The Chemistry Maths Book

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Solutions

Chapter 9. Functions of several variables

Section 9.1

1. Find the value of the function $f(x,y) = 2x^2 + 3xy - y^2 + 2x - 3y + 4$ for

(i) $(x,y) = (0,1)$ **(ii)** $(x,y) = (2,0)$ **(iii)** $(x,y) = (3,2)$

(i) $f(0,1) = 0 + 0 - 1 + 0 - 3 + 4 = 0$

(ii) $f(2,0) = 8 + 0 - 0 + 4 - 0 + 4 = 16$

(iii) $f(3,2) = 18 + 18 - 4 + 6 - 6 + 4 = 36$

2. Find the value of $f(r,\theta,\phi) = r^2 \sin^2\theta \cos^2\phi + 2\cos^2\theta - r^3 \sin 2\theta \sin\phi$ for

(i) $(r,\theta,\phi) = (1, \pi/2, 0)$ **(ii)** $(r,\theta,\phi) = (2, \pi/4, \pi/6)$ **(iii)** $(r,\theta,\phi) = (0, \pi, \pi/3)$.

(i) $f(1, \pi/2, 0) = 1 \times \sin^2 \pi/2 \times \cos^2 0 + 2\cos^2 \pi/2 - 1 \times \sin\pi \times \sin 0$
$= 1 + 0 - 0 = 1$

(ii) $f(2, \pi/4, \pi/6) = 2^2 \times \sin^2 \pi/4 \times \cos^2 \pi/6 + 2\cos^2 \pi/4 - 2^3 \times \sin \pi/2 \times \sin \pi/6$
$= 4 \times \frac{1}{2} \times \frac{3}{4} + 2 \times \frac{1}{2} - 8 \times 1 \times \frac{1}{2} = -\frac{3}{2}$

(iii) $f(0, \pi, \pi/3) = 0 + 2\cos^2 \pi - 0 = 2$

Section 9.3

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

3. $z = 2x^2 - y^2$: $\frac{\partial z}{\partial x} = 4x$, $\frac{\partial z}{\partial y} = -2y$

4. $z = x^2 + 2y^2 - 3x + 2y + 3$: $\frac{\partial z}{\partial x} = 2x - 3$, $\frac{\partial z}{\partial y} = 4y + 2$

5. $z = e^{2x+3y}$: $\frac{\partial z}{\partial x} = 2e^{2x+3y} = 2z$, $\frac{\partial z}{\partial y} = 3e^{2x+3y} = 3z$

6. $z = \sin(x^2 - y^2)$: $\frac{\partial z}{\partial x} = 2x\cos(x^2 - y^2)$, $\frac{\partial z}{\partial y} = -2y\cos(x^2 - y^2)$

7. $z = e^{x^2} \cos xy$: By the product rule,

$$\frac{\partial z}{\partial x} = e^{x^2} \times \frac{\partial}{\partial x}\cos xy + \frac{\partial}{\partial x}(e^{x^2}) \times \cos xy$$
$$= e^{x^2} \times (-y \sin xy) + (2xe^{x^2}) \times \cos xy$$
$$= \left[2x \cos xy - y \sin xy\right] e^{x^2}$$

$$\frac{\partial z}{\partial y} = e^{x^2} \times \frac{\partial}{\partial y}\cos xy = -xe^{x^2} \sin xy$$

Find all the nonzero partial derivatives of

8. $z = x^2 - 3x^2 y + 4xy^2$:

First derivatives: $\frac{\partial z}{\partial x} = 2x - 6xy + 4y^2$, $\frac{\partial z}{\partial y} = -3x^2 + 8xy$

Second : $\frac{\partial^2 z}{\partial x^2} = 2 - 6y$, $\frac{\partial^2 z}{\partial y \partial x} = -6x + 8y = \frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2} = 8x$

Third: $\frac{\partial^3 z}{\partial y \partial x^2} = -6 = \frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial^3 z}{\partial x^2 \partial y}$

$$\frac{\partial^3 z}{\partial y^2 \partial x} = 8 = \frac{\partial^3 z}{\partial y \partial x \partial y} = \frac{\partial^3 z}{\partial x \partial y^2}$$

9. $u = 3x^2 + y^2 + 2xy^3$:

$$u_x = 6x + 2y^3, \quad u_y = 2y + 6xy^2$$
$$u_{xx} = 6, \quad u_{yx} = 6y^2 = u_{xy}, \quad u_{yy} = 2 + 12xy$$
$$u_{yyx} = 12y = u_{yxy} = u_{xyy}$$
$$u_{yyy} = 12x$$
$$u_{xyyy} = 12 = u_{yxyy} = u_{yyxy} = u_{yyyx}$$

Find all the first and second partial derivatives of

10. $z = 2x^2 y + \cos(x + y)$:

$$\frac{\partial z}{\partial x} = 4xy - \sin(x + y), \quad \frac{\partial z}{\partial y} = 2x^2 - \sin(x + y),$$
$$\frac{\partial^2 z}{\partial x^2} = 4y - \cos(x + y), \quad \frac{\partial^2 z}{\partial x \partial y} = 4x - \cos(x + y) = \frac{\partial^2 z}{\partial y \partial x}, \quad \frac{\partial^2 z}{\partial y^2} = -\cos(x + y)$$

11. $z = \sin(x+y)e^{x-y}$: By the product rule,

$$\frac{\partial z}{\partial x} = \left[\sin(x+y) + \cos(x+y)\right]e^{x-y}$$
$$\frac{\partial z}{\partial y} = \left[\cos(x+y) - \sin(x+y)\right]e^{x-y}$$
$$\frac{\partial^2 z}{\partial x^2} = 2\cos(x+y)e^{x-y}, \quad \frac{\partial^2 z}{\partial y^2} = -2\cos(x+y)e^{x-y}$$
$$\frac{\partial^2 z}{\partial x \partial y} = -2\sin(x+y)e^{x-y} = \frac{\partial^2 z}{\partial y \partial x}$$

Show that $f_{xy} = f_{yx}$ for

12. $f = x^3 - 3x^2y + y^3$:

$$\left.\begin{aligned} f_x &= 3x^2 - 6xy, & f_{yx} &= -6x \\ f_y &= -3x^2 + 3y^2, & f_{xy} &= -6x \end{aligned}\right\} \to f_{yx} = f_{xy}$$

13. $f = x^2\cos(y-x)$: By the product rule,

$$\left.\begin{aligned} f_x &= x^2\sin(y-x) + 2x\cos(y-x), & f_{yx} &= x^2\cos(y-x) - 2x\sin(y-x) \\ f_y &= -x^2\sin(y-x), & f_{xy} &= x^2\cos(y-x) - 2x\sin(y-x) \end{aligned}\right\} \to f_{yx} = f_{xy}$$

14. $f = \dfrac{xy}{x^2+y^2}$: By the quotient rule,

$$f_x = \frac{(x^2+y^2)(y) - (xy)(2x)}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

$$f_y = \frac{(x^2+y^2)(x) - (xy)(2y)}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$$

$$f_{yx} = \frac{(x^2+y^2)^2 \times (3y^2 - x^2) - (y^3 - x^2y) \times (4y)(x^2+y^2)}{(x^2+y^2)^4} = -\frac{(x^4 + y^4 - 6x^2y^2)}{(x^2+y^2)^3}$$

$$f_{xy} = \frac{(x^2+y^2)^2 \times (3x^2 - y^2) - (x^3 - xy^2) \times (4x)(x^2+y^2)}{(x^2+y^2)^4} = -\frac{(x^4 + y^4 - 6x^2y^2)}{(x^2+y^2)^3}$$

Therefore $f_{yx} = f_{xy}$

Show that $f_{xyz} = f_{yzx} = f_{zxy}$ for

15. $f = \cos(x+2y+3z)$:

$$f_x = -\sin(x+2y+3z), \quad f_{zx} = -3\cos(x+2y+3z), \quad f_{yzx} = 6\sin(x+2y+3z)$$

$$f_y = -2\sin(x+2y+3z), \quad f_{xy} = -2\cos(x+2y+3z), \quad f_{zxy} = 6\sin(x+2y+3z)$$

$$f_z = -3\sin(x+2y+3z), \quad f_{yz} = -6\cos(x+2y+3z), \quad f_{xyz} = 6\sin(x+2y+3z)$$

Therefore $f_{xyz} = f_{yzx} = f_{zxy}$

16. $f = xye^{yz}$

$$f_x = ye^{yz}, \quad f_{zx} = y^2e^{yz}, \quad f_{yzx} = (2y+y^2z)e^{yz}$$

$$f_y = (x+xyz)e^{yz}, \quad f_{xy} = (1+yz)e^{yz}, \quad f_{zxy} = (2y+y^2z)e^{yz}$$

$$f_z = xy^2e^{yz}, \quad f_{yz} = (2xy+xy^2z)e^{yz}, \quad f_{xyz} = (2y+y^2z)e^{yz}$$

Therefore $f_{xyz} = f_{yzx} = f_{zxy}$

17. If $r = (x^2+y^2+z^2)^{1/2}$ find $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial r}{\partial z}$.

Let $u = x^2+y^2+z^2, \; r = u^{1/2}$.

Then $\dfrac{\partial r}{\partial x} = \dfrac{\partial r}{\partial u}\times\dfrac{\partial u}{\partial x} = \dfrac{1}{2}u^{-1/2}\times 2x = \dfrac{x}{r}$

Simularly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

18. If $\phi = f(x-ct)+g(x+ct)$, where c is a constant, show that $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$.

Let $u = x+ct, \quad v = x-ct$

Then $\phi = f(u)+g(v), \quad \dfrac{\partial\phi}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial f}{\partial u}+\dfrac{\partial g}{\partial v}, \quad \dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial^2 f}{\partial u^2}+\dfrac{\partial^2 g}{\partial v^2}$

$$\frac{\partial\phi}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial t} = c\frac{\partial f}{\partial u}-c\frac{\partial g}{\partial v} \quad \frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2 f}{\partial u^2}+c^2\frac{{}^2\partial g}{\partial v^2}$$

Therefore $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2}$

19. If $xyz + x^2 + y^2 + z^2 = 0$, find $\left(\dfrac{\partial y}{\partial x}\right)_z$.

We have $$\left(\frac{\partial}{\partial x}\right)_z \left[xyz + x^2 + y^2 + z^2\right] = \left[yz + xz\left(\frac{\partial y}{\partial x}\right)_z\right] + \left[2x\right] + \left[2y\left(\frac{\partial y}{\partial x}\right)_z\right] = 0$$

Therefore $$(2y + xz)\left(\frac{\partial y}{\partial x}\right)_z + (2x + yz) = 0 \;\rightarrow\; \left(\frac{\partial y}{\partial x}\right)_z = -\frac{2x + yz}{2y + xz}$$

20. For the van der Waals equation

$$\left(p + \frac{n^2 a}{V^2}\right)(V - nb) - nRT = 0$$

Find **(i)** $\left(\dfrac{\partial V}{\partial T}\right)_{p,n}$, **(ii)** $\left(\dfrac{\partial V}{\partial p}\right)_{T,n}$, **(iii)** $\left(\dfrac{\partial p}{\partial T}\right)_{V,n}$, **(iv)** $\left(\dfrac{\partial p}{\partial V}\right)_{T,n}$.

(i) We have $$\left(\frac{\partial}{\partial T}\right)_{p,n}\left\{\left(p + \frac{n^2 a}{V^2}\right)(V - nb) - nRT\right\}$$

$$= \left[\left(-\frac{2n^2 a}{V^3}\right)(V - nb) + \left(p + \frac{n^2 a}{V^2}\right)\right]\left(\frac{\partial V}{\partial T}\right)_{p,n} - nR$$

$$= \left[p - \frac{n^2 a}{V^2} + \frac{2n^3 ab}{V^3}\right]\left(\frac{\partial V}{\partial T}\right)_{p,n} - nR = 0$$

Therefore $$\left(\frac{\partial V}{\partial T}\right)_{p,n} = nR\bigg/\left(p - \frac{n^2 a}{V^2} + \frac{2n^3 ab}{V^3}\right)$$

(ii) $$\left(\frac{\partial}{\partial p}\right)_{T,n}\left\{\left(p + \frac{n^2 a}{V^2}\right)(V - nb) - nRT\right\}$$

$$= \left[1 - \frac{2n^2 a}{V^3}\left(\frac{\partial V}{\partial p}\right)_{T,n}\right](V - nb) + \left(p + \frac{n^2 a}{V^2}\right)\left(\frac{\partial V}{\partial p}\right)_{T,n}$$

$$= (V - nb) + \left[p - \frac{n^2 a}{V^2} + \frac{2n^3 ab}{V^3}\right]\left(\frac{\partial V}{\partial p}\right)_{T,n} = 0$$

Therefore $$\left(\frac{\partial V}{\partial p}\right)_{T,n} = -(V - nb)\bigg/\left(p - \frac{n^2 a}{V^2} + \frac{2n^3 ab}{V^3}\right)$$

(iii) $$\left(\frac{\partial}{\partial T}\right)_{V,n}\left\{\left(p+\frac{n^2a}{V^2}\right)(V-nb)-nRT\right\}$$

$$=(V-nb)\left(\frac{\partial p}{\partial T}\right)_{V,n}-nR=0$$

Therefore $$\left(\frac{\partial p}{\partial T}\right)_{V,n}=\frac{nR}{V-nb}$$

(iv) $$\left(\frac{\partial}{\partial V}\right)_{T,n}\left\{\left(p+\frac{n^2a}{V^2}\right)(V-nb)-nRT\right\}$$

$$=\left(\left(\frac{\partial p}{\partial V}\right)_{T,n}-\frac{2n^2a}{V^3}\right)(V-nb)+\left(p+\frac{n^2a}{V^2}\right)=0$$

Therefore $$\left(\frac{\partial p}{\partial V}\right)_{T,n}=\frac{2n^2a}{V^3}-\frac{(p+n^2a/V^2)}{V-nb}$$

Section 9.4

Find the stationary points of the following functions:

21. $f(x,y)=3-x^2-xy-y^2+2y$

For a stationary value, $\frac{\partial f}{\partial x}=-2x-y=0$ and $\frac{\partial f}{\partial y}=-x-2y+2=0$

Therefore $x=-2/3,\ y=4/3$

and the single stationary point lies at

$$(x,y)=(-2/3,4/3)$$

22. $f(x,y)=x^3+y^2-3x-4y+2$

For the stationary values, $\frac{\partial f}{\partial x}=3x^2-3=0\ \rightarrow\ x=\pm1$

$\frac{\partial f}{\partial y}=2y-4=0\ \rightarrow\ y=2$

There are two stationary points, at

$$(x,y)=(1,2)\ \text{and}\ (-1,2)$$

23. $4x^3 - 3x^2 y + y^3 - 9y$

For the stationary values,

$$\frac{\partial f}{\partial x} = 12x^2 - 6xy = 0 \;\rightarrow\; 6x(2x - y) = 0 \;\rightarrow\; x = 0 \text{ or } x = y/2$$

and $$\frac{\partial f}{\partial y} = -3x^2 + 3y^2 - 9 = 0; \quad \begin{cases} x = 0 & \rightarrow 3y^2 - 9 = 0 \;\rightarrow\; y = \pm\sqrt{3} \\ x = y/2 & \rightarrow \frac{9}{4}y^2 - 9 = 0 \;\rightarrow\; y = \pm 2 \end{cases}$$

There are four stationary points, at

$$(x, y) = (0, \sqrt{3}), \quad (0, -\sqrt{3}), \quad (1, 2), \text{ and } (-1, -2)$$

Determine the nature of of the stationary points of the functions in Exercises 21 – 23:

24. $3 - x^2 - xy - y^2 + 2y$ (Exercise 21)

We have $f_x = -2x - y, \quad f_y = -x - 2y + 2$

$$\left.\begin{array}{l} f_{xx} = -2 < 0 \\ f_{yy} = -2 < 0 \\ f_{xy} = -1 \end{array}\right\} \rightarrow f_{xx} f_{yy} - f_{xy}{}^2 = 3 > 0$$

The stationary point is therefore a maximum.

25. $x^3 + y^2 - 3x - 4y + 2$ (Exercise 22)

We have $f_x = 3x^2 - 3, \quad f_y = 2y - 4$

$f_{xx} = 6x, \quad f_{yy} = 2, \quad f_{xy} = 0$

Therefore $(x, y) = (1, 2) \;\rightarrow\; f_{xx} = 6 > 0, \; f_{yy} = 2 > 0, \; f_{xx} f_{yy} - f_{xy}{}^2 = 12 > 0$

a minimum

$(x, y) = (-1, 2) \;\rightarrow\; f_{xx} = -6 < 0, \; f_{yy} = 2 > 0, \; f_{xx} f_{yy} - f_{xy}{}^2 = -12 < 0$

a saddle point

26. $4x^3 - 3x^2y + y^3 - 9y$ (Exercise 23)

We have $f_x = 12x^2 - 6xy,\ f_y = -3x^2 + 3y^2 - 9$

$f_{xx} = 24x - 6y,\ f_{yy} = 6y,\ f_{xy} = -6x$

Then

x	y	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	type
0	$\sqrt{3}$	$-6\sqrt{3} < 0$	$6\sqrt{3} < 0$	0	$-108 < 0$	saddle point
0	$-\sqrt{3}$	$6\sqrt{3} > 0$	$-6\sqrt{3} < 0$	0	$-108 < 0$	saddle point
1	2	$12 > 0$	$12 > 0$	-6	$108 > 0$	minimum
-1	-2	$-12 < 0$	$-12 < 0$	6	$108 > 0$	maximum

27. Find the stationary value of the function $f = 2x^2 + 3y^2 + 6z^2$ subject to the constraint $x + y + z = 1$,

(i) by using the constraint to eliminate *z* from the function, **(ii)** by the method of Lagrange multipliers.

(i) We have $z = 1 - x - y$. Then

$$f(x, y, z) = 2x^2 + 3y^2 + 6z^2$$

$$\rightarrow F(x, y) = 2x^2 + 3y^2 + 6(1 - x - y)^2$$
$$= 8x^2 + 9y^2 - 12x - 12y + 12xy + 6$$

Then , for a stationary value,

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 16x - 12 + 12y = 0 \\ \frac{\partial F}{\partial y} &= 18y - 12 + 12x = 0 \end{aligned}\right\} \rightarrow x = \frac{1}{2},\ y = \frac{1}{3} \text{ and } z = 1 - x - y = \frac{1}{6}$$

The function, a quadratic form in *x*, *y* and *z*, has the single stationary value $f(x, y, z) = 1$ at $(x, y, z) = (1/2,\ 1/3,\ 1/6)$. This is a minimum value. Thus

$$f_{xx} = 16 > 0,\ f_{yy} = 18 > 0,\ f_{xy} = 12,$$

$$f_{xx}f_{yy} - f_{xy}{}^2 > 0$$

(ii) Let $\phi = (2x^2 + 3y^2 + 6z^2) - \lambda(x + y + z)$

Then
$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= 4x - \lambda = 0 \\ \frac{\partial\phi}{\partial y} &= 6y - \lambda = 0 \\ \frac{\partial\phi}{\partial z} &= 12z - \lambda = 0 \end{aligned}\right\} \to\ z = \frac{x}{3},\ \ y = \frac{2x}{3},\ \ \text{and}\ \ x + y + z = 1 \to x = \frac{1}{2}$$

and $(x, y, z) = (1/2,\ 1/3,\ 1/6)$, as in (i).

28. Find the maximum value of the function $f = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 = c^2$, **(i)** by using the constraint to eliminate *z* from the function, **(ii)** by the method of Lagrange multipliers.

Let $u = x^2,\ v = y^2,\ w = z^2,$ and $a = c^2$.

Then $f = uvw$ subject to $u + v + w = a$.

(i) We have $w = a - u - v$. Then

$$f = uv(a - u - v) = auv - u^2v - uv^2$$

and , for a stationary value,

$$\frac{\partial f}{\partial u} = av - 2uv - v^2 = 0,\quad \frac{\partial f}{\partial v} = au - u^2 - 2uv = 0$$

The possible solutions are

$(u, v) = (0, 0),\ (0, a),\ (a, 0)\ \to\ f = 0$, the minimum

and $(u, v) = (a/3, a/3) \to\ w = a/3\ \to\ f = a^3/27$.

Therefore $f = c^6/27$ at $x^2 = y^2 = z^2$, the (local) maximum.

(ii) Let $\phi = uvw - \lambda(u + v + w)$.

Then
$$\begin{aligned} \frac{\partial\phi}{\partial u} &= vw - \lambda = 0 \\ \frac{\partial\phi}{\partial v} &= uw - \lambda = 0 \\ \frac{\partial\phi}{\partial w} &= uv - \lambda = 0 \end{aligned}$$

with nonzero solution $u = v = w = a/3$, as in **(i)**.

29. **(i)** Find the stationary points of the function $f=(x-1)^2+(y-2)^2+(z-2)^2$ subject to the constraint $x^2+y^2+z^2=1$. **(ii)** Show that these lie at the shortest and longest distances of the point $(1, 2, 2)$ from the surface of the sphere $x^2+y^2+z^2=1$.

(i) By the method of Lagrange multipliers, let

$$\phi=(x-1)^2+(y-2)^2+(z-2)^2-\lambda(x^2+y^2+z^2)$$

For a stationary value,

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=2(x-1)-2\lambda x=0\ \rightarrow\ x(1-\lambda)=1\\ \frac{\partial\phi}{\partial y}&=2(y-2)-2\lambda y=0\ \rightarrow\ y(1-\lambda)=2\\ \frac{\partial\phi}{\partial z}&=2(z-2)-2\lambda z=0\ \rightarrow\ z(1-\lambda)=2\end{aligned}\right\}\ \rightarrow\ y=z=2x$$

Therefore

$$x^2+y^2+z^2=9x^2=1\ \rightarrow\ x=\pm 1/3\,.$$

There are therefore two stationary points:

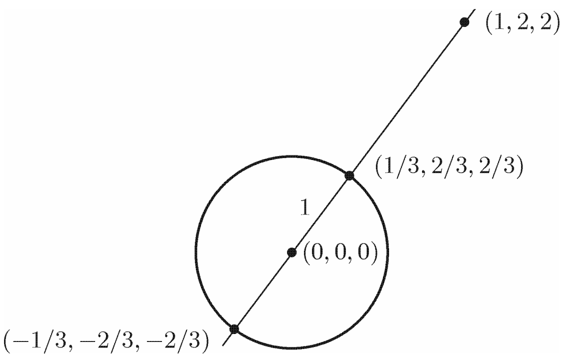
$$(x, y, z)=(1/3,\ 2/3,\ 2/3)\ \rightarrow\ f=(1/3-1)^2+(2/3-2)^2+(2/3-2)^2=4$$
$$(x, y, z)=(-1/3,\ -2/3,\ -2/3)\ \rightarrow\ f=(-1/3-1)^2+(-2/3-2)^2+(-2/3-2)^2=16$$

(ii) Geometrically, the quantity

$$f=(x-1)^2+(y-2)^2+(z-2)^2$$

is the distance of point $(1, 2, 2)$ from a point (x, y, z) on the surface of sphere $x^2+y^2+z^2=1$ with radius $r=1$ and centre at $(0, 0, 0)$. The stationary points are then the maximum and minimum values of this distance. As illustrated in Figure 1, the line defined by $y=z=2x$ passes through the four points $(-1/3,\ -2/3,\ -2/3)$, $(0, 0, 0)$, $(1/3,\ 2/3,\ 2/3)$, and $(1, 2, 2)$.

Figure 1



30. **(i)** Show that the problem of finding the stationary values of the function

$$E(x, y, z) = a(x^2 + y^2 + z^2) + 2b(xy + yz)$$

subject to the constraint $x^2 + y^2 + z^2 = 1$ (a and b are constants) is equivalent to solving the secular equations

$$\begin{aligned} (a-\lambda)x + \quad by \quad\quad\quad &= 0 \\ bx \quad + (a-\lambda)y + \quad bz \quad &= 0 \\ by \quad + (a-\lambda)z &= 0 \end{aligned}$$

These equations have solutions for three possible values of the Lagrangian multiplier:

$$\lambda_1 = a, \quad \lambda_2 = a + \sqrt{2}b, \quad \lambda_3 = a - \sqrt{2}b$$

(ii) Find the stationary point corresponding to each value of λ (assume x is positive). **(iii)** Show that the three stationary values of E are identical to the corresponding values of λ. (This is the Hückel problem for the allyl radical, CH_2CHCH_2; see also Example 17.9).

(i) By the method of Lagrange multipliers, let

$$\begin{aligned} \phi &= E(x, y, z) - \lambda(x^2 + y^2 + z^2) \\ &= (a-\lambda)(x^2 + y^2 + z^2) + 2b(xy + yz) \end{aligned}$$

For stationary values

$$\frac{\partial\phi}{\partial x} = 2(a-\lambda)x + 2by = 0, \quad \frac{\partial\phi}{\partial y} = 2(a-\lambda)y + 2bx + 2bz = 0, \quad \frac{\partial\phi}{\partial z} = 2(a-\lambda)z + 2by = 0$$

and the secular equations are

$$\begin{aligned} (a-\lambda)x + \quad by \quad\quad\quad &= 0 \\ bx \quad + (a-\lambda)y + \quad bz \quad &= 0 \\ by \quad + (a-\lambda)z &= 0 \end{aligned}$$

(ii) For $\lambda = a$,

$$\left.\begin{aligned} 0 + by + 0 &= 0 \\ bx + 0 + bz &= 0 \\ 0 + by + 0 &= 0 \end{aligned}\right\} \rightarrow y = 0, \; z = -x$$

Then, given that $x^2 + y^2 + z^2 = 1$, the corresponding stationary point is

$$(x, y, z) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).$$

For $\lambda = a + \sqrt{2}b$,

$$\left.\begin{aligned} -\sqrt{2}bx + by + 0 &= 0 \\ bx - \sqrt{2}by + bz &= 0 \\ 0 + by - \sqrt{2}bz &= 0 \end{aligned}\right\} \rightarrow y = \sqrt{2}x, \quad z = x$$

and

$$(x, y, z) = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right).$$

Similarly for $\lambda = a - \sqrt{2}b$,

$$(x, y, z) = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right).$$

(iii) We have $E = a(x^2 + y^2 + z^2) + 2b(xy + yz) = a + 2b(xy + yz)$

For $\lambda = a$, $E = a + 2b(0+0) = a$

For $\lambda = a \pm \sqrt{2}b$, $E = a + 2b\left(\pm\frac{1}{2\sqrt{2}} \pm \frac{1}{2\sqrt{2}}\right) = a \pm \sqrt{2}b$

Section 9.5

Find the total differential df :

31. $f(x,y) = x^2 + y^2$: $\frac{\partial f}{\partial x} = 2x, \ \frac{\partial f}{\partial y} = 2y \ \to \ df = 2x\,dx + 2y\,dy$

32. $f(x,y) = 3x^2 + \sin(x-y)$: $\frac{\partial f}{\partial x} = 6x + \cos(x-y), \ \frac{\partial f}{\partial y} = -\cos(x-y)$

$$\to \ df = \left[6x + \cos(x-y)\right]dx - \cos(x-y)\,dy$$

33. $f(x,y) = x^3y^2 + \ln y$: $\frac{\partial f}{\partial x} = 3x^2y^2, \ \frac{\partial f}{\partial y} = 2x^3y + \frac{1}{y}$

$$\to \ df = 3x^2y^2\,dx + \left[2x^3y + \frac{1}{y}\right]dy$$

34. $f(x, y, z) = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$

Let $u = x^2 + y^2 + z^2, \ f = \dfrac{1}{u^{1/2}}$.

Then, for example, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} = \dfrac{-x}{u^{3/2}} = -\dfrac{x}{(x^2 + y^2 + z^2)^{1/2}}$

and $df = -\dfrac{1}{(x^2 + y^2 + z^2)^{3/2}}\left[x\,dx + y\,dy + z\,dz\right]$

35. $f(r, \theta, \phi) = r\sin\theta\sin\phi$**:** $df = \sin\theta\sin\phi\ dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi$

36. Write down the total differential of the volume of a two-component system in terms of changes in temperature *T*, pressure *p*, and amounts n_{A} and n_{B} of the components A and B. Use the full notation with subscripts for constant variables.

$$dV = \left(\frac{\partial V}{\partial T}\right)_{p,n_{\mathrm{A}},n_{\mathrm{B}}} dT + \left(\frac{\partial V}{\partial p}\right)_{T,n_{\mathrm{A}},n_{\mathrm{B}}} dp + \left(\frac{\partial V}{\partial n_{\mathrm{A}}}\right)_{T,p,n_{\mathrm{B}}} dn_{\mathrm{A}} + \left(\frac{\partial V}{\partial n_{\mathrm{B}}}\right)_{T,p,n_{\mathrm{A}}} dn_{\mathrm{B}}$$

Section 9.6

37. Given $z = x^2 + 2xy + 3y^2$, where $x = (1+t)^{1/2}$ and $y = (1-t)^{1/2}$, find $\dfrac{dz}{dt}$ by **(i)** substitution, **(ii)** the chain rule (9.21).

(i) $$z = x^2 + 2xy + 3y^2 \;\to\; z = (1+t) + 2(1+t)^{1/2}(1-t)^{1/2} + 3(1-t)$$
$$= 4 - 2t + 2(1-t^2)^{1/2}$$

Then $$\frac{dz}{dt} = -2 - \frac{2t}{(1-t^2)^{1/2}}$$
$$= -2 - \frac{(y^2 - x^2)}{xy} = -2 + \frac{y}{x} - \frac{x}{y}$$

(ii) $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\times\frac{dx}{dt} + \frac{\partial z}{\partial y}\times\frac{dy}{dt} = (2x+2y)\times\left(\frac{1}{2x}\right) + (2x+6y)\times\left(-\frac{1}{2y}\right)$$
$$= -2 + \frac{y}{x} - \frac{x}{y} \quad \text{as in (i)}$$

38. Given $u = e^{x-y}$, where $x = 2\cos t$ and $y = 3t$, use the chain rule to find $\dfrac{du}{dt}$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\times\frac{dx}{dt} + \frac{\partial u}{\partial y}\times\frac{dy}{dt} = \left(e^{x-y}\right)\times(-2\sin t) + \left(-e^{x-y}\right)\times(3)$$
$$= -(2\sin t + 3)e^{2\cos t - 3t}$$

39. Given $u = \ln(x+y+z)$, where $x = a\cos t$, $y = b\sin t$, and $z = ct$, find $\dfrac{du}{dt}$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\times\frac{dx}{dt} + \frac{\partial u}{\partial y}\times\frac{dy}{dt} + \frac{\partial u}{\partial z}\times\frac{dz}{dt} = \frac{1}{x+y+z}(-a\sin t + b\cos t + c)$$

$$= \frac{-a\sin t + b\cos t + c}{a\cos t + b\sin t + ct}$$

40. Given $z = \ln(2x+3y),\ x = a\cos\theta,\ y = a\sin\theta$, use the chain rule to find $\dfrac{dz}{d\theta}$.

$$\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\times\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\times\frac{dy}{d\theta} = \frac{2}{2x+3y}\times(-a\sin\theta) + \frac{3}{2x+3y}\times(a\cos\theta)$$

$$= \frac{3\cos\theta - 2\sin\theta}{2\cos\theta + 3\sin\theta}$$

41. Given $f = \sin(u+v)$, where $v = \cos u$, **(i)** find $\dfrac{df}{du}$, **(ii)** if $u = e^{-t}$, find $\dfrac{df}{dt}$.

(i)

$$\frac{df}{du} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\times\frac{dv}{du} = \cos(u+v) + \cos(u+v)\times(-\sin u)$$

$$= \left[1-\sin u\right]\cos(u+v) = \left[1-\sin u\right]\cos(u+\cos u)$$

(ii) We have $\dfrac{du}{dx} = -e^{-t} = -u$

Therefore $\dfrac{df}{dt} = \dfrac{df}{du}\dfrac{du}{dt} = -u\left[1-\sin u\right]\cos(u+\cos u)$

42. If $z = x^5 y - \sin y$, find $\left(\dfrac{\partial y}{\partial x}\right)_z$.

We have $\left(\dfrac{\partial z}{\partial x}\right)_y = 5x^4 y,\ \left(\dfrac{\partial z}{\partial y}\right)_x = x^5 - \cos y$

Therefore $\left(\dfrac{\partial y}{\partial x}\right)_z = -\left(\dfrac{\partial z}{\partial x}\right)_y \Big/ \left(\dfrac{\partial z}{\partial y}\right)_x = \dfrac{-5x^4 y}{x^5 - \cos y}$

43. For the van der Waals gas, use the expressions for $\left(\frac{\partial V}{\partial T}\right)_{p,n}$ and $\left(\frac{\partial V}{\partial p}\right)_{T,n}$ from Exercise 20 to find $\left(\frac{\partial p}{\partial T}\right)_{V,n}$

We have $\left(\frac{\partial V}{\partial T}\right)_{p,n} = nR\bigg/\left(p-\frac{n^2a}{V^2}+\frac{2n^3ab}{V^3}\right)$, $\left(\frac{\partial V}{\partial p}\right)_{T,n} = -(V-nb)\bigg/\left(p-\frac{n^2a}{V^2}+\frac{2n^3ab}{V^3}\right)$

Therefore $\left(\frac{\partial p}{\partial T}\right)_{V,n} = -\left(\frac{\partial V}{\partial T}\right)_{p,n}\bigg/\left(\frac{\partial V}{\partial p}\right)_{T,n}$

$$= -\left[nR\bigg/\left(p-\frac{n^2a}{V^2}+\frac{2n^3ab}{V^3}\right)\right]\bigg/\left[-(V-nb)\bigg/\left(p-\frac{n^2a}{V^2}+\frac{2n^3ab}{V^3}\right)\right]$$

$$= \frac{nR}{V-nb}$$

44. If $z = x^2 + y^2$ and $u = xy$, find $\left(\frac{\partial z}{\partial y}\right)_u$ by **(i)** substitution, **(ii)** equation (9.30).

(i) We have $x = \frac{u}{y} \rightarrow z = \frac{u^2}{y^2} + y^2$

Therefore $\left(\frac{\partial z}{\partial y}\right)_u = -\frac{2u^2}{y^3} + 2y = -\frac{2x^2}{y} + 2y = \frac{2(y^2-x^2)}{y}$

(ii) We have $\left(\frac{\partial z}{\partial x}\right)_y = 2x$, $\left(\frac{\partial z}{\partial y}\right)_x = 2y$, $\left(\frac{\partial x}{\partial y}\right)_u = -\frac{u}{y^2} = -\frac{x}{y}$

Therefore $\left(\frac{\partial z}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x + \left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_u$

$$= 2y + 2x\times\left(-\frac{x}{y}\right) = \frac{2(y^2-x^2)}{y}$$

45. Given $z = x\sin y$ and $u = x^2 + 2xy + 3y^2$, find $\left(\dfrac{\partial z}{\partial y}\right)_u$.

We have $\left(\dfrac{\partial z}{\partial x}\right)_y = \sin y$, $\left(\dfrac{\partial z}{\partial y}\right)_x = x\cos y$, $\left(\dfrac{\partial x}{\partial y}\right)_u = -\left(\dfrac{\partial u}{\partial y}\right)_x \bigg/ \left(\dfrac{\partial u}{\partial x}\right)_y = -\dfrac{2x+6y}{2x+2y}$

Therefore
$$\left(\frac{\partial z}{\partial y}\right)_u = \left(\frac{\partial z}{\partial y}\right)_x + \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_u$$
$$= x\cos y - \left(\frac{x+3y}{x+y}\right)\sin y$$

46. If $U = U(V,T)$ and $p = p(V,T)$ are functions of V and T and if $H = U + pV$, show that

$$\left(\frac{\partial H}{\partial T}\right)_p - \left(\frac{\partial U}{\partial T}\right)_V = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p .$$

We have
$$\left(\frac{\partial H}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p$$

and, by equation (9.30),
$$\left(\frac{\partial U}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_V + \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p$$

Therefore
$$\left(\frac{\partial H}{\partial T}\right)_p = \left[\left(\frac{\partial U}{\partial T}\right)_V + \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p\right] + p\left(\frac{\partial V}{\partial T}\right)_p$$
$$\left(\frac{\partial H}{\partial T}\right)_p - \left(\frac{\partial U}{\partial T}\right)_V = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p$$

47. Given $x = au + bv$ and $y = bu - av$, where a and b are constants, **(i)** if f is a function of x and y, express $\left(\frac{\partial f}{\partial u}\right)_v$ and $\left(\frac{\partial f}{\partial v}\right)_u$ in terms of $\left(\frac{\partial f}{\partial x}\right)_y$ and $\left(\frac{\partial f}{\partial y}\right)_x$, **(ii)** if $f = x^2 + y^2$, find $\left(\frac{\partial f}{\partial u}\right)_v$ and $\left(\frac{\partial f}{\partial v}\right)_u$ in terms of u and v.

(i) We have $$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$

Then $$\left(\frac{\partial f}{\partial u}\right)_v = a\left(\frac{\partial f}{\partial x}\right)_y + b\left(\frac{\partial f}{\partial y}\right)_x$$

Similarly, $$\left(\frac{\partial f}{\partial v}\right)_u = b\left(\frac{\partial f}{\partial x}\right)_y - a\left(\frac{\partial f}{\partial y}\right)_x$$

(ii) If $f = x^2 + y^2$ then $\left(\frac{\partial f}{\partial x}\right)_y = 2x$, $\left(\frac{\partial f}{\partial y}\right)_x = 2y$

Therefore $$\left(\frac{\partial f}{\partial u}\right)_v = 2ax + 2by = 2a(au + bv) + 2b(bu - av)$$
$$= 2(a^2 + b^2)u$$

Similarly, $$\left(\frac{\partial f}{\partial v}\right)_u = 2(a^2 + b^2)v$$

48. Given $u = x^n + y^n$ and $v = x^n - y^n$, where *n* is a constant,

(i) show that $\left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y = \frac{1}{2} = \left(\frac{\partial y}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$.

(ii) If *f* is a function of *x* and *y*, express $\left(\frac{\partial f}{\partial x}\right)_y$ and $\left(\frac{\partial f}{\partial y}\right)_x$ in terms of $\left(\frac{\partial f}{\partial u}\right)_v$ and $\left(\frac{\partial f}{\partial v}\right)_u$.

Hence, **(iii)** if $f = u^2 - v^2$, find $\left(\frac{\partial f}{\partial x}\right)_y$ and $\left(\frac{\partial f}{\partial y}\right)_x$ in terms of *x* and *y*.

(i) We have $\left(\frac{\partial u}{\partial x}\right)_y = nx^{n-1}, \quad \left(\frac{\partial v}{\partial y}\right)_x = -ny^{n-1}$

Also

$$x^n = \frac{1}{2}(u+v) \;\rightarrow\; \left(\frac{\partial}{\partial u}\right)_v x^n = nx^{n-1}\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2}$$

$$y^n = \frac{1}{2}(u-v) \;\rightarrow\; \left(\frac{\partial}{\partial v}\right)_u y^n = ny^{n-1}\left(\frac{\partial y}{\partial v}\right)_u = -\frac{1}{2}$$

Therefore

$$\left.\begin{array}{l} \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} \\ \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \frac{1}{2} \end{array}\right\} \rightarrow \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y = \frac{1}{2} = \left(\frac{\partial y}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

(ii)

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial f}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y$$

$$= nx^{n-1}\left[\left(\frac{\partial f}{\partial u}\right)_v + \left(\frac{\partial f}{\partial v}\right)_u\right]$$

Similarly,

$$\left(\frac{\partial f}{\partial y}\right)_x = ny^{n-1}\left[\left(\frac{\partial f}{\partial u}\right)_v - \left(\frac{\partial f}{\partial v}\right)_u\right]$$

(iii) We have $f = u^2 - v^2 \;\rightarrow\; \frac{\partial f}{\partial u} = 2u, \quad \frac{\partial f}{\partial v} = -2v$

Therefore

$$\left(\frac{\partial f}{\partial x}\right)_y = nx^{n-1}\left[\left(\frac{\partial f}{\partial u}\right)_v + \left(\frac{\partial f}{\partial v}\right)_u\right] = nx^{n-1}\left[2u - 2v\right] = 4nx^{n-1}y^n$$

$$\left(\frac{\partial f}{\partial y}\right)_x = ny^{n-1}\left[\left(\frac{\partial f}{\partial u}\right)_v - \left(\frac{\partial f}{\partial v}\right)_u\right] = ny^{n-1}\left[2u + 2v\right] = 4nx^n y^{n-1}$$

49. If $x = au + bv$, $y = bu - av$, and f is a function of x and y (see Exercise 45(i)), show that

(i) $\dfrac{\partial^2 f}{\partial u^2} + \dfrac{\partial^2 f}{\partial v^2} = (a^2 + b^2)\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right)$, **(ii)** $\dfrac{\partial^2 f}{\partial u \partial v} = ab\left(\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2}\right) + (b^2 - a^2)\dfrac{\partial^2 f}{\partial x \partial y}$.

From Exercise 47,

$$\left(\frac{\partial f}{\partial u}\right)_v = a\left(\frac{\partial f}{\partial x}\right)_y + b\left(\frac{\partial f}{\partial y}\right)_x, \quad \left(\frac{\partial f}{\partial v}\right)_u = b\left(\frac{\partial f}{\partial x}\right)_y - a\left(\frac{\partial f}{\partial y}\right)_x$$

(i) We have $\dfrac{\partial^2 f}{\partial u^2} = \dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial u}\right) = \dfrac{\partial}{\partial u}\left(a\dfrac{\partial f}{\partial x} + b\dfrac{\partial f}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\right) \times \frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\right) \times \frac{\partial y}{\partial u}$$

$$= a^2\frac{\partial^2 f}{\partial x^2} + ba\frac{\partial^2 f}{\partial x \partial y} + ab\frac{\partial^2 f}{\partial y \partial x} + b^2\frac{\partial^2 f}{\partial y^2}$$

Similarly, $\dfrac{\partial^2 f}{\partial v^2} = b^2\dfrac{\partial^2 f}{\partial x^2} - ab\dfrac{\partial^2 f}{\partial x \partial y} - ba\dfrac{\partial^2 f}{\partial y \partial x} + a^2\dfrac{\partial^2 f}{\partial y^2}$

Therefore $\dfrac{\partial^2 f}{\partial u^2} + \dfrac{\partial^2 f}{\partial v^2} = (a^2 + b^2)\left(\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}\right)$

(ii) We have $\dfrac{\partial^2 f}{\partial u \partial v} = \dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial v}\right) = \dfrac{\partial}{\partial u}\left(b\left(\dfrac{\partial f}{\partial x}\right)_y - a\left(\dfrac{\partial f}{\partial y}\right)_x\right)$

$$= \frac{\partial}{\partial x}\left(b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y}\right) \times \frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y}\right) \times \frac{\partial y}{\partial u}$$

$$= ba\frac{\partial^2 f}{\partial x^2} - a^2\frac{\partial^2 f}{\partial x \partial y} + b^2\frac{\partial^2 f}{\partial y \partial x} - ab\frac{\partial^2 f}{\partial y^2}$$

Therefore $\dfrac{\partial^2 f}{\partial u \partial v} = ab\left(\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2}\right) + (b^2 - a^2)\dfrac{\partial^2 f}{\partial y \partial x}$

Show that the following functions of position in a plane satisfy Laplace's equation:

50. $f(x, y) = x^5 - 10x^3y^2 + 5xy^4$

$$\frac{\partial f}{\partial x} = 5x^4 - 30x^2y^2 + 5y^4, \quad \frac{\partial^2 f}{\partial x^2} = 20x^3 - 60xy^2$$

$$\frac{\partial f}{\partial y} = -20x^3y + 20xy^3, \quad \frac{\partial^2 f}{\partial y^2} = -20x^3 + 60xy^2$$

Therefore $\quad \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$

51. $f(r, \theta) = \left(Ar + \dfrac{B}{r}\right)\sin\theta$

We have $\quad \nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$

$$\frac{\partial f}{\partial r} = \left(A - \frac{B}{r^2}\right)\sin\theta, \quad \frac{\partial^2 f}{\partial r^2} = \frac{2B}{r^3}\sin\theta, \quad \frac{\partial^2 f}{\partial \theta^2} = -\left(Ar + \frac{B}{r}\right)\sin\theta$$

Therefore
$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$
$$= \left(\frac{2B}{r^3}\sin\theta\right) + \frac{1}{r}\left(A - \frac{B}{r^2}\right)\sin\theta - \frac{1}{r^2}\left(Ar + \frac{B}{r}\right)\sin\theta = 0$$

52. $f(r, \theta) = r^n \cos n\theta, \ n = 1, 2, 3, \ldots$

$$\frac{\partial f}{\partial r} = nr^{n-1}\sin n\theta, \quad \frac{\partial^2 f}{\partial r^2} = n(n-1)r^{n-2}\sin n\theta,$$

$$\frac{\partial f}{\partial \theta} = -nr^n \cos\theta, \quad \frac{\partial^2 f}{\partial \theta^2} = -n^2r^n \sin\theta$$

Therefore
$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$
$$= n(n-1)r^{n-2}\sin\theta + nr^{n-2}\sin\theta - n^2r^{n-2}\sin\theta = 0$$

Section 9.7

Test for exactness:

53. $F\,dx + G\,dy = (4x+3y)\,dx + (3x+8y)\,dy$

We have $\quad \dfrac{\partial F}{\partial y} = 3, \ \dfrac{\partial G}{\partial x} = 3 \ \rightarrow \ (4x+3y)\,dx + (3x+8y)\,dy$ is exact

54. $F\,dx + G\,dy = (6x+5y+7)\,dx + (4x+10y+8)\,dy$

$$\frac{\partial F}{\partial y} = 5, \ \frac{\partial G}{\partial x} = 4 \ \rightarrow \ (6x+5y+7)\,dx + (4x+10y+8)\,dy \text{ is not exact}$$

55. $F\,dx + G\,dy = y\cos x\,dx + \sin x\,dy$

$$\frac{\partial F}{\partial y} = \cos x, \ \frac{\partial G}{\partial x} = \cos x \ \rightarrow \ y\cos x\,dx + \sin x\,dy \text{ is exact}$$

56. Given the total differential $dG = -S\,dT + V\,dp$, show that $\left(\dfrac{\partial S}{\partial p}\right)_T = -\left(\dfrac{\partial V}{\partial T}\right)_p$.

The total differential of $G = G(T, p)$ is

$$dG = \left(\frac{\partial G}{\partial T}\right)_p dT + \left(\frac{\partial G}{\partial p}\right)_T dp$$

Therefore $\quad -S = \left(\dfrac{\partial G}{\partial T}\right)_p, \ V = \left(\dfrac{\partial G}{\partial p}\right)_T$

$$-\left(\frac{\partial S}{\partial p}\right)_T = \left(\frac{\partial}{\partial p}\right)_T \left(\frac{\partial G}{\partial T}\right)_p = \frac{\partial^2 G}{\partial p \partial T},$$

$$\left(\frac{\partial V}{\partial T}\right)_p = \left(\frac{\partial}{\partial T}\right)_p \left(\frac{\partial G}{\partial p}\right)_T = \frac{\partial^2 G}{\partial T \partial p} = \frac{\partial^2 G}{\partial p \partial T}$$

and $\quad \left(\dfrac{\partial S}{\partial p}\right)_T = -\left(\dfrac{\partial V}{\partial T}\right)_p$

Section 9.8

57. Evaluate the line integral $\int_C \left[xy\,dx + 2y\,dy\right]$ on the line $y = 2x$ from $x = 0$ to $x = 2$.

$$\int_C \left[xy\,dx + 2y\,dy\right] = \int_0^2 \left[2x^2 + 8x\right]dx = \left[\frac{2}{3}x^3 + 4x^2\right]_0^2 = \frac{64}{3}$$

58. When the path of integration is given in parametric form $x = x(t)$, $y = y(t)$ from $t = t_A$ to $t = t_B$, the line integral can be evaluated as

$$\int_C \left[F\,dx + G\,dy\right] = \int_{t_A}^{t_B} \left[F\frac{dx}{dt} + G\frac{dy}{dt}\right]dt\,.$$

Evaluate $\int_C \left[(x^2 + 2y)dx + (y^2 + x)dy\right]$ on the curve with parametric equations $x = t$, $y = t^2$ from $A(0, 0)$ to $B(1, 1)$.

$$\int_C \left[(x^2 + 2y)dx + (y^2 + x)dy\right] = \int_0^1 \left[(t^2 + 2t^2) + (t^4 + t)\times 2t\right]dt$$

$$= \int_0^1 \left[2t^5 + 5t^2\right]dt = \left[\frac{t^6}{3} + \frac{5t^3}{3}\right]_0^1 = 2$$

59. Evaluate $\int_C \left[xy\,dx + 2y\,dy\right]$ on the curve $y = x^2$ from $x = 0$ to $x = 2$ (see Exercise 57).

$$\int_C \left[xy\,dx + 2y\,dy\right] = \int_0^2 \left[x^3 + 4x^3\right]dx = \left[\frac{5x^4}{4}\right]_0^2 = 20$$

60. Evaluate the line integral $\int_C \left[(x^2 + 2y)dx + (y^2 + x)dy\right]$ on the curve with parametric equations $x = t^2$, $y = t$ from $A(0, 0)$ to $B(1, 1)$ (see Exercise 58).

$$\int_C \left[(x^2 + 2y)dx + (y^2 + x)dy\right] = \int_0^1 \left[(t^4 + 2t)\times 2t + (t^2 + t^2)\right]dt$$

$$= \int_0^1 \left[2t^5 + 6t^2\right]dt = \left[\frac{t^6}{3} + 2t^3\right]_0^1 = \frac{7}{3}$$

61. The total differential of entropy as a function of T and p is (Example 9.27)

$$dS = \left(\frac{\partial S}{\partial T}\right)_p dT + \left(\frac{\partial S}{\partial p}\right)_T dp$$

Given that, $\left(\frac{\partial S}{\partial T}\right)_p = \frac{C_p}{T} = \frac{5R}{2T}$ and $\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p = -\frac{R}{p}$ for 1 mole of ideal gas, show that the (reversible) heat absorbed by the ideal gas round the closed path shown in Figure 2 is equal to the work done by the gas; that is, $\oint T\,dS = \oint p\,dV$ (see Example 9.25).

We have

$$dS = \left(\frac{\partial S}{\partial T}\right)_p dT + \left(\frac{\partial S}{\partial p}\right)_T dp$$

$$= \frac{5R}{2T}dT - \frac{R}{p}dp$$

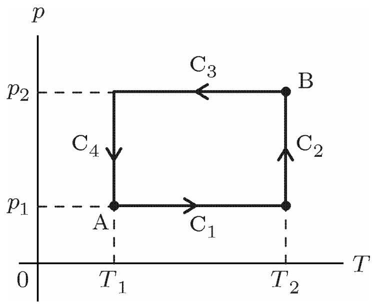


Figure 2

The line integral round the closed path is the sum of the lines integrals along the four sides:

$$\oint T\,dS = \int_{C_1} T\,dS + \int_{C_2} T\,dS + \int_{C_3} T\,dS + \int_{C_4} T\,dS$$

Then C_1 (constant $p = p_1$): $\int_{C_1} T\,dS = \int_{T_1}^{T_2}\left(\frac{5R}{2}\right)dT = \frac{5R}{2}(T_2 - T_1)$

C_2 (constant $T = T_2$): $\int_{C_2} T\,dS = \int_{p_1}^{p_2}\left(-\frac{RT_2}{p}\right)dT = -RT_2 \ln\frac{p_2}{p_1}$

C_3 (constant $p = p_2$): $\int_{C_3} T\,dS = \int_{T_2}^{T_1}\left(\frac{5R}{2}\right)dT = \frac{5R}{2}(T_1 - T_2)$

C_4 (constant $T = T_1$): $\int_{C_2} T\,dS = \int_{p_2}^{p_1}\left(-\frac{RT_1}{p}\right)dp = -RT_1 \ln\frac{p_1}{p_2}$

Therefore

$$\oint T\,dS = \frac{5R}{2}(T_2 - T_1) - RT_2 \ln\frac{p_2}{p_1} + \frac{5R}{2}(T_1 - T_2) - RT_1 \ln\frac{p_1}{p_2} = -R(T_2 - T_1)\ln\frac{p_2}{p_1}$$

and this is equal to the work $\oint p\,dV$ done by (one mole of) the gas round the closed path, as demonstrated in Example 9.25.

62. (i) Show that $F\,dx + G\,dy$ for $F = 9x^2 + 4y^2 + 4xy$ and $G = 8xy + 2x^2 + 3y^2$ is an exact differential. **(ii)** By choosing an appropriate path, evaluate $\int_C [F\,dx + G\,dy]$ from $(x, y) = (0,0)$ to $(1, 2)$.

(iii) Show that the result in (ii) is consistent with the differential as the total differential of

$$z(x, y) = 3x^3 + 4xy^2 + 2x^2y + y^3 .$$

(i) $\left(\frac{\partial F}{\partial y}\right)_x = 8y + 4x = \left(\frac{\partial G}{\partial x}\right)_y$

(ii) For the path shown in Figure 3,

$$\int_C [F\,dx + G\,dy] = \int_{C_1} F\,dx + \int_{C_2} G\,dy$$

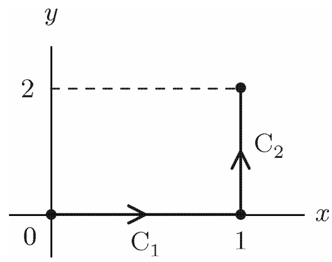


Figure 3

where

$$C_1: \quad \int_{C_1} (F)_{y=0}\,dx = \int_0^1 9x^2\,dx = 3$$

$$C_2: \quad \int_{C_2} (G)_{x=1}\,dx = \int_0^2 (8y + 2 + 3y^2)\,dy = \left[4y^2 + 2y + y^3\right]_0^2 = 28$$

Therefore

$$\int_C [F\,dx + G\,dy] = 31$$

(iii) If $\quad z(x, y) = 3x^3 + 4xy^2 + 2x^2y + y^3$

then $\quad \left(\frac{\partial z}{\partial x}\right)_y = 9x^2 + 4y^2 + 4xy = F$

$$\left(\frac{\partial z}{\partial y}\right)_x = 8xy + 2x^2 + 3y^2 = G$$

and $\quad dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy = F\,dx + G\,dy$

63. Evaluate $\int_C \left[2xy\,dx + (x^2 - y^2)\,dy\right]$ on the circle with parametric equations $x = \cos\theta$, $y = \sin\theta$,

(i) from A(1, 0) to B(0, 1) and

(ii) around a complete circle ($\theta = 0 \to 2\pi$).

(iii) Confirm that the differential $2xy\,dx + (x^2 - y^2)\,dy$ is exact.

We have $x = \cos\theta,\quad dx = -\sin\theta\,d\theta$

$y = \sin\theta,\quad dy = \cos\theta\,d\theta$

Then

$$\int_C \left[2xy\,dx + (x^2 - y^2)\,dy\right] = \int \left[(2\cos\theta\sin\theta \times (-\sin\theta) + (\cos^2\theta - \sin^2\theta) \times \cos\theta\right] d\theta$$

$$= \int \left[\cos 2\theta \cos\theta - \sin 2\theta \sin\theta\right] d\theta$$

$$= \int \cos 3\theta\,d\theta$$

The paths of integration are anticlockwise, as illustrated in Figure 4.

(i) $\theta = 0 \to \pi/2$: $\displaystyle\int_0^{\pi/2} \cos 3\theta\,d\theta = \left[\frac{1}{3}\sin 3\theta\right]_0^{\pi/2} = -\frac{1}{3}$

(ii) $\theta = 0 \to 2\pi$ $\displaystyle\int_0^{2\pi} \cos 3\theta\,d\theta = \left[\frac{1}{3}\sin 3\theta\right]_0^{2\pi} = 0$

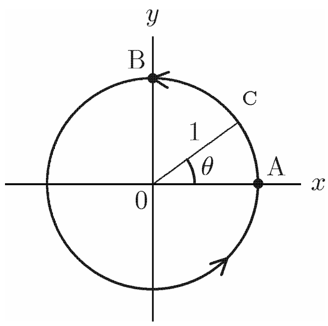


Figure 4

(iii) The zero result of integration round the closed loop in (ii) suggests that the differential is exact. Thus,

if $2xy\,dx + (x^2 - y^2)\,dy = F\,dx + G\,dy$

then $\dfrac{\partial F}{\partial y} = 2x = \dfrac{\partial G}{\partial x}$

Section 9.9

64. Evaluate the integral $\int_0^3 \int_1^2 (x^2 y + xy^2)\, dxdy$ and show that the result is independent of the order of integration.

$$\int_0^3 \int_1^2 (x^2 y + xy^2)\, dxdy = \int_0^3 \left[\int_1^2 (x^2 y + xy^2)\, dx \right] dy = \int_0^3 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2} \right]_1^2 dy$$

$$= \int_0^3 \left[\frac{7}{3} y + \frac{3}{2} y^2 \right] dy = \left[\frac{7}{6} y^2 + \frac{1}{2} y^3 \right]_0^3 = 24$$

Interchanging the order of integration,

$$\int_1^2 \int_0^3 (x^2 y + xy^2)\, dy\, dx = \int_1^2 \left[\int_0^3 (x^2 y + xy^2)\, dy \right] dx = \int_1^2 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_0^3 dx$$

$$= \int_1^2 \left[\frac{9}{2} x^2 + 9x \right] dx = \left[\frac{3}{2} x^3 + \frac{9}{2} x^2 \right]_1^2 = 24$$

and the integration is independent of order.

65. Evaluate the integral $\int_0^\pi \int_0^R e^{-r} \cos^2\theta \sin\theta \, dr\, d\theta$

$$\int_0^\pi \int_0^R e^{-r} \cos^2\theta \sin\theta \, dr\, d\theta = \int_0^\pi \cos^2\theta \sin\theta \, d\theta \times \int_0^R e^{-r} dr$$

$$= \left[-\frac{\cos^3\theta}{3} \right]_0^\pi \times \left[-e^{-r} \right]_0^R$$

$$= \frac{2}{3}(1 - e^{-R})$$

Section 9.10

Evaluate the integral and sketch the region of integration:

66. $$\int_0^2\int_x^{2x}(x^2+y^2)\,dy\,dx=\int_0^2\left[\int_x^{2x}(x^2+y^2)\,dy\right]dx$$

$$=\int_0^2\left[x^2y+\frac{y^3}{3}\right]_x^{2x}dx$$

$$=\int_0^2\frac{10x^3}{3}dx=\frac{40}{3}$$

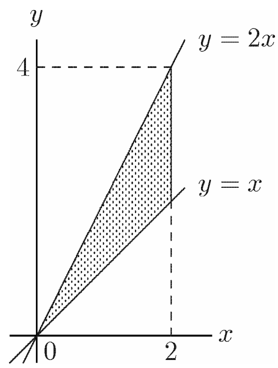


Figure 5

The region of integration, shown in Figure 5, lies between $x=0$ and $x=2$ and, from the limits of integration for *y*, between the lines $y=x$ and $y=2x$.

67. $$\int_0^a\int_0^{\sqrt{a^2-x^2}}xy^2\,dy\,dx=\int_0^a x\left[\int_0^{\sqrt{a^2-x^2}}y^2\,dy\right]dx$$

$$=\int_0^a x\left[\frac{1}{3}(a^2-x^2)^{3/2}\right]dx$$

Let $u=a^2-x^2$, $du=-2x\,dx$.

Then $u=a^2$ when $x=0$, and $u=0$ when $x=a$.

Therefore $$\int_0^a\int_0^{\sqrt{a^2-x^2}}xy^2\,dy\,dx=-\frac{1}{6}\int_a^0 u^{3/2}\,du=\frac{a^5}{15}$$

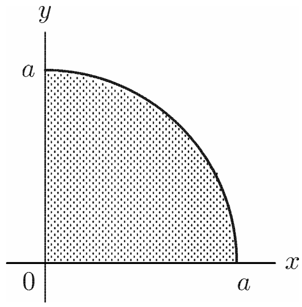


Figure 6

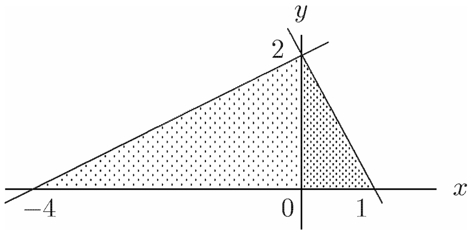
The region of integration, shown in Figure 6, lies between $x=0$ and $x=a$, and between $y=0$ and $y=\sqrt{a^2-x^2}$. Comparison with equation (9.59) shows that the integration is over that quarter of the circle of radius *a* that lies in the first quadrant.

68. **(i)** Show from a sketch of the region of integration that

$$\int_0^2\int_{2y-4}^{(2-y)/2} x^3\,dx\,dy = \int_0^1\int_0^{2-2x} x^3\,dy\,dx + \int_{-4}^0\int_0^{(x+4)/2} x^3\,dy\,dx\,,$$

(ii) evaluate the integral.

(i) **Figure 7**



The region of integration lies between $y=0$ and $y=2$ and, from the limits of integration for *x*, between the lines $x = 2y-4$ and $x=(2-y)/2$. Interchange of the order of integration gives Figure 7.

(ii) $$\int_0^1\int_0^{2-2x} x^3\,dy\,dx = \int_0^1 x^3\left[\int_0^{2-2x} dy\right]dx = \int_0^1 x^3(2-2x)\,dx = \frac{1}{10}$$

$$\int_{-4}^0\int_0^{(x+4)/2} x^3\,dy\,dx = \int_{-4}^0 x^3\left[\int_0^{(x+4)/2} dy\right]dx = \frac{1}{2}\int_{-4}^0 x^3(x+4)\,dx = -\frac{256}{10}$$

Therefore $$\int_0^2\int_{2y-4}^{(2-y)/2} x^3\,dx\,dy = -\frac{51}{2}$$

Section 9.11

Transform to polar coordinates and evaluate:

69. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+2xy)\,dy\,dx$

Integration is that illustrated in Figure 6, with $a=1$.

Then $$\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+2xy)\,dy\,dx = \int_{r=0}^1\int_{\theta=0}^{\pi/2}\left[r^2\cos^2\theta + 2r^2\cos\theta\sin\theta\right]r\,dr\,d\theta$$

$$= \int_0^1 r^3\,dr \times \int_0^{\pi/2}\left[\cos^2\theta + 2\cos\theta\sin\theta\right]d\theta$$

$$= \int_0^1 r^3\,dr \times \int_0^{\pi/2}\left[\frac{1}{2}(1+\cos 2\theta) + \sin 2\theta\right]d\theta = \frac{1}{4}\left(\frac{\pi}{4}+1\right)$$

70. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2(x^2+y^2)^{1/2}}\,(x^2+y^2)^3\,dxdy$

The integration is over the whole plane. Therefore

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2(x^2+y^2)^{1/2}}\,(x^2+y^2)^3\,dxdy = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi} e^{-2r}r^6\times r\,dr\,d\theta$$

$$= \int_{r=0}^{\infty} e^{-2r}r^7\,dr\int_{\theta=0}^{\pi} d\theta = \frac{7!}{2^8}\times 2\pi = \frac{315\pi}{8}$$

71. $\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)}x^2\,dx\,dy$

The integration is over one quarter of the *xy*–plane. Therefore

$$\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)}x^2\,dx\,dy = \frac{1}{4}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2}r^2\cos^2\theta\times r\,dr\,d\theta$$

$$= \frac{1}{4}\int_{0}^{\infty} e^{-r^2}r^3\,dr\times\int_{0}^{2\pi}\cos^2\theta\,d\theta = \frac{1}{4}\times\frac{1}{2}\times\pi = \frac{\pi}{8}$$