \\
&\quad + \boldsymbol{j}(a_x c_x + a_y c_y + a_z c_z)b_y - \boldsymbol{j}(a_x b_x + a_y b_y + a_z b_z)c_y \\
&\quad + \boldsymbol{k}(a_x c_x + a_y c_y + a_z c_z)b_z - \boldsymbol{k}(a_x b_x + a_y b_y + a_z b_z)c_z \\
&= (a_x c_x + a_y c_y + a_z c_z)(\boldsymbol{i}b_x + \boldsymbol{j}b_y + \boldsymbol{k}b_z) \\
&\quad - (a_x b_x + a_y b_y + a_z b_z)(\boldsymbol{i}c_x + \boldsymbol{j}c_y + \boldsymbol{k}c_z) \\
&= (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}
\end{aligned}$$

(ii) From Exercise 36, $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = -14\boldsymbol{j} - 21\boldsymbol{k}$

Also $\quad \boldsymbol{a}\cdot\boldsymbol{c} = -7, \quad \boldsymbol{a}\cdot\boldsymbol{b} = 7 \quad \to \quad (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} = -7(\boldsymbol{b}+\boldsymbol{c})$

$$= -14\boldsymbol{j} + 21\boldsymbol{k}$$

Therefore

$$\mathbf{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\bullet\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\bullet\boldsymbol{b})\boldsymbol{c}$$

40. Find the area of the parallelogram whose vertices (in the *xy*-plane) have coordinates $(1, 2)$, $(4, 3)$, $(8, 6)$, $(5, 5)$.

If A, B, C, and D are the points $(1, 2)$, $(4, 3)$, $(5, 5)$, and $(8, 6)$, respectively, in the *xy*-plane, then, as in Figure 4,

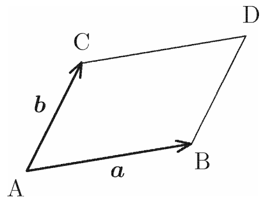
$$\overrightarrow{AB} = (3,1) = \boldsymbol{a}\,,\ \overrightarrow{AC} = (4,3) = \boldsymbol{b}\,,\ \overrightarrow{BD} = (4,3) = \boldsymbol{b}\,,\ \text{and}\ \overrightarrow{CD} = (3,1) = \boldsymbol{a}$$

Then $\boldsymbol{a}\times\boldsymbol{b} = (3\boldsymbol{i}+\boldsymbol{j})\times(4\boldsymbol{i}+3\boldsymbol{j}) = 5\boldsymbol{k}$

and the area of the parallelogram ABCD is

$$|\,\boldsymbol{a}\times\boldsymbol{b}\,| = 5$$

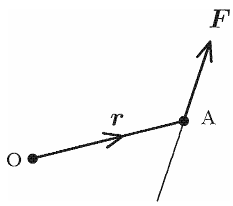
Figure 4



41. The force $\boldsymbol{F}$ acts on a line through the point A. Find the moment of the force about the point O for
(i) $\boldsymbol{F} = (1, -3, 0)$, $A(2, 1, 0)$, $O(0, 0, 0)$ **(ii)** $\boldsymbol{F} = (0, 1, -1)$, $A(1, 1, 0)$, $O(1, 0, 2)$
(iii) $\boldsymbol{F} = (1, 0, -2)$, $A(0, 0, 0)$, $O(1, 0, 3)$

In each case, let $\boldsymbol{r} = \overrightarrow{OA}$

Figure 5



(i) $\boldsymbol{r} = (2,1,0)$ and $\boldsymbol{F} = (1,-3,0)$ both lie in the *xy*-plane, and the torque

$$\boldsymbol{r}\times\boldsymbol{F} = \left[2\times(-3) - 1\times 1\right]\boldsymbol{k} = -7\boldsymbol{k}$$

lies along the *z*-direction.

(ii) $\boldsymbol{r} = (1,1,0) - (1,0,2) = (0,1,-2)$ and $\boldsymbol{F} = (0, 1, -1)$ lie in the *yz*-plane, and

$$\boldsymbol{r}\times\boldsymbol{F} = \left[1\times(-1) - (-2)\times 1\right]\boldsymbol{i} = \boldsymbol{i}$$

lies along the *x*-direction

(ii) $\boldsymbol{r} = (0,0,0) - (1,0,3) = (-1,0,-3)$ and $\boldsymbol{F} = (1, 0, -2)$ lie in the *xz*-plane, and

$$\boldsymbol{r}\times\boldsymbol{F} = \left[(-3)\times 1) - (-1)\times(-2)\right]\boldsymbol{j} = -5\boldsymbol{j}$$

lies along the *y*-direction

42. Calculate the torque experienced by the system of charges $q_1 = 2$, $q_2 = -3$ and $q_3 = 1$ at positions $\boldsymbol{r}_1 = (3, -2, 1)$, $\boldsymbol{r}_2 = (0, 1, 2)$ and $\boldsymbol{r}_3 = (0, 2, 1)$, respectively, in the electric field $\boldsymbol{E} = -\boldsymbol{k}$.

The dipole moment of the neutral system is

$$\boldsymbol{\mu} = \sum_i q_i \boldsymbol{r}_i = 2(3, -2, 1) - 3(0, 1, 2) + (0, 2, 1)$$
$$= 6\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k}$$

and the torque acting on the system in the presence of electric field $\boldsymbol{E} = -\boldsymbol{k}$ is

$$\boldsymbol{T} = \boldsymbol{\mu} \times \boldsymbol{E} = -(6\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{k}) \times \boldsymbol{k} = 5\boldsymbol{i} + 6\boldsymbol{j}$$

43. A charge *q* moving with velocity $\boldsymbol{v}$ in the presence of an electric field ***E*** and a magnetic field **B** experiences a total force $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ called the **Lorentz force**. Calculate the force acting on the charge $q = 3$ moving with velocity $\boldsymbol{v} = (2, 3, 1)$ in the presence of the electric field $\boldsymbol{E} = 2\boldsymbol{i}$ and magnetic field $\boldsymbol{B} = 3\boldsymbol{j}$.

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = 3\left[(2\boldsymbol{i}) + (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) \times (3\boldsymbol{j})\right]$$
$$= -3\boldsymbol{i} + 18\boldsymbol{k}$$

44. Use the property of the triple vector product (Exercise 39) to derive equation (16.60) from equations (16.58) and (16.59).

Equation (16.59) for angular momentum is

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}$$

and equation (16.58) for velocity is

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

Therefore $\quad \boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$.

This is triple vector product and, by Exercise 39, $\mathbf{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$. Therefore

$$\boldsymbol{l} = m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = m(\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{\omega} - m(\boldsymbol{r} \cdot \boldsymbol{\omega})\boldsymbol{r}$$
$$= mr^2\boldsymbol{\omega} - m(\boldsymbol{r} \cdot \boldsymbol{\omega})\boldsymbol{r}$$

and this is equation (16.60)

45. The position of a particle of mass m moving in a circle of radius R about the z-axis with angular speed ω is given by the vector function $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z\boldsymbol{k}$ where $x(t) = R\cos\omega t$, $y(t) = R\sin\omega t$, $z = \text{constant}$. **(i)** What is the angular velocity $\boldsymbol{\omega}$ about the z-axis? **(ii)** Find the velocity $\boldsymbol{v}$ of the particle in terms of ω, x, y, and z. **(iii)** Find the angular momentum of the particle in terms of ω, x, y, and z. **(iv) C**onfirm equation (16.60) in this case

(i) $\boldsymbol{\omega} = \omega\boldsymbol{k}$

(ii)
$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k} = -\omega R\sin\omega t\,\boldsymbol{i} + \omega R\cos\omega t\,\boldsymbol{j}$$
$$= -\omega y\,\boldsymbol{i} + \omega x\,\boldsymbol{j}$$

(iii)
$$\boldsymbol{l} = \boldsymbol{r}\times\boldsymbol{p} = m(x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\boldsymbol{k})\times(-\omega y\,\boldsymbol{i} + \omega x\,\boldsymbol{j})$$
$$= m\omega\Big[(x^2 + y^2)\boldsymbol{k} - xz\boldsymbol{i} - yz\,\boldsymbol{j}\Big]$$

(iv) We have, from (iii),

$$\boldsymbol{l} = m\omega\Big[(x^2 + y^2 + z^2)\boldsymbol{k} - xz\boldsymbol{i} - yz\,\boldsymbol{j} - z^2\boldsymbol{k}\Big]$$
$$= m\omega\Big[r^2\,\boldsymbol{k} - z(x\boldsymbol{i} + y\,\boldsymbol{j} + z\boldsymbol{k})\Big] = mr^2\boldsymbol{\omega} - m(z\omega)\boldsymbol{r}$$
$$= mr^2\boldsymbol{\omega} - m(\boldsymbol{r}\cdot\boldsymbol{\omega})\boldsymbol{r}$$

which is equation (16.60).

46. For the system in Exercise 45, show that $\boldsymbol{l} = I\boldsymbol{\omega}$ when $z = 0$, where I is the moment of inertia about the axis of rotation.

From (iii) in Exercise 45,

$$\boldsymbol{l} = m\omega\Big[(x^2 + y^2)\boldsymbol{k} - xz\boldsymbol{i} - yz\,\boldsymbol{j}\Big]$$
$$= m(x^2 + y^2)\,\omega\boldsymbol{k} \quad \text{when } z = 0$$

But $I = m(x^2 + y^2)$ is the moment of inertia with respect to the axis of rotation.

Therefore $\boldsymbol{l} = I\boldsymbol{\omega}$.

47. The total angular momentum of a system of particles is the sum of the angular momenta of the individual particles. If the system in Exercise 45 is replaced by a system of two particles of mass m with positions $\boldsymbol{r}_1 = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{r}_2 = -x(t)\boldsymbol{i} - y(t)\boldsymbol{j} + z\boldsymbol{k}$, find the total angular momentum $\boldsymbol{l} = \boldsymbol{l}_1 + \boldsymbol{l}_2$, and show that $\boldsymbol{l} = I\boldsymbol{\omega}$ where I is the total moment of inertia about the axis of rotation. This example demonstrates that $\boldsymbol{l} = I\boldsymbol{\omega}$ when the axis of rotation is an axis of symmetry of the system.

From (iii) in Exercise 45,

$$\left.\begin{aligned} \boldsymbol{l}_1 &= m\boldsymbol{\omega}\Big[(x^2+y^2)\boldsymbol{k} - xz\boldsymbol{i} - yz\boldsymbol{j}\Big] \\ \boldsymbol{l}_2 &= m\boldsymbol{\omega}\Big[(x^2+y^2)\boldsymbol{k} + xz\boldsymbol{i} + yz\boldsymbol{j}\Big] \end{aligned}\right\} \boldsymbol{l} = \boldsymbol{l}_1 + \boldsymbol{l}_2 = 2m(x^2+y^2)\boldsymbol{\omega}\boldsymbol{k} = I\boldsymbol{\omega}$$

where $I = 2m(x^2+y^2)$ is the moment of inertia of the system of two particles.

Section 16.8

Find the gradient ∇f for

48. $f = 2x^2 + 3y^2 - z^2$

$$\nabla f = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)(2x^2 + 3y^2 - z^2) = 4x\boldsymbol{i} + 6y\boldsymbol{j} - 2z\boldsymbol{k}$$

49. $f = xy + zx + yz$

$$\nabla(xy + zx + yz) = (y+z)\boldsymbol{i} + (x+z)\boldsymbol{j} + (x+y)\boldsymbol{k}$$

50. $f = (x^2 + y^2 + z^2)^{-1/2}$

We have $\dfrac{\partial f}{\partial x} = -x(x^2 + y^2 + z^2)^{-3/2}$

and similarly for the other partial derivatives. Then

$$\nabla(x^2 + y^2 + z^2)^{-1/2} = -(x^2 + y^2 + z^2)^{-3/2}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

Section 16.9

Find div $\boldsymbol{v}$ and curl $\boldsymbol{v}$ for

We have div $\boldsymbol{v} = \nabla \cdot \boldsymbol{v}$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}k\right)\cdot(v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

curl $\boldsymbol{v} = \nabla \times \boldsymbol{v}$

$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{k}$$

51. $\boldsymbol{v} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$$\text{div}\ \boldsymbol{v} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}k\right)\cdot(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

$$= \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 1 + 1 + 1 = 3$$

$$\text{curl}\ (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\boldsymbol{k} = \boldsymbol{0}$$

52. $\boldsymbol{v} = z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}$

$$\text{div}\ (z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}) = \left(\frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z}\right) = 0$$

$$\text{curl}\ (z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}) = \left(\frac{\partial y}{\partial y} - \frac{\partial x}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial z}{\partial z} - \frac{\partial y}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial z}{\partial y}\right)\boldsymbol{k}$$

$$= \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

53. $\boldsymbol{v} = yz\boldsymbol{i} + zx\boldsymbol{j} + xy\boldsymbol{k}$

$$\text{div}\ (yz\boldsymbol{i} + zx\boldsymbol{j} + xy\boldsymbol{k}) = \frac{\partial yz}{\partial x} + \frac{\partial zx}{\partial y} + \frac{\partial xy}{\partial z} = 0$$

$$\text{curl}\ (yz\boldsymbol{i} + zx\boldsymbol{j} + xy\boldsymbol{k}) = \left(\frac{\partial xy}{\partial y} - \frac{\partial zx}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial yz}{\partial z} - \frac{\partial xy}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial zx}{\partial x} - \frac{\partial yz}{\partial y}\right)\boldsymbol{k}$$

$$= (x - x)\,\boldsymbol{i} + (y - y)\,\boldsymbol{j} + (z - z)\,\boldsymbol{k} = \boldsymbol{0}$$

54. Show that curl $\boldsymbol{v} = \mathbf{0}$ if $\boldsymbol{v} = \text{grad}\, f$.

We have $\boldsymbol{v} = \text{grad}\, f = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k}$

Therefore $\text{curl}\,\boldsymbol{v} = \left(\dfrac{\partial f_z}{\partial y} - \dfrac{\partial f_y}{\partial z}\right)\boldsymbol{i} + \left(\dfrac{\partial f_x}{\partial z} - \dfrac{\partial f_z}{\partial x}\right)\boldsymbol{j} + \left(\dfrac{\partial f_y}{\partial x} - \dfrac{\partial f_x}{\partial y}\right)\boldsymbol{k}$

$$= \left(f_{yz} - f_{zy}\right)\boldsymbol{i} + \left(f_{zx} - f_{xz}\right)\boldsymbol{j} + \left(f_{xy} - f_{yx}\right)\boldsymbol{k} = \mathbf{0}$$

55. Show that div $\boldsymbol{v} = 0$ if $\boldsymbol{v} = \text{curl}\,\boldsymbol{w}$.

We have $\text{curl}\,\boldsymbol{w} = \left(\dfrac{\partial w_z}{\partial y} - \dfrac{\partial w_y}{\partial z}\right)\boldsymbol{i} + \left(\dfrac{\partial w_x}{\partial z} - \dfrac{\partial w_z}{\partial x}\right)\boldsymbol{j} + \left(\dfrac{\partial w_y}{\partial x} - \dfrac{\partial w_x}{\partial y}\right)\boldsymbol{k}$

Therefore $\text{div}\,\boldsymbol{v} = \left(\dfrac{\partial^2 w_z}{\partial x\partial y} - \dfrac{\partial^2 w_y}{\partial x\partial z}\right) + \left(\dfrac{\partial^2 w_x}{\partial y\partial z} - \dfrac{\partial^2 w_z}{\partial y\partial x}\right) + \left(\dfrac{\partial^2 w_y}{\partial z\partial x} - \dfrac{\partial^2 w_x}{\partial z\partial y}\right) = 0$

because all the third derivatives are equal.