The Chemistry Maths Book

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Solutions

Chapter 8. Complex numbers,

8.1 Concepts
8.2 Algebra of complex numbers
8.3 Graphical representation
8.4 Complex functions
8.5 Euler's formula
8.6 Periodicity
8.7 Evaluation of integrals

Section 8.2

Express as a single complex number:

1. $(2+3i)+(4-5i)=(2+4)+(3-5)i=6-2i$

2. $(2+3i)+(2-3i)=(2+2)+(3-3)i=4$

3. $(2+3i)-(2-3i)=(2-2)+(3+3)i=6i$

4. $$\begin{aligned}(5+3i)(3-i)&=15-5i+9i-3i^2\\&=15-5i+9i+3\\&=18+4i\end{aligned}$$

5. $(1-3i)^2=1-6i+(3i)^2=-8-6i$

6. $$\begin{aligned}(1+2i)^5&=1+5(2i)+10(2i)^2+10(2i)^3+5(2i)^5+(2i)^5\\&=1+10i-40-80i+80+32i\\&=41-38i\end{aligned}$$

7. $(1-3i)(1+3i)=1-(3i)^2=1+9=10$

8. If $z=3-2i$, find

(i) z^* and **(ii)** zz^*. **(iii)** Express the real and imaginary parts of z in terms of z and z^*.

$z=3-2i$ **(i)** $z^*=3+2i$

(ii) $zz^*=(3-2i)(3+2i)=3^2+2^2=13$

(iii) $\text{Re}(z)=\dfrac{1}{2}\left(z+z^*\right)=3,\quad \text{Im}(z)=\dfrac{1}{2i}\left(z-z^*\right)=-2$

9. Find z such that $zz^*+4(z-z^*)=5+16i$.

Let $z=x+iy$

Then $$\begin{aligned}zz^*+4(z-z^*)&=x^2+y^2+4(x+iy-x+iy)\\&=x^2+y^2+8yi\\&=5+16i\ \text{ when }\ x^2+y^2=5\ \text{ and }\ 8y=16\end{aligned}$$

Therefore $x=\pm1,\ y=2$ and $z=\pm1+2i$

Solve the equations:

10. $z^2 - 2z + 4 = 0$

By equation (2.20) for the solution of a quadratic equation,

$$z = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{-3}}{2} = 1 \pm i\sqrt{3}$$

11. $z^3 + 8 = 0 \ \rightarrow \ z^3 = -8$

One root is $z = -2$. Then, as in the discussion of the factorization of cubic functions in Section 2.5,

let $\quad z^3 + 8 = (z+2)(az^2 + bz + c) \rightarrow a = 1,\ b = -2, c = 4$

Therefore $\quad z^3 + 8 = (z+2)(z^2 - 2z + 4) = 0$ when $z = -2$ and $1 \pm i\sqrt{3}$

Express as a single complex number:

12. $\dfrac{1-i}{1+i} = \dfrac{(1-i)^2}{(1+i)(1-i)} = \dfrac{1-2i+i^2}{2} = \dfrac{-2i}{2} = -i$

13. $\dfrac{1}{5+3i} = \dfrac{5-3i}{(5+3i)(5-3i)} = \dfrac{5-3i}{25+9} = \dfrac{5}{34} - \dfrac{3}{34}i$

14. $\dfrac{3+2i}{3-2i} = \dfrac{(3+2i)^2}{(3-2i)(3+2i)} = \dfrac{5+12i}{9+4} = \dfrac{5}{13} + \dfrac{12}{13}i$

15. $\dfrac{1}{5} - \dfrac{3-4i}{3+4i} = \dfrac{1}{5} - \dfrac{(3-4i)^2}{9+16} = \dfrac{5}{25} + \dfrac{7+24i}{25} = \dfrac{12}{25} + \dfrac{24}{25}i$

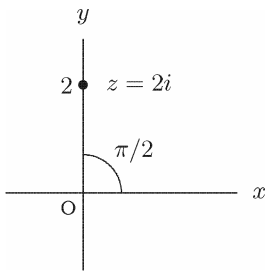
Section 8.3

(i) Plot as a point in the complex plane, **(ii)** find the modulus and argument, **(iii)** express in polar form $r(\cos\theta + i\sin\theta)$:

Let $z = x + iy = r(\cos\theta + i\sin\theta)$.

16. $z = 2i$: $x = 0,\ \ y = 2$

(i) The point lies on the imaginary axis, at $y = 2$ →



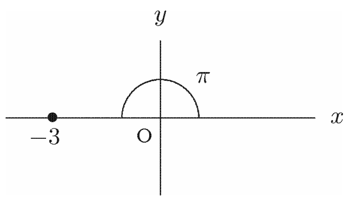
(ii) $r = |z| = \sqrt{x^2 + y^2} = 2$,

$\theta = \arg z = \pi/2$

(iii) $z = 2\left[\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right]$

17. $z = -3$ **:** $x = -3,\ \ y = 0$

(i) The point lies on the real axis, at $x = -3$ →



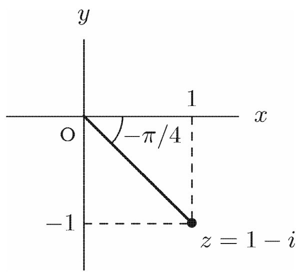
(ii) $r = |z| = 3$,

$\theta = \arg z = \pi$

(iii)

18. $z = 1 - i$ **:** $x = 1,\ \ y = -1$

(i) The point lies in the fourth quadrant →



(ii) $r = |z| = \sqrt{x^2 + y^2} = \sqrt{2}$,

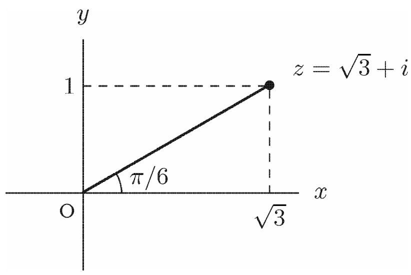
$\theta = \arg z = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}(-1) = -\pi/4$

(iii) $z = 2(\cos(-\pi/4) + i\sin(-\pi/4))$

$= 2\left[\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right]$

19. $z = \sqrt{3} + i$ **:** $x = \sqrt{3},\ \ y = 1$

(i) The point lies in the first quadrant →



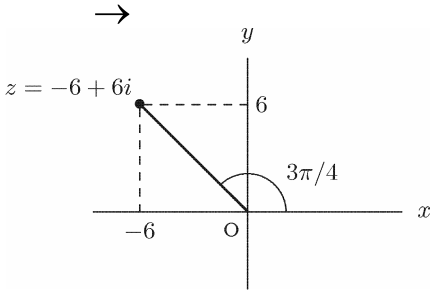
(ii) $r = |z| = \sqrt{3+1} = 2$,

$\theta = \arg z = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \pi/6$

(iii) $z = 2\left[\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right]$

20. **:** $x = -6,\ y = 6$

(i) The point lies in the second quadrant →



(ii) $r = |z| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$ **,**

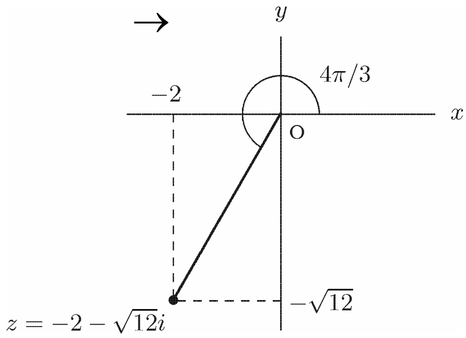
$\theta = \arg z = \tan^{-1}(-1) + \pi \ \ (x < 0)$

$= -\pi/4 + \pi = 3\pi/4$

(iii) $z = 6\sqrt{2}\left[\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right]$

21. $z = -2 - \sqrt{12}i$ **:** $x = -2,\ y = -\sqrt{12} = -2\sqrt{3}$

(i) The point lies in the third quadrant →



(ii) $r = |z| = \sqrt{4+12} = 4$ **,**

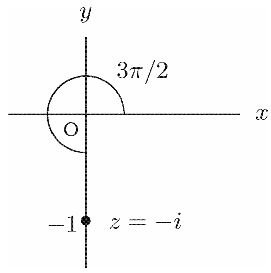
$\theta = \arg z = \tan^{-1}\left(\sqrt{3}\right) + \pi \ \ (x < 0)$

$= \pi/3 + \pi = 4\pi/3$

(iii) $z = 4\left[\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right]$

22. $z = 1/i = -i$: $x = 0,\ y = -1$

(i) The point lies in the imaginary axis at $y = -1$ →



(ii) $r = |z| = 1$ **,**

$\theta = \arg z = 3\pi/2$

(iii) $z = \cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}$

Given z_1 and z_2, express **(i)** $z_1 z_2$, **(ii)** z_1/z_2, **(iii)** z_2/z_1 as a single complex number for

23. $z_1 = 2\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right),\quad z_2 = 3\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

(i) $z_1 z_2 = (2\times 3)\left[\cos(\pi/2 + \pi/3) + i\sin(\pi/2 + \pi/3)\right]$ by equation (8.19),

$= 6\left[\cos 5\pi/6 + i\sin 5\pi/6\right]$

(ii) $z_1/z_2 = (2/3)\left[\cos(\pi/2 - \pi/3) + i\sin(\pi/2 - \pi/3)\right]$ by equation (8.22),

$= \dfrac{2}{3}\left[\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right]$

(iii) $z_2/z_1 = (z_1/z_2)^{-1} = \dfrac{3}{2}\left[\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right]$ by equation (8.24),

24. $z_1 = 5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \quad z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

(i) $z_1 z_2 = (5\times 1)\left[\cos(3\pi/4 + 2\pi/3) + i\sin(3\pi/4 + 2\pi/3)\right]$

$$= 5\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right]$$

(ii) $z_1/z_2 = 5\left[\cos(3\pi/4 - 2\pi/3) + i\sin(3\pi/4 - 2\pi/3)\right]$

$$= 5\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$$

(iii) $z_2/z_1 = \frac{1}{5}\left[\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right]$

25. For $z = 3\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ find **(i)** z^4, **(ii)** z^{-4}.

$$z^4 = 3^4\left[\cos\left(4\times\frac{\pi}{8}\right) + i\sin 4\times\left(\frac{\pi}{8}\right)\right] = 81\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 81i$$

$$z^{-4} = \frac{1}{81}\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right] = -\frac{i}{81}$$

26. Use de Moivre's formula to show that

(i) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) $\sin 4\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

We have

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta \\ &= \left[\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right] \\ &\quad + i\left[4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)\right]\end{aligned}$$

Therefore **(i)** $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) $\sin 4\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

27. Use de Moivre's formula to expand $\cos 8x$ as a polynomial in $\cos x$.

We have $\cos 8x + i\sin 8x = (\cos x + i\sin x)^8$

$$= \cos^8 x + 8i\cos^7 x\sin x + +28i^2\cos^6 x\sin^2 x$$

$$+ 56i^3\cos^5 x\sin^3 x + 70i^4\cos^4 x\sin^4 x + 54i^5\cos^3 x\sin^5 x$$

$$+ 28i^6\cos^2 x\sin^6 x + 8i^7\cos x\sin^7 x + i^8\sin^8 x$$

Then $\cos 8x = \mathrm{Re}\left[\cos 8x + i\sin 8x\right]$

$$= \cos^8 x - 28\cos^6 x\sin^2 x + 70\cos^4 x\sin^4 x - 28\cos^2 x\sin^6 x + \sin^8 x$$

$$= \cos^8 x - 28\cos^6 x \times (1-\cos^2 x) + 70\cos^4 x \times (1-\cos^2 x)^2$$

$$- 28\cos^2 x \times (1-\cos^2 x)^3 + (1-\cos^2 x)^4$$

$$= 128\cos^8 x - 256\cos^6 x + 160\cos^4 x - 32\cos^2 x + 1$$

Section 8.4

28. **(i)** Express the complex function $f(x) = 3x^2 + (1+2i)x + 2(i-1)$ in the form $f(x) = g(x) + ih(x)$, where $g(x)$ and $h(x)$ are real. **(ii)** solve $g(x) = 0$, $h(x) = 0$, then $f(x) = 0$. **(iii)** find $|f(x)|^2$.

(i) $f(x) = (3x^2 + x - 2) + 2(x+1)i = g(x) + ih(x)$

(ii) We have $g(x) = 3x^2 + x - 2 = (3x-2)(x+1) = 0$ when $x = 2/3$ or $x = -1$

$h(x) = 2(x+1) = 0$ when $x = -1$

Therefore $f(x) = 0$ when $x = -1$

(iii) $|f(x)|^2 = g(x)^2 + h(x)^2 = \left[(3x-2)(x+1)\right]^2 + \left[2(x+1)\right]^2$

$$= (x+1)^2(9x^2 - 12x + 8)$$

29. **(i)** Express the complex function $f(z)=z^2-2z+3$ in the form $f(z)=g(x,y)+ih(x,y)$ where $g(x,y)$ and $h(x,y)$ are real functions of the real variables *x* and *y*. **(ii)** Find the (real) solutions of the pair of equations $g(x,y)=0$ and $h(x,y)=0$, and hence of $f(z)=0$, **(iii)** Solve $f(z)=0$ directly in terms of *z* to confirm the results of **(ii)**.

(i) $$f(z)=z^2-2z+3=(x+iy)^2-2(x+iy)+3$$
$$=\left[x^2-y^2-2x+3\right]+i\left[2y(x-1)\right]=g(x,y)+ih(x,y)$$

(ii) We have $h(x,y)=2y(x-1)=0$ when $y=0$ or $x=1$

If $y=0$, $g(x,0)=x^2-2x+3=0$ when $x=\dfrac{2\pm\sqrt{4-12}}{2}$, not real

If $x=1$, $g(1,y)=2-y^2=0$ when $y=\pm\sqrt{2}$

Therefore $f(z)=0$ when $z=1\pm i\sqrt{2}$.

(iii) $$f(z)=z^2-2z+3=0 \text{ when } z=\frac{2\pm\sqrt{-8}}{2}=1\pm i\sqrt{2}$$

Section 8.5

Express **(i)** z, **(ii)** z^*, **(iii)** z^{-1} in exponential form $re^{i\theta}$:

30. $z=1-i$: $r=\sqrt{2}$, $\theta=\tan^{-1}(-1)=-\dfrac{\pi}{4}$

Then **(i)** $z=\sqrt{2}(\cos\pi/4-i\sin\pi/4)=\sqrt{2}\,e^{-i\pi/4}$ **(ii)** $z^*=\sqrt{2}\,e^{i\pi/4}$ **(iii)** $z^{-1}=\dfrac{1}{\sqrt{2}}e^{i\pi/4}$

31. $z=\sqrt{3}+i$: $r=2$, $\theta=\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)=\dfrac{\pi}{6}$

Then **(i)** $z=2e^{i\pi/6}$ **(ii)** $z^*=2e^{-i\pi/6}$ **(iii)** $z^{-1}=\dfrac{1}{2}e^{-i\pi/6}$

32. $z=2i$: $r=2$, $\theta=\dfrac{\pi}{2}$

Then **(i)** $z=2e^{i\pi/2}$ **(ii)** $z^*=2e^{-i\pi/2}$ **(iii)** $z^{-1}=\dfrac{1}{2}e^{-i\pi/2}$

33. $z=-3$: $r=3$, $\theta=\pi$

Then **(i)** $z=3e^{i\pi}$ **(ii)** $z^*=3e^{-i\pi}$ **(iii)** $z^{-1}=\dfrac{1}{3}e^{-i\pi}$

Express in cartesian form $x+iy$:

34. $3e^{i\pi/4} = 3(\cos\pi/4 + i\sin\pi/4) = \dfrac{3}{\sqrt{2}} + i\dfrac{3}{\sqrt{2}}$

35. $e^{-i\pi/3} = \cos\pi/3 - i\sin\pi/3 = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

36. $2e^{\pi i/6} = 2(\cos\pi/6 + i\sin\pi/6) = \sqrt{3} + i$

37. $e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i = i$

38. $e^{3\pi i/2} = \cos 3\pi/2 + i\sin 3\pi/2 = 0 - i = -i$

39. $e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1 + 0 = -1$

40. Use Euler's formulas for $\cos x$ and $\sin x$ to show that
(i) $\cos ix = \cosh x$, **(ii)** $\sin ix = i\sinh x$, **(iii)** $\tan ix = i\tanh x$

By equations (8.35) and (8.36),

(i) $\cos ix = \dfrac{1}{2}\left[e^{i(ix)} + e^{-i(ix)}\right] = \dfrac{1}{2}\left[e^{-x} + e^{+x}\right] = \cosh x$ Equation (3.47)

(ii) $\sin ix = \dfrac{1}{2i}\left[e^{i(ix)} - e^{-i(ix)}\right] = \dfrac{1}{2i}\left[e^{-x} - e^{+x}\right] = \dfrac{i}{2}\left[e^{x} - e^{-x}\right] = i\sinh x$

(iii) $\tan ix = \dfrac{\sin ix}{\cos ix} = \dfrac{i\sinh x}{\cosh x} = i\tanh x$

41. Express $\cos(a+ib)$ in the form $x+iy$.

We have
$$\cos(a+ib) = \frac{1}{2}\left[e^{i(a+ib)} + e^{-i(a+ib)}\right] = \frac{1}{2}\left[e^{ia}\times e^{-b} + e^{-ia}\times e^{b}\right]$$
$$= \frac{1}{2}\left[e^{-b}(\cos a + i\sin a) + e^{b}(\cos a - i\sin a)\right]$$
$$= \frac{1}{2}\left(e^{b} + e^{-b}\right)\cos a - \frac{i}{2}\left(e^{b} - e^{-b}\right)\sin a$$

Therefore $\cos(a+ib) = \cos a\cosh b - i\sin a\sinh b$

42. Show that $\ln z = \ln|z| + i\arg z$

We have $z = re^{i\theta}$, where $r = |z|$, $\theta = \arg z$

Then $\ln z = \ln r + i\theta = \ln|z| + i\arg z$

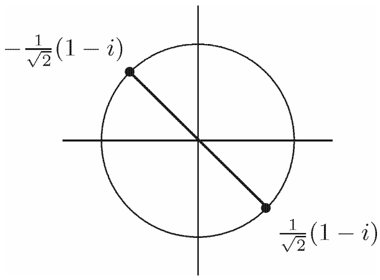
43. Use de Moivre's formula to find the square roots of $-i$. Locate them on the complex plane.

We have $e^{3\pi i/2} = \cos 3\pi/2 + i\sin 3\pi/2 = -i$

Then $\sqrt{-i} = \left(e^{3\pi i/2}\right)^{1/2} = \pm e^{3\pi i/4}$

$$= \pm(\cos 3\pi/4 + i\sin 3\pi/4) = \pm\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \pm\frac{1}{\sqrt{2}}(1-i)$$



44. Find the number obtained from $z = 3 + 2i$ by
(i) anticlockwise rotation through 30º,
(ii) clockwise rotation through 30º about the origin of the complex plane.

Number $z = 3 + 2i$ represents point $(x, y) = (3, 2)$ in the complex plane. Then, by equations (8.41):

(i) For rotation through angle $\pi/6$ about the origin, $(x, y) \to (x', y')$,

where
$$x' = x\cos\pi/6 - y\sin\pi/6 = \frac{\sqrt{3}}{2}x - \frac{1}{2}y = \frac{3\sqrt{3}}{2} - 1$$
$$y' = x\sin\pi/6 + y\cos\pi/6 = \frac{1}{2}x + \frac{\sqrt{3}}{2}y = \frac{3}{2} + \sqrt{3}$$

Therefore
$$z = 3 + 2i \to z' = \left(\frac{3\sqrt{3}}{2} - 1\right) + i\left(\frac{3}{2} + \sqrt{3}\right)$$

(ii) For rotation through angle $-\pi/6$ about the origin, as in (i) with

$$\sin\pi/6 \to \sin(-\pi/6) = -\sin\pi/6$$
$$\cos\pi/6 \to \cos(-\pi/6) = \cos\pi/6$$

Then
$$z = 3 + 2i \to z'' = \left(\frac{3\sqrt{3}}{2} + 1\right) + i\left(-\frac{3}{2} + \sqrt{3}\right)$$

Section 8.6

Find all the roots and plot them in the complex plane:

By equation (8.44):

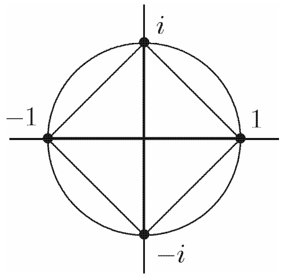
45. $\sqrt[4]{1} = \left(e^{2\pi ki}\right)^{1/4} = e^{\pi ki/2}, \;\; k = 0, \pm 1, 2$

The four fourth roots are

$k = 0, \quad e^0 = 1$

$k = \pm 1, \quad e^{\pm i\pi/2} = \cos \pi/2 \pm i \sin \pi/2 = \pm i$

$k = 2, \quad e^{i\pi} = \cos \pi + i \sin \pi = -1$



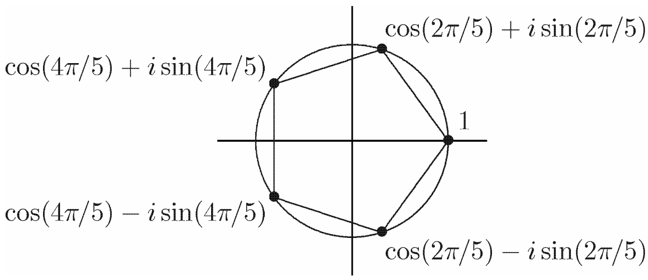
46. $\sqrt[5]{1} = e^{2\pi ki/5}, \;\; k = 0, \pm 1, \pm 2$

The five fifth roots are

$k = 0, \quad e^0 = 1$

$k = \pm 1, \quad e^{\pm 2\pi i/5} = \cos 2\pi/5 \pm i \sin 2\pi/5$

$k = \pm 2, \quad e^{\pm 4\pi i/5} = \cos 4\pi/5 \pm i \sin 4\pi/5$



47. $\sqrt[8]{1} = e^{\pi ki/4}, \;\; k = 0, \pm 1, \pm 2, \pm 3, 4$

The eight eighth roots are

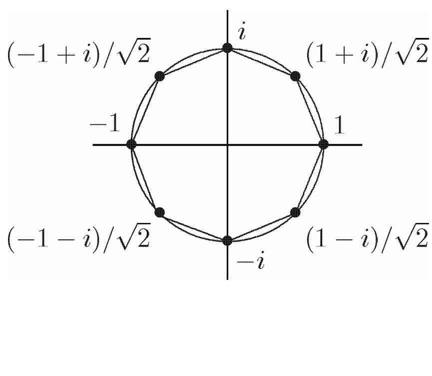
$k = 0, \quad e^0 = 1$

$k = \pm 1, \quad e^{\pm \pi i/4} = \cos \pi/4 \pm i \sin \pi/4 = \dfrac{1}{\sqrt{2}}(1 \pm i)$

$k = \pm 2, \quad e^{\pm \pi i/2} = \cos \pi/2 \pm i \sin \pi/2 = \pm i$

$k = \pm 3, \quad e^{\pm 3\pi i/4} = \cos 3\pi/4 \pm i \sin 3\pi/4 = \dfrac{1}{\sqrt{2}}(-1 \pm i)$

$k = 4 \quad e^{\pi i} = \cos \pi + i \sin \pi = -1$



48. The wave functions for the quantum mechanical rigid rotor in a plane are

$$\psi_n(\theta) = Ce^{in\theta},\ n = 0, \pm 1,\ \pm 2, \ldots$$

(i) Calculate the "normalization constant" C for which $\int_0^{2\pi} |\psi_n(\theta)|^2\, d\theta = 1$.

(ii) Show that $\int_0^{2\pi} \psi_m^*(\theta)\psi_n(\theta)\, d\theta = 0$ if $m \neq n$.

(i) We have

$$\int_0^{2\pi} |\psi_n(\theta)|^2\, d\theta = \int_0^{2\pi} \psi_n{}^*(\theta)\psi_n(\theta)\, d\theta$$

$$= C^*C\int_0^{2\pi} e^{-in\theta} \times e^{in\theta}\, d\theta = C^*C\int_0^{2\pi} d\theta = 2\pi\, C^*C$$

$$= 1 \text{ when } C^*C = 1/2\pi$$

The conventional choice is $C = 1/\sqrt{2\pi}$, and the normalized functions are $\psi_n(\theta) = \dfrac{1}{\sqrt{2\pi}}e^{in\theta}$.

(ii) We have

$$\int_0^{2\pi} \psi_m^*(\theta)\psi_n(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta} \times e^{in\theta} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(m-n)\theta} d\theta = \frac{1}{2\pi}\left[-\frac{e^{-i(m-n)\theta}}{(m-n)i}\right]_0^{2\pi} \qquad (m \neq n)$$

$$= \frac{1}{2\pi(m-n)i}\left[1 - e^{-2\pi(m-n)i}\right]$$

Because m and n are integers, $m - n$ is also integer so that, when $m \neq n$, $e^{-2\pi(m-n)i} = 1$.

Therefore $\int_0^{2\pi} \psi_m^*(\theta)\psi_n(\theta)\, d\theta = 0$ if $m \neq n$.

Section 8.7

Use complex numbers to integrate:

49. $\int_0^\infty e^{-x}\cos 2x\,dx$

By Euler's formula, equation (8.33),

$$\cos 2x = \text{Re}\,(e^{2ix})$$

so that $$\int_0^\infty e^{-x}\cos 2x\,dx = \text{Re}\int_0^\infty e^{-x}e^{2ix}dx$$

We have $$\int_0^\infty e^{-x}e^{2ix}dx = \int_0^\infty e^{-(1-2i)x}dx = \left[-\frac{e^{-(1-2i)x}}{1-2i}\right]_0^\infty = \frac{1}{1-2i} = \frac{1+2i}{(1-2i)(1+2i)} = \frac{1}{5}(1+2i)$$

Therefore $$\int_0^\infty e^{-x}\cos 2x\,dx = \text{Re}\int_0^\infty e^{-x}e^{2ix}dx = \frac{1}{5}$$

50. $\int_0^\infty e^{-2x}\sin^3 x\,dx$

By Euler's formula, equation (8.36),

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Then $$\sin^3 x = \left[\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right]^3 = -\frac{1}{8i}\left[e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}\right]$$
$$= -\frac{1}{4}\text{Im}\left[e^{3ix} - 3e^{ix}\right]$$

and $$\int_0^\infty e^{-2x}\sin^3 x\,dx = -\frac{1}{4}\text{Im}\int_0^\infty e^{-2x}\left[e^{3ix} - 3e^{ix}\right]dx$$

We have $$\int_0^\infty e^{-2x}\left[e^{3ix} - 3e^{ix}\right]dx = \int_0^\infty \left[e^{-(2-3i)x} - 3e^{-(2-i)x}\right]dx$$
$$= \left[-\frac{e^{-(2-3i)x}}{2-3i} + 3\frac{e^{-(2-i)x}}{2-i}\right]_0^\infty$$
$$= \frac{1}{2-3i} - \frac{3}{2-i} = \frac{2+3i}{13} - \frac{3(2+i)}{5} = \frac{-68-24i}{65}$$

Therefore $$\int_0^\infty e^{-2x}\sin^3 x\,dx = -\frac{1}{4}\text{Im}\int_0^\infty e^{-2x}\left[e^{3ix} - 3e^{ix}\right]dx = \left(-\frac{1}{4}\right)\times\left(\frac{-24}{65}\right) = \frac{6}{65}$$