The Chemistry Maths Book

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Solutions

Chapter 11 First-order differential equations

Section 11.2

State the order of the differential equation and verify that the given function is a solution:

1. $\frac{dy}{dx} - 2y = 2; \quad y = e^{2x} - 1$

First order.

$$y = e^{2x} - 1 \;\rightarrow\; \frac{dy}{dx} = 2e^{2x}$$

Therefore $\frac{dy}{dx} - 2y = \cancel{2e^{2x}} - 2(\cancel{e^{2x}} - 1) = 2$

2. $\frac{d^2y}{dx^2} + 4y = 0; \quad y = A\cos 2x + B\sin 2x$

Second order.

$$y = A\cos 2x + B\sin 2x \;\rightarrow\; \frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$$

$$\rightarrow\; \frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x = -4y$$

Therefore $\frac{d^2y}{dx^2} + 4y = 0$

3. $\frac{d^3y}{dx^3} = 12; \quad y = 2x^3 + 3x^2 + 4x + 5$

Third order.

$$y = 2x^3 + 3x^2 + 4x + 5 \;\rightarrow\; \frac{dy}{dx} = 6x^2 + 6x + 4$$

$$\rightarrow\; \frac{d^2y}{dx^2} = 12x + 6$$

Therefore $\frac{d^3y}{dx^3} = 12$

4. $\frac{dy}{dx} + \frac{3y}{x} = 3x^2; \quad y = \frac{x^3}{2} + \frac{c}{x^3}$

First order

$$y = \frac{x^3}{2} + \frac{c}{x^3} \;\rightarrow\; \frac{dy}{dx} = \frac{3x^2}{2} - \frac{3c}{x^4}$$

Therefore $\frac{dy}{dx} + \frac{3y}{x} = \frac{3x^2}{2} - \frac{3c}{x^4} + \frac{3}{x}\left(\frac{x^3}{2} + \frac{c}{x^3}\right) = \frac{3x^2}{2} - \cancel{\frac{3c}{x^4}} + \frac{3x^2}{2} + \cancel{\frac{3c}{x^4}}$

$$= 3x^2$$

Find the general solution of the differential equation:

5. $\dfrac{dy}{dx} = x^2$

Integrate: $$\int \frac{dy}{dx}\,dx = \int x^2\,dx \quad \to \quad y = \frac{1}{3}x^3 + c$$

6. $\dfrac{dy}{dx} = e^{-3x}$

Integrate: $$\int \frac{dy}{dx}\,dx = \int e^{-3x}\,dx \quad \to \quad y = -\frac{1}{3}e^{-3x} + c$$

7. $\dfrac{d^2y}{dx^2} = \cos 3x$

Integrate twice: $$\int \frac{d^2y}{dx^2}\,dx = \int \cos 3x\,dx \quad \to \quad \frac{dy}{dx} = \frac{1}{3}\sin 3x + a$$

$$\int \frac{dy}{dx}\,dx = \int \left[\frac{1}{3}\sin 3x + a\right]dx \to \quad y = -\frac{1}{9}\cos 3x + ax + b$$

8. $\dfrac{d^3y}{dx^3} = 24x$

Integrate three times:

$$\int \frac{d^3y}{dx^3}\,dx = \int 24x\,dx \quad \to \frac{d^2y}{dx^2} = 12x^2 + a$$

$$\int \frac{d^2y}{dx^2}\,dx = \int \left[12x^2 + a\right]dx \quad \to \quad \frac{dy}{dx} = 4x^3 + ax + b$$

$$\int \frac{dy}{dx}\,dx = \int \left[4x^3 + ax + b\right]dx \quad \to \quad y = x^4 + a'x^2 + bx + c \quad (a' = a/2)$$

9. A body moves along the x direction under the influence of the force $F(t) = \cos 2\pi t$, where t is the time. **(i)** Write down the equation of motion. **(ii)** Find the solution that satisfies the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$.

(i) By Newton's second law of motion, for a body of mass m,

$$F(t) = m\frac{d^2x}{dt^2} = \cos 2\pi t$$

(ii) Integrating twice with respect to time t,

$$\int \frac{d^2x}{dt^2}\,dt = \frac{1}{m}\int \cos 2\pi t\,dt \qquad \to \quad \frac{dx}{dt} = \frac{1}{2\pi m}\sin 2\pi t + a$$

$$\int \frac{dx}{dt}\,dt = \int\left[\frac{1}{2\pi m}\sin 2\pi t + a\right]dt \quad \to \quad x(t) = -\frac{1}{4\pi^2 m}\cos 2\pi t + at + b$$

By the initial conditions:

$$x(0) = 0 = -\frac{1}{4\pi^2 m} + b \quad \to \quad b = \frac{1}{4\pi^2 m}$$

$$\dot{x}(0) = \left(\frac{dx}{dt}\right)_{t=0} = 1 = a$$

Therefore $\quad x(t) = \dfrac{1}{4\pi^2 m}\left(1 - \cos 2\pi t\right) + t$

Verify that the given function is a solution of the differential equation, and determine the particular solution for the given initial condition:

10. $x\dfrac{dy}{dx} = 2y;\quad y = cx^2;\quad y = 24$ when $x = 2$

$$y = cx^2 \quad \to \quad \frac{dy}{dx} = 2cx \quad \to \quad x\frac{dy}{dx} = 2cx^2 = 2y$$

The initial condition is $y = 24$ when $x = 2$.

Therefore $\quad 24 = 4c \quad \to \quad c = 6$

and $\quad y = 6x^2$

11. $\dfrac{dy}{dx} + 2xy = 0; \quad y = ce^{-x^2}; \quad y = 2$ when $x = 2$

$$y = ce^{-x^2} \quad \to \quad \frac{dy}{dx} = -2cxe^{-x^2} = -2xy \quad \to \quad \frac{dy}{dx} + 2cxe^{-x^2} = 0$$

The initial condition is $y = 2$ when $x = 2$.

Therefore $\quad 2 = ce^{-4} \quad \to \quad c = 2e^4$

and $\quad y = 2e^{4-x^2}$

12. $\dfrac{dy}{dx} + 2y + 2 = 0; \quad y = ce^{-2x} - 1; \quad y(0) = 4$

$$y = ce^{-2x} - 1 \quad \to \quad \frac{dy}{dx} = -2ce^{-2x} = -2(y+1) \quad \to \quad \frac{dy}{dx} + 2y + 2 = 0$$

The initial condition is $y = 4$ when $x = 0$.

Therefore $\quad 4 = c - 1 \quad \to \quad c = 5$

and $\quad y = 5e^{-2x} - 1$

Section 11.3

Find the general solution of the differential equation:

13. $\dfrac{dy}{dx} = \dfrac{3x^2}{y}$: Put $y\,dy = 3x^2\,dx$.

Then $\quad \displaystyle\int y\,dy = \int 3x^2\,dx \quad \to \quad \frac{1}{2}y^2 = x^3 + c'$

and $\quad y^2 = 2x^3 + c$

14. $\dfrac{dy}{dx} = 4xy^2$: Put $\dfrac{dy}{y^2} = 4x\,dx$

Then $\quad \displaystyle\int \frac{dy}{y^2} = \int 4x\,dx \quad \to \quad -\frac{1}{y} = 2x^2 + c$

and $\quad y = \dfrac{-1}{2x^2 + c}$

15. $\dfrac{dy}{dx} = 3x^2\, y$: Put $\dfrac{dy}{y} = 3x^2\, dx$.

Then $$\int \frac{dy}{y} = \int 3x^2\, dx \quad \rightarrow \quad \ln y = x^3 + c$$

and $$y = e^{x^3+c} = ae^{x^3} \qquad (a = e^c)$$

16. $y^2 \dfrac{dy}{dx} = e^x$: Put $y^2\, dy = e^x\, dx$.

Then $$\int y^2\, dy = \int e^x\, dx \quad \rightarrow \quad \frac{1}{3} y^3 = e^x + c'$$

and $$y^3 = 3e^x + c \qquad (c = 3c')$$

17. $\dfrac{dy}{dx} = y(y-1)$

We have $$\frac{dy}{y(y-1)} = dx \quad \rightarrow \quad \int \frac{dy}{y(y-1)} = \int dx = x + c$$

Now $$\int \frac{1}{y(y-1)} dy = \int \left[\frac{1}{(y-1)} - \frac{1}{y} \right] dy = \ln(y-1) - \ln y = \ln\left(\frac{y-1}{y} \right)$$

Therefore $$\ln\left(\frac{y-1}{y} \right) = x + c \quad \rightarrow \quad \frac{y-1}{y} = ae^x$$

Then, solving for *y*,

$$\frac{y-1}{y} = a \quad \rightarrow \quad y = \frac{1}{1 - ae^x}$$

18. $\dfrac{dy}{dx} = \dfrac{y}{x}$: Put $\dfrac{dy}{y} = \dfrac{dx}{x}$

Then $$\int \frac{dy}{y} = \int \frac{dx}{x} \quad \rightarrow \quad \ln y = \ln x + a \quad \rightarrow \quad \ln \frac{y}{x} = a$$

and $$\frac{y}{x} = e^a = c \quad \rightarrow \quad y = cx$$

Solve the initial value problems:

19. $\frac{dy}{dx} = \frac{y+2}{x-3}$; $y(0) = 1$

Separation of variables and integration gives

$$\frac{dy}{y+2} = \frac{dx}{x-3} \rightarrow \int \frac{dy}{y+2} = \int \frac{dx}{x-3} \rightarrow \ln(y+2) = \ln(x-3) + c$$

Then, putting, $c = \ln a$,

$$\ln(y+2) = \ln a(x-3) \rightarrow y+2 = a(x-3)$$

The initial condition is $y = 1$ when $x = 0$.

Therefore $3 = -3a \rightarrow a = -1$

and $y = 1 - x$

20. $\frac{dy}{dx} = \frac{x^2-1}{2y+1}$; $y(0) = -1$

Separation of variables and integration gives

$$\int (2y+1)\,dy = \int (x^2-1)\,dx \rightarrow y^2 + y = \frac{1}{3}x^3 - x + c$$

The initial condition is $y = -1$ when $x = 0$. Then $c = 0$

and $y^2 + y = \frac{1}{3}x^3 - x$

21. $\frac{dy}{dx} = \frac{y(y+1)}{x(x-1)}$; $y(2) = 1$

We have $\int \frac{dy}{y(y+1)} = \int \frac{dx}{x(x-1)}$

and, by the method of partial fractions,

$$\int \left[\frac{1}{y} - \frac{1}{y+1}\right] dy = \int \left[\frac{1}{x-1} - \frac{1}{x}\right] dx \rightarrow \ln \frac{y}{y+1} = \ln a\left(\frac{x-1}{x}\right)$$

Then $\frac{y}{y+1} = a\left(\frac{x-1}{x}\right)$.

The initial condition is $y = 1$ when $x = 2$.. Then $a = 1$

and $\frac{y}{y+1} = \frac{x-1}{x} \rightarrow xy = xy - y + x - 1 \rightarrow y = x - 1$

22. $\dfrac{dy}{dx} = e^{x+y}; \quad y(0) = 0$

Separation of variables and integration gives

$$e^{-y}\,dy = e^{x}\,dx \quad \to \quad \int e^{-y}\,dy = \int e^{x}\,dx$$

$$\to \quad -e^{-y} = e^{x} + c$$

The initial condition is $y = 0$ when $x = 0$.

Then $\quad -e^{-y} = e^{x} + c \quad \to \quad -1 = 1 + c \quad \to \quad c = -2$

Therefore $\quad e^{-y} = 2 - e^{x}$

and, taking the log of each side,

$$y = -\ln(2 - e^{x})$$

Solve the initial value problems:

23. $\dfrac{dy}{dx} = \dfrac{x+y}{x}; \quad y(1) = 2$

Let $\quad u = y/x$

Then $\quad \dfrac{dy}{dx} = \dfrac{x+y}{x} = 1 + \dfrac{y}{x} \quad \to \quad 1 + u = f(u)$

The general solution is given by equation (11.19),

$$\int \frac{du}{f(u) - u} = \ln x + c$$

The left side is $\quad \displaystyle\int \frac{du}{f(u) - u} = \int du = u = \frac{y}{x}$

Therefore $\quad \dfrac{y}{x} = \ln x + c \quad \to \quad y = x(\ln x + c)$

The initial condition, $y = 2$ when $x = 1$, gives $c = 2$. The particular solution of the differential equation is therefore

$$y = x(\ln x + 2)$$

24. $2xy\dfrac{dy}{dx} = -(x^2 + y^2); \quad y(1) = 0$

We have $\quad 2xy\dfrac{dy}{dx} = -(x^2 + y^2) \;\to\; \dfrac{dy}{dx} = -\dfrac{(x^2 + y^2)}{2xy} = -\dfrac{1}{2}\left(\dfrac{x}{y} + \dfrac{y}{x}\right)$

Let $\quad u = y/x$

Then $\quad \dfrac{dy}{dx} = -\dfrac{1}{2}\left(\dfrac{1}{u} + u\right) = f(u); \;\; f(u) - u = -\dfrac{1}{2}\left(3u + \dfrac{1}{u}\right)$

By equation (11.19), the general solution is

$$\int \frac{du}{f(u) - u} = \ln x + c$$

The left side is $\quad \displaystyle\int \frac{du}{f(u) - u} = -2\int \frac{du}{3u + 1/u} = 2\int \frac{u\,du}{3u^2 + 1} = -\frac{1}{3}\ln(3u^2 + 1)$

Therefore $\quad -\dfrac{1}{3}\ln\left(\dfrac{3y^2}{x^2} + 1\right) = \ln x + c \;\to\; \ln(3y^2 + x^2) = -\ln ax \qquad (\ln a = 3c)$

and the general solution of the differential equation is

$$\ln(3y^2 + x^2) = \ln\frac{1}{ax} \;\to\; 3y^2 + x^2 = \frac{1}{ax} \;\to\; y^2 = \frac{1}{3}\left[\frac{1}{ax} - x^2\right]$$

The initial condition, $y = 0$ when $x = 1$, gives $a = 1$. The particular solution of the homogeneous differential equation is therefore

$$y^2 = \frac{1}{3}\left[\frac{1}{x} - x^2\right]$$

25. $xy^3\dfrac{dy}{dx} = x^4 + y^4; \quad y(2) = 0$

We have $\quad xy^3\dfrac{dy}{dx} = x^4 + y^4 \;\to\; \dfrac{dy}{dx} = \dfrac{x^3}{y^3} + \dfrac{y}{x} = \dfrac{1}{u^3} + u = f(u) \quad$ where $u = \dfrac{y}{x}$

Then $\quad \displaystyle\int \frac{du}{f(u) - u} = \int u^3\,du = \frac{1}{4}u^4$

and the general solution of the differential equation is

$$\frac{y^4}{4x^4} = \ln x + c$$

The initial condition, $y = 0$ when $x = 2$, gives $c = -\ln 2$.

Therefore $\quad y^4 = 4x^4 \ln\dfrac{x}{2}$

26. Show that a differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

is reduced to separable form by means of the substitution $u = ax + by$.

Let $u = ax + by$

Then $\frac{du}{dx} = a + b\frac{dy}{dx} = a + f(u + c)$

Therefore, by separation of variables,

$$\frac{du}{a + f(u + c)} = dx \quad \to \quad \int \frac{du}{a + f(u + c)} = x + c$$

Use the method of Exercise 26 to find the general solution:

27. $\frac{dy}{dx} = 2x + y + 3$

Let $u = 2x + y$

Then $\frac{du}{dx} = 2 + \frac{dy}{dx} = u + 5$

Then, by separation of variables,

$$\int \frac{du}{u + 5} = \int dx \quad \to \quad \ln(u + 5) = x + c \quad \to \quad u + 5 = e^{x+c} = ae^x \quad (a = e^c)$$

Therefore $y = ae^x - 2x - 5$

28. $\frac{dy}{dx} = \frac{x - y}{x - y + 2}$

Let $u = x - y$

Then $\frac{du}{dx} = 1 - \frac{dy}{dx} = 1 - \frac{x - y}{x - y + 2} = 1 - \frac{u}{u + 2} = \frac{2}{u + 2}$

Then, by separation of variables,

$$\frac{1}{2}\int (u + 2)\,du = \int dx \quad \to \quad \frac{1}{2}\left(\frac{u^2}{2} + 2u\right) = x + c$$

and $\frac{1}{2}\left(\frac{(x - y)^2}{2} + 2(x - y)\right) = x + c \quad \to \quad (x - y)^2 - 4y = a \quad (a = 4c)$

Section 11.4

29. Find the interval $\tau_{1/n}$ in which the amount of reactant in a first-order decay process is reduced by factor *n*.

As for the half-life $(n = 2)$,

$$x(t + \tau_{1/n}) = \frac{1}{n}x(t)$$

and, for first-order exponential decay,

$$x(t) = ae^{-kt}$$

$$x(t + \tau_{1/n}) = ae^{-k(t+\tau_{1/n})} = ae^{-kt} \times e^{-k\tau_{1/n}}$$

$$= e^{-k\tau_{1/n}}x(t)$$

Then $\quad \frac{1}{n} = e^{-k\tau_{1/n}} \quad \to \quad \ln\frac{1}{n} = -k\tau_{\tau_{1/n}} \quad \to \quad \ln n = k\tau_{\tau_{1/n}}$

and $\quad \tau_{1/n} = \frac{\ln n}{k}$

30. Solve the initial value problem for the *n*th-order kinetic process A → products

$$\frac{dx}{dt} = -kx^n \qquad x(0) = a \qquad (n > 1)$$

By separation of variables and integration,

$$-\int\frac{dx}{x^n} = k\int dt \quad \to \quad \frac{1}{(n-1)x^{n-1}} = kt + c$$

The initial condition, $x = a$ when $t = 0$, gives $c = \frac{1}{(n-1)a^{n-1}}$

Therefore $\quad \frac{1}{x^{n-1}} - \frac{1}{a^{n-1}} = (n-1)kt$

31. The reversible reaction A $\rightleftharpoons$ B, first-order in both directions, has rate equation

$$\frac{dx}{dt} = k_1(a-x) - k_{-1}x$$

Find $x(t)$ for initial condition $x(0) = 0$.

We have $$\frac{dx}{dt} = k_1(a-x) - k_{-1}x = k_1 a - (k_1 + k_{-1})x$$

Then, by separation of variables and integration,

$$\int \frac{dx}{k_1 a - (k_1+k_{-1})x} = \int dt \;\rightarrow\; \frac{-1}{k_1+k_{-1}} \ln\left[k_1 a - (k_1+k_{-1})x\right] = t + c$$

The initial condition, $x = 0$ when $t = 0$ gives $c = \dfrac{-1}{k_1+k_{-1}} \ln\left[k_1 a\right]$.

Therefore $$\ln\left[\frac{k_1 a}{k_1 a - (k_1+k_{-1})x}\right] = (k_1+k_{-1})t \;\rightarrow\; \frac{k_1 a}{k_1 a - (k_1+k_{-1})x} = e^{(k_1+k_{-1})t}$$

and $$x = \frac{k_1 a}{k_1+k_{-1}}\left[1 - e^{(k_1+k_{-1})t}\right]$$

32. A third-order process A + 2B → products has rate equation

$$\frac{dx}{dt} = k(a-x)(b-2x)^2$$

where *a* and *b* are the initial amounts of A and B, respectively. Show that the solution that satisfies the initial condition $x(0) = 0$ is given by

$$kt = \frac{1}{(2a-b)^2} \ln \frac{a(b-2x)}{b(a-x)} + \frac{2x}{b(2a-b)(b-2x)}$$

Separation of variables and integration gives

$$\int \frac{dx}{(a-x)(b-2x)^2} = k\int dt = kt + c$$

Now, by the method of partial fractions,

$$\frac{1}{(a-x)(b-2x)^2} = \frac{1}{(2a-b)^2}\left[\frac{1}{(a-x)} - \frac{2}{b-2x} + \frac{2(2a-b)}{(b-2x)^2}\right]$$

Therefore $$\int \frac{dx}{(a-x)(b-2x)^2} = \frac{1}{(2a-b)^2}\int\left[\frac{1}{(a-x)} - \frac{2}{b-2x} + \frac{2(2a-b)}{(b-2x)^2}\right]dx$$

$$= \frac{1}{(2a-b)^2}\left[-\ln(a-x) + \ln(b-2x) + \frac{2a-b}{b-2x}\right] = kt + c$$

The initial condition, $x = 0$ when $t = 0$, gives

$$c = \frac{1}{(2a-b)^2}\left[-\ln a + \ln b + \frac{2a-b}{b}\right] = \frac{1}{(2a-b)^2}\ln\frac{b}{a} + \frac{1}{b(2a-b)}$$

Therefore $$kt = \frac{1}{(2a-b)^2}\left[\ln\frac{a(b-2x)}{b(a-x)}\right] + \frac{2x}{b(2a-b)(b-2x)}$$

Section 11.5

Find the general solution:

33. $\dfrac{dy}{dx} + 2y = 4$

The linear differential equation has the form (11.47) with $p(x) = 2,\ r(x) = 4$. Then

$\int p(x)\,dx = 2x$, and the integrating factor is

$$F(x) = e^{\int p(x)\,dx} = e^{2x}$$

Then, by formula,

$$e^{2x}y = 4\int e^{2x}\,dx + c = 2e^{2x} + c$$

and $$y = 2 + ce^{-2x}$$

34. $\dfrac{dy}{dx} - 4xy = x$

We have $p(x) = -4x,\ r(x) = x$, and integrating factor $F(x) = e^{\int -4x\,dx} = e^{-2x^2}$.

Therefore, by formula,

$$e^{-2x^2}y = \int e^{-2x^2}x\,dx + c = -\frac{1}{4}e^{-2x^2} + c$$

and $$y = ce^{2x^2} - \frac{1}{4}$$

35. $\dfrac{dy}{dx}+3y=e^{-3x}$

We have $p(x)=3,\ r(x)=e^{-3x}$, and integrating factor $F(x)=e^{\int 3\,dx}=e^{3x}$.

Then, by formula,

$$e^{3x}y=\int e^{3x}\times e^{-3x}\,dx=\int dx=x+c$$

and $\quad y=e^{-3x}(x+c)$

36. $\dfrac{dy}{dx}+\dfrac{2y}{x}=2\cos x$

We have $p(x)=2/x,\ r(x)=2\cos x$. Then $\displaystyle\int p(x)\,dx=2\int\frac{dx}{x}=2\ln x=\ln x^2$ and the integrating factor is $F(x)=e^{\int p(x)\,dx}=e^{\ln x^2}=x^2$. Then, by formula,

$$x^2y=2\int x^2\cos x\,dx+c$$

and integration by parts, as in Example 6.11, gives

$$x^2y=2\left[x^2\sin x+2x\cos x-2\sin x+c\right]$$

$$y=\frac{2}{x^2}\left[x^2\sin x+2x\cos x-2\sin x+c\right]$$

37. $\dfrac{dy}{dx}-\dfrac{y}{x^2}=\dfrac{4}{x^2}$

$$p(x)=-1/x^2,\ r(x)=4/x^2,$$

and the integrating factor is $F(x)=e^{\int p\,dx}=e^{1/x}$. Then, by formula,

$$e^{1/x}y=4\int e^{1/x}\times\frac{1}{x^2}dx+c$$

Let $\quad u=\dfrac{1}{x},\quad du=-\dfrac{1}{x^2}dx$.

Then $\quad e^{1/x}y=4\displaystyle\int e^{1/x}\times\frac{1}{x^2}dx+c=-4\int e^u\,du+c=-4e^u+c=-4e^{1/x}+c$

and $\quad e^{1/x}y=-4e^{1/x}+c\ \ \rightarrow\ \ y=ce^{-1/x}-4$

38. $\dfrac{dy}{dx} + (2\tan x)y = \sin x$

We have $p(x) = 2\tan x,\ r(x) = \sin x$.

Then $\displaystyle\int p(x)\,dx = 2\int \tan x\,dx = -2\ln(\cos x) = \ln(1/\cos^2 x)$

and the integrating factor is $F(x) = e^{\ln(1/\cos^2 x)} = \dfrac{1}{\cos^2 x}$.

Then $\displaystyle\frac{1}{\cos^2 x}y = \int \frac{1}{\cos^2 x}\times \sin x\,dx + c = \frac{1}{\cos x} + c$

and $y = \cos x + c\cos^2 x$

39. $\dfrac{dy}{dx} + ax^n y = bx^n, \quad (n \neq -1)$

We have $p(x) = ax^n,\ r(x) = bx^n$, and

$F(x) = e^{\int ax^n\,dx} = e^{ax^{n+1}/(n+1)}$.

Then $\displaystyle e^{ax^{n+1}/(n+1)}y = \int e^{ax^{n+1}/(n+1)} \times bx^n dx + c$

Let $t = x^{n+1}/(n+1),\quad dt = x^n$.

Then $\displaystyle e^{ax^{n+1}/(n+1)}y = b\int e^{at}\,dt + c = \frac{b}{a}e^{at} + c = \frac{b}{a}e^{ax^{n+1}/(n+1)} + c$

and $y = \dfrac{b}{a} + ce^{-ax^{n+1}/(n+1)}$

40. $\dfrac{dy}{dx} + a\dfrac{y}{x} = x^n$

We have $p(x) = a/x,\ r(x) = x^n$, and integrating factor $F(x) = e^{\int a/x\,dx} = e^{a\ln x} = x^a$.

Then $\displaystyle x^a y = \int x^a \times x^n\,dx + c = \frac{x^{a+n+1}}{a+n+1} + c$

and $y = \dfrac{x^{n+1}}{a+n+1} + cx^{-a}$

Section 11.6

41. The system of three consecutive first-order processes $A \xrightarrow{k_1} B \xrightarrow{k_2} C \xrightarrow{k_3} D$ is modelled by the set of equations

$$\frac{d(a-x)}{dt} = -k_1(a-x), \quad \frac{dy}{dt} = k_1(a-x) - k_2 y, \quad \frac{dz}{dt} = k_2 y - k_3 z$$

where $(a-x)$, y, and z are the amounts of A, B, and C, respectively, at time *t*. Given the initial conditions $x = y = z = 0$ at $t = 0$, find the amount of C as a function of *t*. Assume $k_1 \neq k_2,\ k_1 \neq k_3,\ k_2 \neq k_3$.

For the first-order step $A \to B$,

$$\frac{d(a-x)}{dt} = -k_1(a-x)$$

Separation of variables and integration gives

$$\int \frac{d(a-x)}{(a-x)} = -\int k_1\, dt \quad \to \quad \ln(a-x) = -k_1 t + c \quad \to \quad a - x = Ae^{-k_1 t} \quad (A = e^c)$$

The initial condition for this step is $x = 0$ when $t = 0$. Therefore $A = a$ and

$$a - x = ae^{-k_1 t} \qquad \text{(equation 1)}$$

For step $B \to C$, making use of equation 1 above,

$$\frac{dy}{dt} = k_1(a-x) - k_2 y = ak_1 e^{-k_1 t} - k_2 y \quad \to \quad \frac{dy}{dt} + k_2 y = ak_1 e^{-k_1 t}$$

We have a linear equation with $p(t) = k_2$, $r(t) = ak_1 e^{-k_1 t}$ and integrating factor

$F(t) = e^{\int k_2\, dt} = e^{k_2 t}$.

Then $$e^{k_2 t} y = \int e^{k_2 t} \times ak_1 e^{-k_1 t}\, dt = ak_1 \int e^{(k_2 - k_1)t}\, dt = \frac{ak_1}{k_2 - k_1} e^{(k_2 - k_1)t} + c$$

and $$y = \frac{ak_1}{k_2 - k_1} e^{-k_1 t} + ce^{-k_2 t}$$

The initial condition for this step is $y = 0$ when $t = 0$. Therefore $c = -\dfrac{ak_1}{k_2 - k_1}$ and

$$y = \frac{ak_1}{k_2 - k_1}\left[e^{-k_1 t} - e^{-k_2 t}\right] \qquad \text{(equation 2)}$$

For step $C \to D$, making use of equation 2,

$$\frac{dz}{dt} = k_2 y - k_3 z \quad \to \quad \frac{dz}{dt} + k_3 z = k_2 y = \frac{ak_1k_2}{k_2 - k_1}\left[e^{-k_1t} - e^{-k_2t}\right]$$

This is a linear equation with $p(t) = k_3$, $r(t) = \frac{ak_1k_2}{k_2 - k_1}\left[e^{-k_1t} - e^{-k_2t}\right]$ and integrating factor e^{k_3t}.

Then $$e^{k_3t}z = \frac{ak_1k_2}{k_2 - k_1}\int\left[e^{(k_3-k_1)t} - e^{(k_3-k_2)t}\right]dt = \frac{ak_1k_2}{k_2 - k_1}\left[\frac{e^{(k_3-k_1)t}}{k_3 - k_1} - \frac{e^{(k_3-k_2)t}}{k_3 - k_2}\right] + c$$

and $$z = \frac{ak_1k_2}{k_2 - k_1}\left[\frac{e^{-k_1}}{k_3 - k_1} - \frac{e^{-k_2t}}{k_3 - k_2}\right] + ce^{-k_3t}$$

The initial condition for this step is $z = 0$ when $t = 0$. Therefore $c = -\frac{ak_1k_2}{k_2 - k_1}\left[\frac{1}{k_3 - k_1} - \frac{1}{k_3 - k_2}\right]$,

and

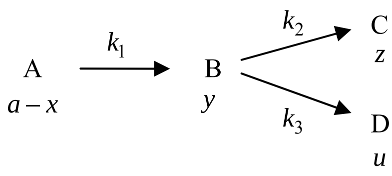
the amount of substance C at time *t* is

$$z = \frac{ak_1k_2}{k_2 - k_1}\left[\frac{e^{-k_1t} - e^{-k_3t}}{k_3 - k_1} - \frac{e^{-k_2t} - e^{-k_3t}}{k_3 - k_2}\right]$$

42. The first-order process $A \xrightarrow{k_1} B$ is followed by the parallel first-order processes $B \xrightarrow{k_2} C$ and $B \xrightarrow{k_3} D$, and the system is modelled by the equations

$$\frac{d(a-x)}{dt} = -k_1(a-x), \quad \frac{dy}{dt} = k_1(a-x) - (k_2+k_3)y,$$
$$\frac{dz}{dt} = k_2 y, \quad \frac{du}{dt} = k_3 y$$

where $(a-x)$, y, z and u are the amounts of A, B, C, and D, respectively, at time *t*. Given the initial conditions $x = y = z = u = 0$ at $t = 0$ find the amount of C as a function of time.



$A \to B$: as in Exercise 41.

$$\frac{d(a-x)}{dt} = -k_1(a-x) \quad \to \quad a - x = ae^{-k_1 t}$$

$B \to C + D$: as in Exercise 41, but with k_2 replaced by $k_2 + k_3$.

$$\frac{dy}{dt} = k_1(a-x) - (k_2+k_3)y \quad \to \quad y = \frac{ak_1}{k_2+k_3-k_1}\left[e^{-k_1 t} - e^{-(k_2+k_3)t}\right]$$

$B \to D$:

$$\frac{dz}{dt} = k_2 y = \frac{ak_1 k_2}{k_2+k_3-k_1}\left[e^{-k_1 t} - e^{-(k_2+k_3)t}\right]$$

Integration with respect to *t* gives

$$z = \frac{ak_1 k_2}{k_2+k_3-k_1}\left[-\frac{e^{-k_1 t}}{k_1} + \frac{e^{-(k_2+k_3)t}}{k_2+k_3}\right] + c$$

The initial condition, $z = 0$ when $t = 0$, gives $c = \dfrac{ak_1 k_2}{k_2+k_3-k_1}\left[\dfrac{1}{k_1} - \dfrac{1}{k_2+k_3}\right]$. The amount of substance C at time *t* is therefore,

$$z = \frac{ak_1 k_2}{k_2+k_3-k_1}\left[\frac{1-e^{-k_1 t}}{k_1} - \frac{1-e^{-(k_2+k_3)t}}{k_2+k_3}\right]$$

Section 11.7

43. The current in an RL–circuit containing one resistor and one inductor is given by the equation

$$L\frac{dI}{dt} + RI = E$$

Solve the equation for a periodic e.m.f. $E(t) = E_0 \sin \omega t$, with initial condition $I(0) = 0$.

This is the problem discussed in Example 11.7, but with a periodic e.m.f. Thus, by equation (11.66), the solution of the linear differential equation is

$$I(t) = e^{-Rt/L}\left[\int e^{Rt/L}\frac{E}{L}dt + c\right]$$

and, inserting $E = E_0 \sin \omega t$,

$$I(t) = e^{-Rt/L}\left[\frac{E_0}{L}\int e^{Rt/L}\sin \omega t dt + c\right]$$

The integral is evaluated by parts, as described in Example 6.13. Let $a = R/L$ and $b = \omega$, then (see also Exercises 49 and 50 in Chapter 6)

$$\begin{aligned}\int e^{at}\sin bt\,dt &= \frac{1}{a}e^{at}\sin bt - \frac{b}{a}\int e^{at}\cos bt\,dt\\ &= \frac{1}{a}e^{at}\sin bt - \frac{b}{a^2}e^{at}\cos bt - \frac{b^2}{a^2}\int e^{at}\sin bt\,dt\end{aligned}$$

Solving for $\int e^{at}\sin bt\,dt$,

$$\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}\left[a\sin bt - b\cos bt\right] .$$

Then

$$\begin{aligned}I(t) &= e^{-Rt/L}\left[\frac{E_0}{L}\int e^{Rt/L}\sin \omega t dt + c\right]\\ &= e^{-Rt/L}\left\{\frac{e^{Rt/L}E_0L}{R^2+\omega^2L^2}\left[\frac{R}{L}\sin \omega t - \omega\cos \omega t\right] + c\right\}\end{aligned}$$

The initial condition, $I = 0$ when $t = 0$, gives $c = \dfrac{E_0\omega L}{R^2+\omega^2L^2}$.

Therefore

$$I(t) = \frac{E_0}{R^2+\omega^2L^2}\left[\omega L e^{-Rt/L} + R\sin \omega t - \omega L\cos \omega t\right]$$

44. Show that the current in an RC–circuit containing one resistor and one capacitor is given by the equation

$$R\frac{dI}{dt}+\frac{I}{C}=\frac{dE}{dt}$$

Solve the equation for **(i)** a constant e.m.f., $E=E_0$, **(ii)** a periodic e.m.f., $E(t)=E_0\sin\omega t$.

By Kirchoff's law and equations (11.62) and (11.64),

$$RI+\frac{Q}{C}=E$$

Then, because $\frac{dQ}{dt}=I$

we have $R\frac{dI}{dt}+\frac{1}{C}\frac{dQ}{dt}=\frac{dE}{dt} \quad\rightarrow\quad R\frac{dI}{dt}+\frac{I}{C}=\frac{dE}{dt}$

(i) For constant e.m.f., $\frac{dE}{dt}=0$ and the differential equation is separable. Thus

$$R\frac{dI}{dt}+\frac{I}{C}=0 \quad\rightarrow\quad \int\frac{dI}{I}=-\frac{1}{RC}\int dt \quad\rightarrow\quad \ln I=-\frac{t}{RC}+c$$

Therefore $I=I_0e^{-t/RC}$, where I_0 is the current at time $t=0$.

(ii) We have $E=E_0\sin\omega t \quad\rightarrow\quad \frac{dE}{dt}=E_0\omega\cos\omega t$.

Then $$\frac{dI}{dt}+\frac{I}{RC}=\frac{E_0\omega}{R}\cos\omega t$$

is a linear equation with $p=1/RC$, $r=\frac{E_0\omega}{R}\cos\omega t$ and integrating factor $F(t)=e^{t/RC}$.

Then $$I(t)=e^{-t/RC}\left\{\frac{E_0\omega}{R}\int e^{t/RC}\cos\omega t\,dt+c\right\}$$

The integral is evaluated by parts, as in Exercise 43 above and Exercise 50 in Chapter 6:

$$\int e^{at}\cos bt\,dt=\frac{e^{at}}{a^2+b^2}\big[a\cos bt+b\sin bt\big]$$

where $a=1/RC$ and $b=\omega$. Therefore

$$\int e^{t/RC}\cos\omega t\,dt=\frac{RCe^{t/RC}}{q+(\omega RC)^2}\big[\cos\omega t+\omega RC\sin\omega t\big]$$

and $$I(t)=\frac{\omega E_0C}{1+(\omega RC)^2}\big[\cos\omega t+\omega RC\sin\omega t\big]+ce^{-t/RC}$$