The Chemistry Maths Book

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Solutions

Chapter 13 Second-order differential equations.
Some special functions

Section 13.2

Use the power-series method to solve:

1. $y' - 3x^2 y = 0$

By equation (13.2), we express the solution as the power series

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Then $$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$= a_1 + 2a_2 x + \sum_{m=0}^{\infty} (m+3) a_{m+3} x^{m+2}$$

Also $$3x^2 y = \sum_{m=0}^{\infty} 3a_m x^{m+2} = 3a_0 x^2 + 3a_1 x^3 + 3a_2 x^4 + \cdots$$

Therefore $$y' - 3x^2 y = a_1 + 2a_2 x + \sum_{m=0}^{\infty} \left[(m+3)a_{m+3} - 3a_m\right] x^{m+2}$$

$= 0$ when the coefficient of each power of x is zero.

Then $a_1 = a_2 = 0$

and $(m+3)a_{m+3} - 3a_m = 0$ for $m = 0, 1, 2, \ldots$

$\to$ $3a_3 - 3a_0 = 0 \;\to\; a_3 = a_0$ (a_0 arbitrary)

$4a_4 - 3a_1 = 0 \;\to\; a_4 = 0$

$5a_5 - 3a_2 = 0 \;\to\; a_5 = 0$

$6a_6 - 3a_3 = 0 \;\to\; a_6 = \frac{1}{2} a_3 = \frac{1}{2!} a_0$

$7a_7 - 3a_4 = 0 \;\to\; a_7 = 0$

$8a_8 - 3a_5 = 0 \;\to\; a_8 = 0$

$9a_9 - 3a_6 = 0 \;\to\; a_9 = \frac{1}{3} a_6 = \frac{1}{3!} a_0$

and so on.

Therefore $a_{3n+1} = a_{3n+2} = 0$

$a_{3n} = \frac{1}{n!} a_0$ for $n = 0, 1, 2, \ldots$

and the power series expansion is

$$y = a_0\left[1 + x^3 + \frac{1}{2!}x^6 + \frac{1}{3!}x^9 + \cdots\right] = a_0\sum_{n=0}^{\infty}\frac{1}{n!}(x^3)^n$$

We recognise the sum as the power-series expansion of the function e^{x^3}. Therefore

$$y = a_0 e^{x^3}$$

where a_0 is an arbitrary constant.

2. $(1-x)y' - y = 0$. Confirm the solution can be expressed as $y = a/(1-x)$ when $|x| < 1$.

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Then $$(1-x)y' = \sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=1}^{\infty} m a_m x^m$$

$$= a_1 + (2a_2 - a_1)x + (3a_3 - 2a_2)x^2 + \cdots$$

$$= a_1 + \sum_{m=1}^{\infty}\left[(m+1)a_{m+1} - m a_m\right]x^m$$

Therefore $$(1-x)y' - y = a_1 - a_0 + \sum_{m=1}^{\infty}\left[(m+1)a_{m+1} - m a_m\right]x^m - \sum_{m=1}^{\infty} a_m x^m$$

$$= a_1 - a_0 + \sum_{m=1}^{\infty}(m+1)(a_{m+1} - a_m)x^m$$

$= 0$ when $a_{m+1} = a_m$ for all values of *m*.

All the coefficients are therefore equal, to arbitrary a_0 say, and the power series expansion is

$$y = a_0\sum_{m=0}^{\infty} x^m$$

and this is recognized as the geometric series expansion of $\frac{a_0}{1-x}$, convergent when $|x| < 1$.

3. $y'' - 9y = 0$. Confirm that the solution can be expressed as $y = ae^{3x} + be^{-3x}$.

Let $$y = \sum_{m=0}^{\infty} a_m x^m$$

Then $$y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m$$

and $$y'' - 9y = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} 9a_m x^m$$

$$= 0 \text{ when } (m+2)(m+1)a_{m+2} = 9a_m$$

The recurrence relation for the coefficients gives rise to two independent series:

(i) m even $m = 0 \rightarrow 2a_2 = 9a_0 \rightarrow a_2 = \frac{9}{2}a_0$

$m = 2 \rightarrow 3\times 4a_4 = 9a_2 \rightarrow a_4 = \frac{9^2}{4!}a_0$

$m = 4 \rightarrow 5\times 6a_6 = 9a_4 \rightarrow a_6 = \frac{9^3}{6!}a_0$ and so on

Therefore, for even powers of *x*,

$$y_1(x) = a_0\left[1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \frac{(3x)^6}{6!} + \cdots\right]$$

(ii) m odd $m = 1 \rightarrow 2\times 3a_3 = 9a_1 \rightarrow a_3 = \frac{9}{3!}a_1$

$m = 3 \rightarrow 4\times 5a_5 = 9a_3 \rightarrow a_5 = \frac{9^2}{5!}a_1$

$m = 5 \rightarrow 6\times 7a_7 = 9a_5 \rightarrow a_6 = \frac{9^3}{7!}a_1$ and so on

Therefore, for odd powers of *x*,

$$y_2(x) = \frac{a_1}{3}\left[(3x) + \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \frac{(3x)^7}{7!} + \cdots\right]$$

We recognize y_1 and y_2 as the hyperbolic functions

$$y_1(x) = a_0 \cosh 3x = \frac{a_0}{2}\left[e^{3x} + e^{-3x}\right], \quad y_2(x) = \frac{a_1}{3}\sinh 3x = \frac{a_1}{6}\left[e^{3x} - e^{-3x}\right]$$

Therefore $$y(x) = y_1(x) + y_2(x) = a_0 \cosh 3x + (a_1/3)\sinh 3x$$

$$= ae^{3x} + be^{-3x} \quad \text{where } a_0 = a + b,\ a_1/3 = a - b$$

4. $(1-x^2)y''-2xy'+2y=0$ (The Legendre equation (13.13) for $l=1$).

Show that the solution can be written as $y=a_1x+a_0\left[1+\dfrac{x}{2}\ln\left(\dfrac{1-x}{1+x}\right)\right]$ when $|x|<1$.

Let $\quad y=\sum_{m=0}^{\infty}a_m x^m, \quad y'=\sum_{m=1}^{\infty}ma_m x^{m-1}, \quad y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$

Then $\quad (1-x^2)y''-2xy'+2y$

$$=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty}m(m-1)a_m x^m-2\sum_{m=1}^{\infty}ma_m x^m+2\sum_{m=0}^{\infty}a_m x^m$$

$$=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m-\sum_{m=0}^{\infty}m(m-1)a_m x^m-2\sum_{m=0}^{\infty}ma_m x^m+2\sum_{m=0}^{\infty}a_m x^m$$

$$=\sum_{m=0}^{\infty}(m+2)\big[(m+1)a_{m+2}-(m-1)a_m\big]x^m$$

and $\quad (1-x^2)y''-2xy'+2y=0$ when $a_{m+2}=\left(\dfrac{m-1}{m+1}\right)a_m$

The recurrence relation for the coefficients gives rise to two independent series:

(i) m even: $a_2=-a_0$, $a_4=\dfrac{a_2}{3}=-\dfrac{a_0}{3}$, $a_6=\dfrac{3a_4}{5}=-\dfrac{a_0}{5}$, $\cdots$ with a_0 arbitrary

$\rightarrow \quad a_n=-\dfrac{a_0}{n-1}$ for $n=2,4,6,\ldots$

(ii) m odd: a_1 arbitrary,

$a_3=0, a_5=0 \quad\rightarrow\quad a_n=0$ for odd $n>1$

Therefore $\quad y(x)=a_1x+a_0\left[1-x^2-\dfrac{x^4}{3}-\dfrac{x^6}{5}-\dfrac{x^8}{7}-\cdots\right]$

Now

$$1+\frac{x}{2}\ln\left(\frac{1-x}{1+x}\right)=1+\frac{x}{2}\big[\ln(1-x)=\ln(1+x)\big]$$

$$=1+\frac{x}{2}\left[\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\cdots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right)\right]$$

$$=1+\frac{x}{2}\left[-2x-\frac{2x^3}{3}-\frac{2x^5}{5}\cdots\right]=1-x^2-\frac{x^4}{3}-\frac{x^6}{5}-\cdots \quad \text{when } |x|<1$$

Therefore $\quad y=a_1x+a_0\left[1+\dfrac{x}{2}\ln\left(\dfrac{1-x}{1+x}\right)\right]$ when $|x|<1$

5. $y'' - xy = 0$ (Airy equation).

Let $y = \sum_{m=0}^{\infty} a_m x^m \quad \to \quad xy = \sum_{m=0}^{\infty} a_m x^{m+1}$

Then $y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + \sum_{m=0}^{\infty} (m+3)(m+2)a_{m+3} x^{m+1}$

and
$$y'' - xy = 2a_2 + \sum_{m=0}^{\infty} (m+3)(m+2)a_{m+3} x^{m+1} - \sum_{m=0}^{\infty} a_m x^{m+1}$$
$$= 2a_2 + \sum_{m=0}^{\infty} \Big[(m+3)(m+2)a_{m+3} - a_m\Big] x^{m+1}$$
$$= 0 \text{ when } a_2 = 0 \text{ and } a_{m+3} = \frac{1}{(m+3)(m+2)} a_m$$

The recurrence relation for the coefficients gives rise to three independent series:

$$a_3 = \frac{a_0}{2\times 3} = \frac{1}{3!}a_0, \quad a_6 = \frac{a_3}{5\times 6} = \frac{1\times 4}{6!}a_0, \quad a_9 = \frac{a_6}{8\times 9} = \frac{1\times 4\times 7}{9!}a_0, \ \cdots$$
$$a_4 = \frac{a_1}{3\times 4} = \frac{2}{4!}a_1, \quad a_7 = \frac{a_4}{6\times 7} = \frac{2\times 5}{7!}a_1, \quad a_{10} = \frac{a_6}{9\times 10} = \frac{2\times 5\times 8}{9!}a_1, \cdots$$
$$a_5 = \frac{a_2}{4\times 5} = 0 = a_8 = a_{11} = \cdots$$

Therefore
$$y(x) = a_0\left[1 + \frac{1}{3!}x^3 + \frac{1\cdot 4}{6!}x^6 + \frac{1\cdot 4\cdot 7}{9!}x^9 + \cdots\right] + a_1\left[x + \frac{2}{4!}x^4 + \frac{2\cdot 5}{7!}x^7 + \frac{2\cdot 5\cdot 8}{10!}x^{10} + \cdots\right]$$

Section 13.3

For each of the following, find and solve the indicial equation

6. $x^2 y'' + 3xy' + y = 0$

We have $b_0 = 3,\ c_0 = 1$

and the indicial equation is

$$r^2 + (b_0 - 1)r + c_0 = 0 \quad \to \quad r^2 + 2r + 1 = (r+1)^2 = 0 \text{ when } r = -1$$

Therefore $r_1 = r_2 = -1$ (double root)

7. $x^2y'' + xy' + (x^2 - n^2)y = 0$ (Bessel equation)

We have $b_0 = 1,\ c_0 = -n^2$

and the indicial equation is

$$r^2 + (b_0 - 1)r + c_0 = 0 \quad \to \quad r^2 - n^2 = (r-n)(r+n) = 0 \text{ when } r = \pm n$$

8. $xy'' + (1-2x)y' + (x-1)y = 0$

We write the equation as $x^2y'' + (x - 2x^2)y' + (x^2 - x)y = 0$

Therefore $b_0 = 1,\ c_0 = 0 \quad \to \quad r^2 = 0$

Therefore $r_1 = r_2 = 0$

9. $x^2y'' + 6xy' + (6 - x^2)y = 0$

We have $b_0 = 6,\ c_0 = 6 \quad \to \quad r^2 + 5r + 6 = (r+2)(r+3) = 0$

Therefore $r_1 = -2,\ r_2 = -3$

10. **(i)** Find the general solution of the Euler-Cauchy equation $x^2y'' + b_0xy' + c_0y = 0$ for distinct indicial roots, $r_1 \neq r_2$. **(ii)** Show that for a double initial root *r*, the general solution is $y = (a + b\ln x)x^r$.

Let $y(x) = x^r(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) = x^r\sum_{m=0}^{\infty} a_mx^m$

Then, by equation (13.7)

$$\sum_{m=0}^{\infty}\Big[(r+m)^2 + (b_0 - 1)(r+m) + c_0\Big]a_mx^{m+r} = 0$$

The equation is satisfied if, for every power of *x*, either $a_m = 0$ or the term in square brackets is zero. For $m = 0$, the latter is the indicial equation (13.8), so that a particular solution of the differential equation is
$y = x^r$, where *r* is an indicial root. There are two possible types of solution.

(i) Distinct indicial roots, $r_1 \neq r_2$. Then by equations (13.10), x^{r_1} and x^{r_2} are two independent particular solutions, and the general solution of the Euler-Cauchy equation is

$$y(x) = ax^{r_1} + bx^{r_2}$$

(ii) For a double root, by equations (13.11), one particular solution is $y_1 = x^r$. In the present case, the second particular is just the first term in (13.11b), $y_2 = x^r \ln x$.

Thus $\quad y_2' = rx^{r-1} \ln x + x^{r-1}$

$$y_2'' = r(r-1)x^{r-2} \ln x + (2r-1)x^{r-2}$$

Then $\quad x^2 y_2'' + b_0 x y_2' + c_0 y_2 = \left[r^2 + (b_0 - 1)r + c_0\right] x^r \ln x + \left[2r + b_0 - 1\right] x^r$

The first set of terms in square brackets is zero be cause r is an indicial root, a solution of the indicial equation $r^2 + (b_0 - 1)r + c_0 = 0$. The second set of terms is also zero because r is a double root. Thus,

$$r^2 + (b_0 - 1)r + c_0 = 0 \text{ when } r = \frac{1}{2}\left[-(b_0 - 1) \pm \sqrt{(b_0 - 1)^2 - 4c_0}\right]$$

$$= -(b_0 - 1)/2 \quad \text{for a double root}$$

$$\rightarrow 2r + b_0 - 1 = 0$$

The general solution of the Euler-Cauchy equation for a double indicial root is therefore

$$y(r) = ay_1 + by_2 = (a + b \ln x)x^r$$

Solve the differential equations:

11. $x^2 y'' - \frac{1}{2} xy' + \frac{1}{2} y = 0$

This is an Euler-Cauchy equation with $b_0 = -1/2$, $c_0 = 1/2$. The indicial equation is

$$r^2 - 3r/2 + 1/2 = (r - 1/2)(r - 1)$$

$$= 0 \text{ when } r = 1/2 \text{ and } r = 1$$

Therefore $\quad y(x) = ax^{1/2} + bx$

12. $x^2 y'' - xy' + y = 0$

This is an Euler-Cauchy equation with $b_0 = -1$, $c_0 = 1$. The indicial equation is

$$r^2 - 2r + 1 = (r - 1)^2$$

$$= 0 \text{ when } r = 1 \text{ (double)}$$

Therefore $\quad y(x) = (a + b \ln x)x$

13. (i) Solve the Bessel equation $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ for indicial root $r = -1/2$ (see Example 13.4 for $r = 1/2$). **(ii)** Confirm that the solution can be written as

$$y(x) = \frac{a_0}{\sqrt{x}}\cos x + \frac{a_1}{\sqrt{x}}\sin x = a\,J_{-1/2}(x) + b\,J_{1/2}(x)$$

(i) As in Example 13.4, but with $r = -1/2$ instead of $r = +1/2$:

We have $\quad y = x^{-1/2}\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m-1/2}$

Then $\quad y' = \sum_{m=0}^{\infty}\left(m - \frac{1}{2}\right)a_m x^{m-3/2}, \quad y'' = \sum_{m=0}^{\infty}\left(m - \frac{1}{2}\right)\left(m - \frac{3}{2}\right)a_m x^{m-5/2}$

and

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = x^{-1/2}\sum_{m=0}^{\infty} a_m\left[m(m-1)x^m + x^{m+2}\right]$$

$$= x^{-1/2}\sum_{m=0}^{\infty}\left[(m+1)(m+2)a_{m+2} + a_m\right]x^{m+2}$$

$$= 0 \text{ when } a_{m+2} = -\frac{a_m}{(m+1)(m+2)}$$

The recurrence relation for the coefficients gives rise to two independent series:

(a) m even: $a_2 = -\frac{1}{1\times 2}a_0 = -\frac{1}{2!}a_0, \quad a_4 = -\frac{1}{3\times 4}a_2 = +\frac{1}{4!}a_0, \quad a_6 = -\frac{1}{6\times 7}a_4 = -\frac{1}{6!}a_0, \cdots$

(b) m odd: $a_3 = -\frac{1}{2\times 3}a_0 = -\frac{1}{3!}a_0, \quad a_5 = -\frac{1}{4\times 5}a_3 = +\frac{1}{5!}a_1, \quad a_7 = -\frac{1}{6\times 7}a_5 = -\frac{1}{7!}a_0, \cdots$

Therefore, $\quad y(x) = a_0x^{-1/2}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] + a_1x^{-1/2}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$

(ii) We recognize the two set of terms in square brackets as the power series expansions of $\cos x$ and $\sin x$. Therefore

$$y(x) = a_0x^{-1/2}\cos x + a_1x^{-1/2}\sin x = a\,J_{-1/2}(x) + b\,J_{-1/2}(x)$$

where $\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ and $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

are the Bessel functions of order $n = -1/2$ and $n = +1/2$. But, as shown in Example 13.4, $J_{1/2}(x)$ is the particular solution of the Bessel equation for indicial root $r = +1/2$. Both particular solutions of the Bessel equation for $n = \pm 1/2$ have therefore been obtained. This is a common feature of type 3 solutions, equations (13.12), when no logarithmic term is present and the lower of the two values of r is used to find a solution of the equation.

14. (i) Use the expansion method to find a particular solution $y_1(x)$ of $xy''+(1-2x)y'+(x-1)y=0$.
(ii) confirm that $y_2(x)=y_1(x)\ln x$ is a second solution.

As in Exercise 8, the indicial equation has double root $r=0$. The solutions are therefore of type 2, equations (13.11).

(i) Let $$y=\sum_{m=0}^{\infty}a_m x^m,\quad y'=\sum_{m=1}^{\infty}ma_m x^{m-1},\quad y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$$

Then

$$\begin{aligned}xy''+(1-2x)y'+(x-1)y&=\sum_{m=0}^{\infty}\Big[m(m-1)a_m x^{m-1}+m(1-2x)a_m x^{m-1}+(x-1)a_m x^m\Big]\\&=\sum_{m=0}^{\infty}\Big[m^2a_m x^{m-1}-(2m+1)a_m x^m+a_m x^{m+1}\Big]\\&=(a_1-a_0)+\sum_{m=1}^{\infty}\Big[(m+1)^2a_{m+1}-(2m+1)a_m+a_{m-1}\Big]x^m\\&=0\end{aligned}$$

Therefore $a_1=a_0$

and $(m+1)^2a_{m+1}-(2m+1)a_m+a_{m-1}=0$

Then, by considering $m=1,2,3,\ \dots$ in turn, we obtain $a_m=a_0/m!$. Alternatively, we can write

$$\begin{aligned}(m+1)^2a_{m+1}-(2m+1)a_m+a_{m-1}&=(m+1)\Big[(m+1)a_{m+1}-a_m\Big]-\Big[ma_m-a_{m-1}\Big]\\&=0\ \text{when}\ a_m=\frac{1}{m}a_{m-1}\ \to\ a_m=\frac{1}{m!}a_0\end{aligned}$$

Therefore $$y_1(x)=a_0\sum_{m=0}^{\infty}\frac{x^m}{m!}=a_0e^x$$

(ii) Let $$y_2=y_1\ln x,\quad y_2'=y_1'\ln x+\frac{1}{x}y_1,\quad y_2''=y_1''\ln x+\frac{2}{x}y_1'-\frac{1}{x^2}y_1$$

Then $$xy_2''+(1-2x)y_2'+(x-1)y_2=\Big[xy_1''+(1-2x)y_1'+(x-1)y_1\Big]\ln x+\Big[2y_1'-2y_1\Big]$$

The first set of terms is zero because y_1 is a solution. The second set of terms is zero because $y_1{}'=y_1=a_0e^x$. Therefore $y_2(x)=y_1(x)\ln x$ is a second solution of the differential equation.

15. Find the general solution of $xy'' + 2y' + 4xy = 0$. Assume that there is no logarithmic term in the solution.

By Example 13.3(iii), the indicial roots are $r_1 = 0$ and $r_2 = -1$ and the solutions are nominally of type 3, equations (13.12). In the present case, however, there is no logarithmic term. Thus, for $r = -1$,

let $$y = x^{-1}\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m-1}$$

Then $$y' = \sum_{m=0}^{\infty} (m-1)a_m x^{m-2}, \quad y'' = \sum_{m=0}^{\infty}(m-1)(m-2)a_m x^{m-3}$$

and $$\begin{aligned} xy'' + 2y' + 4xy &= \sum_{m=0}^{\infty}\Big[m(m-1)a_m x^{m-2} + 4a_m x^m\Big] \\ &= \sum_{m=0}^{\infty}\Big[(m+1)(m+2)a_{m+2} + 4a_m\Big]x^m \\ &= 0 \text{ when } a_{m+2} = \frac{-4}{(m+1)(m+2)}a_m \end{aligned}$$

The recurrence relation for the coefficients gives rise to two independent series:

(a) m even: $a_2 = \frac{-4}{2!}a_0, \quad a_4 = \frac{-4}{3\times 4}a_2 = \frac{(-4)^2}{4!}a_0, \ \ldots \ \rightarrow \ a_{2n} = \frac{(-4)^n}{(2n)!}a_0 \quad n = 1, 2, 3, \ldots$

(b) m odd: $a_3 = \frac{-4}{3!}a_1, \quad a_5 = \frac{-4}{4\times 5}a_2 = \frac{(-4)^2}{5!}a_1, \ \ldots \ \rightarrow \ a_{2n+1} = \frac{(-4)^n}{(2n+1)!}a_1 \quad n = 1, 2, 3, \ldots$

The corresponding solution of the differential equation is therefore

$$y(x) = \frac{a_0}{x}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right] + \frac{a_1}{2x}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots\right]$$

We recognize the terms in square brackets as $\cos 2x$ and $\sin 2x$, so that

$$y(x) = a\,\frac{\cos 2x}{x} + b\,\frac{\sin 2x}{x}$$

The solution for $r = -1$ therefore consists of a combination of the two independent particular solutions, $y_1(x) = (1/x)\cos 2x$ containing odd powers of x, and $y_2(x) = (1/x)\sin 2x$ containing even powers (compare Example 13.2). These two particular solutions form a basis for the general solution, without a logarithmic term. The solution for indicial parameter $r = 0$ is therefore redundant; in fact it merely duplicates the particular solution $y_2(x)$ (see also Exercise 13).

Section 13.4

16. Show that the polynomial $P_l(x)$ is a solution of the Legendre equation, Table 13.1 for **(i)** $l = 2$ and **(ii)** $l = 5$.

We have $$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

(i) $l = 2$: $$y = P_2(x) = \frac{1}{2}(3x^2 - 1), \quad y' = 3x, \quad y'' = 3$$

Therefore $$(1-x^2)y'' - 2xy' + 6y = (1-x^2)\times 3 - 2x \times 3x + 6 \times \frac{1}{2}(3x^2-1)$$

$$= \cancel{3} - \cancel{3x^3} - \cancel{6x^2} + \cancel{9x^2} - \cancel{3} = 0$$

(i) $l = 5$:

$$y = P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \quad y' = \frac{1}{8}(315x^4 - 210x^2 + 15), \quad y'' = \frac{1}{8}(1260x^3 - 420x)$$

Therefore $$(1-x^2)y'' - 2xy' + 30y = \frac{1}{8}\Big\{\cancel{1260x^3} - \cancel{420x} - \cancel{1260x^5} + \cancel{420x^3} - \cancel{630x^5}$$
$$+\cancel{420x^3} - \cancel{30x} + \cancel{1890x^5} - \cancel{2100x^3} + \cancel{450x}\Big\} = 0$$

17. Find the Legendre polynomial $P_6(x)$ **(i)** by means of the recurrence relation (13.21), **(ii)** from the general expression (13.19) for $P_l(x)$.

We have $$(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0$$

(i) Put $l = 5$**:** $$P_6 = \frac{1}{6}(11xP_5 - 5P_4)$$

$$= \frac{11x}{48}\Big[63x^5 - 70x^3 + 15x\Big] - \frac{5}{48}\Big[35x^4 - 30x^2 + 3\Big]$$

$$= \frac{1}{48}\Big[693x^6 - 945x^4 + 315x^2 - 15\Big] = \frac{1}{16}\Big[231x^6 - 315x^4 + 105x^2 - 5\Big]$$

(ii) By equation (13.19),

$$P_6 = \frac{1\times3\times5\times7\times9\times11}{6\times5\times4\times3\times2}$$
$$\times\left\{x^6 - \frac{6\times5}{2\times11}x^4 + \frac{6\times5\times4\times3}{2\times4\times11\times9}x^2 - \frac{6\times5\times4\times3\times2}{2\times4\times6\times11\times9\times7}\right\}$$
$$= \frac{231}{16}\left[x^6 - \frac{15}{11}x^4 + \frac{5}{11}x^2 - \frac{5}{231}\right]$$
$$= \frac{1}{16}\Big[231x^6 - 315x^4 + 105x^2 - 5\Big]$$

18. Use the formula (13.24) to find the associated Legendre functions **(i)** $P_1^1(x)$, **(ii)** $P_4^m(x)$ for $m = 1, 2, 3, 4$. Express the functions in terms of $\cos\theta = x$ and $\sin\theta = (1-x^2)^{1/2}$.

(i) We have $P_1(x) = x$

Then $$P_1^1(x) = (1-x^2)^{1/2}\frac{dP_1}{dx} = (1-x^2)^{1/2} = \sin\theta$$

(ii) $$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Therefore $$P_4^1 = (1-x^2)^{1/2}\frac{dP_4}{dx} = \frac{5}{2}(1-x^2)^{1/2}(7x^3 - 3x) = \frac{5}{2}\sin\theta(7\cos^3\theta - 3\cos\theta)$$

$$P_4^2 = (1-x^2)\frac{d^2P_4}{dx^2} = \frac{15}{2}(1-x^2)(7x^2-1) = \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1)$$

$$P_4^3 = (1-x^2)^{3/2}\frac{d^3P_4}{dx^3} = 105(1-x^2)^{3/2}x = 105\sin^3\theta\cos\theta$$

$$P_4^4 = (1-x^2)^2\frac{d^4P_4}{dx^4} = 105(1-x^2)^2 = 105\sin^4\theta$$

19. Show that **(i)** P_1 is orthogonal to P_4 and P_5, **(ii)** P_2 is orthogonal to P_0 and P_3.

(i) $$\int_{-1}^{+1} P_1(x)P_4(x)\,dx = \frac{1}{8}\int_{-1}^{+1} x(35x^4 - 30x^2 + 3)\,dx$$

$= 0$ because the integrand is an odd function of x

(ii) $$\int_{-1}^{+1} P_1(x)P_5(x)\,dx = \frac{1}{8}\int_{-1}^{+1} x(63x^5 - 70x^3 + 15x)\,dx$$

$$= \frac{1}{8}\int_{-1}^{+1}(63x^6 - 70x^4 + 15x^2)\,dx$$

$$= \frac{1}{8}\Big[9x^7 - 14x^5 + 5x^3\Big]_{-1}^{+1} = \frac{1}{8}\big[(9-14+5) - (-9+14-5)\big]$$

$$= 0$$

20. Show that P_2^1 is orthogonal to P_1^1 and P_4^1.

We have $P_1^1(x) = (1-x^2)^{1/2}$, $P_2^1(x) = 3x(1-x^2)^{1/2}$, $P_4^1(x) = \dfrac{5}{2}(1-x^2)^{1/2}(7x^3-3x)$.

(i)
$$\int_{-1}^{+1} P_2^1(x)P_1^1(x)\,dx = 3\int_{-1}^{+1} x(1-x^2)\,dx$$
$$= 0 \text{ because the integrand is an odd function of } x$$

(ii)
$$\int_{-1}^{+1} P_2^1(x)P_4^1(x)\,dx = \frac{15}{2}\int_{-1}^{+1} x(1-x^2)(7x^3-3x)\,dx$$
$$= 15\int_{-1}^{+1} (-7x^6+10x^4-3x^2)\,dx$$
$$= 15\Big[-x^7+2x^5-x^3\Big]_{-1}^{+1} = 0$$

Section 13.5

21. **(i)** Use the series expansion (13.31) to find $H_5(x)$. **(ii)** Verify by substitution in (13.30) that $H_5(x)$ is a solution of the Hermite equation. **(iii)** Use the recurrence relation (13.33) to find $H_6(x)$.

(i)
$$H_5(x) = (2x)^5 - \frac{5\times 4}{1!}(2x)^3 + \frac{5\times 4\times 3\times 2}{2!}(2x) = 32x^5 - 160x^3 + 120x$$

(ii) We have $H_5'(x) = 160x^4 - 480x^2 + 120, \quad H_5''(x) = 640x^3 - 960x$

Then

$$H_5'' - 2xH_5' + 10H_5 = \Big[640x^3 - 960x\Big] - 2x\Big[160x^4 - 480x^2 + 120\Big] + 10\Big[32x^5 - 160x^3 + 120x\Big]$$
$$= \cancel{640x^3} - \cancel{960x} - \cancel{320x^5} + \cancel{960x^3} - \cancel{240x} + \cancel{320x^5} - \cancel{1600x^3} + \cancel{1200x}$$
$$= 0$$

(iii) The recurrence relation for $n = 5$ is $H_6 - 2xH_5 + 10H_4 = 0$. Then

$$H_6 = 2xH_5 - 10H_4 = 2x\Big[32x^5 - 160x^3 + 120x\Big] - 10\Big[16x^4 - 48x^2 + 12\Big]$$
$$= 64x^6 - 480x^4 + 720x^2 - 120$$

22. Sketch the graph of the Hermite function $e^{-x^2/2}H_3(x)$.

We have $y = e^{-x^2/2}H_3(x) = e^{-x^2/2}(8x^3 - 12x)$.

For the nodes, $y = 0$ when $8x^3 - 12x = 0 \rightarrow x = 0, \pm\sqrt{3/2}$.

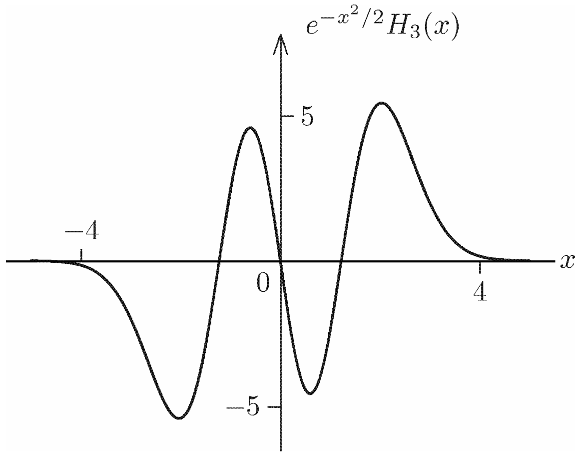
For the stationary values (maxima and minima),

$$y' = -4e^{-x^2/2}(2x^4 - 9x^2 + 3x)$$

$$= 0 \text{ when } x^2 = \frac{9 \pm \sqrt{57}}{4} \rightarrow x = \pm\sqrt{\frac{9 \pm \sqrt{57}}{4}} \approx \pm 0.60, \pm 2.03$$

x	y
-5	–0.004
-4	–0.16
-3	–2.00
-2	–5.41
$-\sqrt{3/2}$	0
-1	+2.43
-0.6	+4.57
0	0
0.6	-4.57
1	-2.43
$\sqrt{3/2}$	0
2	5.41
3	2.00
4	0.16
5	0.004

The sketch of the Hermite function should look like:



Section 13.6

23. (i) Use the power series method to find a solution of the Laguerre equation (13.38). **(ii)** Show that this solution reduces to the polynomial (13.39) when n is a positive integer or zero (and when the arbitrary constant is given its conventional value $n!$.

The Laguerre equation is $xy'' + (1-x)y' + ny = 0$.

(i) By the power-series method,

Let $$y = \sum_{m=0}^{\infty} a_m x^m, \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

Then $$xy'' + (1-x)y' + ny = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-1} + \sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} n a_m x^m$$

$$= \sum_{m=0}^{\infty} \Big[(m+1)m a_{m+1} + (m+1)a_{m+1} - (m-n)a_m\Big] x^m$$

$$= 0 \quad \text{when} \quad a_{m+1} = \frac{(m-n)}{(m+1)^2} a_m$$

Therefore

$$a_1 = -\frac{n}{1^2} a_0, \quad a_2 = \frac{(1-n)}{2^2} a_1 = \frac{n(n-1)}{(2!)^2} a_0, \quad a_3 = \frac{(2-n)}{3^2} a_2 = -\frac{n(n-1)(n-2)}{(3!)^2} a_0, \ \ldots$$

and a solution of the Laguerre equation is

$$y = a_0 \left[1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} x^2 - \frac{n(n-1)(n-2)}{(3!)^2} x^3 + \cdots \right] \qquad \text{(equation A)}$$

(ii) If n is an nonzero integer then the expansion terminates at term x^n . Giving the arbitrary coefficient its conventional value $a_0 = n!$, we obtain

$$y(x) = (-1)^n \left[x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^2 - \cdots + (-1)^n n! \right] = L_n(x) \qquad \text{(equation B)}$$

Thus, the general term in equation A above is, with $a_0 = n!$,

$$I_r = (-1)^r \frac{n!^2}{r!^2 (n-r)!} x^r$$

$$I_n = (-1)^n x^n, \quad I_{n-1} = (-1)^{n-1} \frac{n^2}{1!} x^{n-1}, \quad I_{n-2} = (-1)^{n-2} \frac{n^2(n-1)^2}{2!} x^{n-1}, \quad I_1 = n!$$

Hence equation B

24. Find $L_4(x)$ **(i)** from equation (13.39), **(ii)** from $L_2(x)$ and $L_3(x)$ by means of the recurrence relation (13.41).

(i) From (13.39) $L_4(x) = (-1)^4\left[x^4 - \frac{4^2}{1!}x^3 + \frac{4^2\times 3^2}{2!}x^2 - \frac{4^2\times 3^2\times 2^2}{3!}x + \frac{4^2\times 3^2\times 2^2\times 1^2}{4!}!\right]$

$$= x^4 - 16x^3 + 72x^2 - 96x + 24$$

(ii) From equations (13.40),

$$L_2(x) = 2 - 4x + x^2,\ L_3(x) = 6 - 18x + 9x^2 - x^3$$

Then, with $n = 3$ in (13.41),

$$L_4(x) = (7 - x)(6 - 18x + 9x^2 - x^3) - 9(2 - 4x + x^2)$$

$$= 24 - 96x + 72x^2 - 16x^3 + x^4$$

Section 13.7

25. **(i)** Find the Bessel function $J_2(x)$ **(i)** from the series expansion (13.50). **(ii)** from $J_0(x)$ and $J_1(x)$ by means of the recurrence relation (13.56).

(i)

$$J_2(x) = \left(\frac{x}{2}\right)^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(2+m)!}\left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^2\left[\frac{1}{2!} - \frac{1}{1!3!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!4!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!5!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

(ii)

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

$$= \frac{2}{x}\left[\frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^7 + \frac{1}{4!5!}\left(\frac{x}{2}\right)^9 - \cdots\right]$$

$$-\left[1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \frac{1}{(4!)^2}\left(\frac{x}{2}\right)^8 - \cdots\right]$$

$$= \left[1 - 1\right] + \left[\frac{1}{(1!)^2} - \frac{1}{1!2!}\right]\left(\frac{x}{2}\right)^2 - \left[\frac{1}{(2!)^2} - \frac{1}{2!3!}\right]\left(\frac{x}{2}\right)^4 + \left[\frac{1}{(3!)^2} - \frac{1}{3!4!}\right]\left(\frac{x}{2}\right)^6$$

$$-\left[\frac{1}{(4!)^2} - \frac{1}{4!5!}\right]\left(\frac{x}{2}\right)^8 + \ldots$$

$$= \left(\frac{x}{2}\right)^2\left[\frac{1}{2!} - \frac{1}{1!3!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!4!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!5!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

26. Use the recurrence relation (13.56) to find **(i)** $J_{5/2}(x)$ and **(ii)** $J_{-5/2}(x)$.

(i) For $n = 3/2$: $$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x}\right) - \sin x\right]$$
$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3\cos x}{x}\right]$$

(ii) For $n = -3/2$: $$J_{-5/2}(x) = -\frac{3}{x} J_{-3/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3\cos x}{x^2} + \frac{3\sin x}{x}\right) - \cos x\right]$$
$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\cos x + \frac{3\sin x}{x}\right]$$

27. Confirm that the spherical Bessel function $j_l(x)$ satisfies equation (13.60).

We have $$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) = \sqrt{\frac{\pi}{2}}\, y(x), \text{ say}$$

Then $$y(x) = x^{-1/2} J_{l+1/2}(x)$$
$$y'(x) = -\frac{1}{2} x^{-3/2} J_{l+1/2}(x) + x^{-1/2} J'_{l+1/2}(x)$$
$$y''(x) = \frac{3}{4} x^{-5/2} J_{l+1/2}(x) - x^{-3/2} J'_{l+1/2}(x) + x^{-1/2} J''_{l+1/2}(x)$$

Therefore

$$x^2 y'' + 2xy' + \left[x^2 - l(l+1)\right] y = \left[\frac{3}{4} x^{-1/2} J_{l+1/2}(x) - x^{1/2} J'_{l+1/2}(x) + x^{3/2} J''_{l+1/2}(x)\right]$$
$$+ \left[-x^{-1/2} J_{l+1/2}(x) + 2x^{1/2} J'_{l+1/2}(x)\right]$$
$$+ \left[x^{3/2} J_{l+1/2}(x) - l(l+1) x^{-1/2} J_{l+1/2}(x)\right]$$
$$= x^{-1/2}\left\{x^2 J''_{l+1/2}(x) + x J'_{l+1/2}(x) + \left[x^2 - l(l+1)\right] J_{l+1/2}(x)\right\}$$

$= 0$ because $J_{l+1/2}(x)$ is a solution of the Bessel equation (13.47)