The Chemistry Maths Book

Erich Steiner

University of Exeter

Second Edition 2008

Solutions

Chapter 7. Sequences and series

Section 7.2

Find (a) the general term and (b) the recurrence relation for the sequences:

1. $1, 4, 7, 10, \dots$

(a) $u_r = 1 + 3r; \quad r = 0, 1, 2, \dots$ (b) $u_r = u_{r-1} + 3; \quad u_0 = 1$

2. $1, 3, 9, 27, \dots$

(a) $u_r = 3^r; \quad r = 0, 1, 2, \dots$ (b) $u_r = 3u_{r-1}; \quad u_0 = 1$

3. $1, -\frac{1}{5}, \frac{1}{25}, -\frac{1}{125}, \dots$

(a) $u_r = (-1/5)^r; \quad r = 0, 1, 2, \dots$ (b) $u_r = (-1/5)u_{r-1}; \quad u_0 = 1$

Find the first 6 terms of the sequences:

4. $u_{r+1} = u_r + \frac{1}{2}; \quad u_1 = 0$

$$0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$$

5. $v_n = \left(\frac{2}{3}\right)^n; \quad n = 0, 1, 2, \dots$

$$1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}$$

6. $u_x = \frac{1}{x(x+2)}; \quad x = 1, 2, 3, \dots$

$$\frac{1}{1\cdot 3}, \frac{1}{2\cdot 4}, \frac{1}{3\cdot 5}, \frac{1}{4\cdot 6}, \frac{1}{5\cdot 7}, \frac{1}{6\cdot 8} \rightarrow \frac{1}{3}, \frac{1}{8}, \frac{1}{15}, \frac{1}{24}, \frac{1}{35}, \frac{1}{48}$$

7. $w_{n+1} = \frac{w_n}{n}; \quad w_1 = 1$

$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{710}$$

8. $u_{n+2} = u_{n+1} + 2u_n; \quad u_0 = 1, \ u_1 = 3$

$$u_0 = 1, \ u_1 = 3, \ u_2 = u_1 + 2u_0 = 5, \ u_3 = u_2 + 2u_1 = 11,$$

$$u_4 = u_3 + 2u_2 = 21, \ u_5 = u_4 + 2u_3 = 43$$

9. $u_{n+2} = 3u_{n+1} - 2u_n$; $u_0 = 1,\ u_1 = 1/2$

$$u_0 = 1,\ u_1 = 1/2,\ u_2 = 3u_1 - 2u_0 = -1/2,\ u_3 = 3u_2 - 2u_1 = -5/2,$$
$$u_4 = 3u_3 - 2u_2 = -13/2,\ u_5 = 3u_4 - 2u_3 = -29/2$$

10. $u_{n+2} = 3u_{n+1} - 2u_n$; $u_0 = u_1$

$$u_0,\ u_1 = u_0,\ u_2 = 3u_0 - 2u_0 = u_0, \ldots$$
$$u_n = 3u_0 - 2u_0 = u_0, \text{ all } n$$

Find the limit $r \to \infty$ for:

11. $\dfrac{1}{3^r}$

$$\frac{1}{3^1} = \frac{1}{3},\ \frac{1}{3^2} = \frac{1}{9},\ \frac{1}{3^3} = \frac{1}{27},\ \frac{1}{3^4} = \frac{1}{81},\ \ldots \text{ and } \lim_{r\to\infty}\left(\frac{1}{3^r}\right) = 0$$

12. 2^r

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ \ldots \text{ and } 2^r \to \infty \text{ as } r \to \infty$$

13. $\dfrac{1}{r+2}$

$$\frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \ldots \text{ and } \lim_{r\to\infty}\left(\frac{1}{r+2}\right) = 0$$

14. $\dfrac{r}{r+2}$

$$\frac{1}{3},\ \frac{2}{4},\ \frac{3}{5},\ \frac{4}{6},\ \ldots \text{ and } \frac{r}{r+2} \to \frac{r}{r} = 1 \text{ as } r \to \infty$$

15. $\dfrac{r}{r^2+r+1}$

Divide top and bottom by r: $\dfrac{r}{r^2+r+1} = \dfrac{1}{r+1+1/r} \to 0$ as $r \to \infty$

16. $\dfrac{3r^2+3r+1}{5r^2-6r-1}$

Divide top and bottom by r^2: $\dfrac{3r^2+3r+1}{5r^2-6r-1} = \dfrac{3+3/r+1/r^2}{5-6/r-1/r^2} \to \dfrac{3}{5}$ as $r \to \infty$

17. Find the limit of the sequence $\{u_{n+1}/u_n\}$ for $u_{n+2} = u_{n+1} + 2u_n$; $u_0 = 1$, $u_1 = 3$ (see Exercise 8).

We have $u_{n+2} = u_{n+1} + 2u_n \ \to\ \dfrac{u_{n+2}}{u_{n+1}} = 1 + 2\dfrac{u_n}{u_{n+1}}$

Let $\dfrac{u_{n+2}}{u_{n+1}} \to x$ as $n \to \infty$

Then $\dfrac{u_n}{u_{n+1}} = 1\Big/\left(\dfrac{u_{n+1}}{u_n}\right) \to \dfrac{1}{x}$ as $n \to \infty$

and $\dfrac{u_{n+2}}{u_{n+1}} = 1 + 2\dfrac{u_n}{u_{n+1}} \ \to\ x = 1 + \dfrac{2}{x}$

Solving for *x*,

$$x = 1 + \frac{2}{x} \ \to\ x^2 - x - 2 = (x-2)(x+1)$$
$$= 0 \text{ when } x = -1 \text{ or } x = 2$$

But the terms of the series are positive.

Therefore $x = \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 2$

Section 7.3

Find the sum of **(i)** the first *n* terms, **(ii)** the first 10 terms:

18. $1 + 5 + 9 + 13 + \cdots$

(i) This is the arithmetic series with $a = 1$, $d = 4$, and sum $S_n = \dfrac{n}{2}\left[2 + 4(n-1)\right] = n(2n-1)$

(ii) $S_{10} = 10 \times 19 = 190$

19. $3 - 2 - 7 - 12 - \cdots$

(i) Arithmetic series with $a = 3$, $d = -5$, and sum $S_n = \dfrac{n}{2}\left[6 - 5(n-1)\right] = \dfrac{n}{2}\left[11 - 5n\right]$

(ii) $S_{10} = -5 \times 39 = -195$

20. $1 + 3 + 9 + 27 + \cdots$

(i) This is the geometric series with $a = 1$, $x = 3$, and sum $S_n = \dfrac{1-3^n}{1-3} = \dfrac{1}{2}(3^n - 1)$

(ii) $S_{10} = \dfrac{1}{2}(3^{10} - 1) = 29524$

21. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

(i) Geometric series with $a=1$, $x=1/3$, and sum $S_n=\frac{1-(1/3)^n}{1-1/3}=\frac{3}{2}\left[1-(1/3)^n\right]$

(ii) $S_{10}==\frac{3}{2}\left[1-(1/3)^{10}\right]=1.5-\frac{1}{2\times 3^9}\approx 1.499975$

Find the sum of the first n terms:

22. $x^3+x^5+x^7+\cdots=x^3\left[1+x^2+x^4+\cdots\right]$

Therefore $S_n=x^3\left[\frac{1-x^{2n}}{1-x^2}\right]$

23. $x+2x^2+4x^3+\cdots=x\left[1+(2x)+(2x)^2+\cdots\right]$

Therefore $S_n=x\left[\frac{1-(2x)^n}{1-2x}\right]$

Use equation (7.11) to expand in powers of x:

24. $(1+x)^5=1+5x+\frac{5\times 4}{2}x^2+\frac{5\times 4\times 3}{3!}x^3+\frac{5\times 4\times 3\times 2}{4!}x^4+\frac{5\times 4\times 3\times 2\times 1}{5!}x^5$

$=1+5x+10x^2+10x^3+5x^4+x^5$

25. $(1+x)^7=1+7x+\frac{7\times 6}{2}x^2+\frac{7\times 6\times 5}{3!}x^3+\frac{7\times 6\times 5\times 4}{4!}x^4+\frac{7\times 6\times 5\times 4\times 3}{5!}x^5+\frac{7\times 6\times 5\times 4\times 3\times 2}{6!}x^6+\frac{7!}{7!}x^7$

$=1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$

Calculate the binomial coefficients $\binom{n}{r}$, $r=0,1,\ldots,n$, for

26. $n=3$

$$\binom{3}{0}=\frac{3!}{0!3!}=1,\quad \binom{3}{1}=\frac{3!}{1!2!}=3,\quad \binom{3}{2}=\frac{3!}{2!1!}=3,\quad \binom{3}{3}=\frac{3!}{3!0!}=1$$

27. $n = 4$

We have $\binom{4}{n} = \dfrac{4!}{n!(4-n)!}$

Therefore $\binom{4}{0} = \binom{4}{4} = \dfrac{4!}{0!4!} = 1,\quad \binom{4}{1} = \binom{4}{3} = \dfrac{4!}{1!3!} = 4,\quad \binom{4}{2} = \dfrac{4!}{2!2!} = 6$

28. $n = 7$

$$\binom{7}{0} = \binom{7}{7} = 1,\quad \binom{7}{1} = \binom{7}{6} = \frac{7!}{1!6!} = 7,\quad \binom{7}{2} = \binom{7}{5} = \frac{7!}{2!5!} = 21,\quad \binom{7}{3} = \binom{7}{4} = \frac{7!}{3!4!} = 35$$

Use equation (7.13) or (7.14) to expand in powers of *x*:

29. $(1-x)^3 = \binom{3}{0} + \binom{3}{1}(-x) + \binom{3}{2}(-x)^2 + \binom{3}{3}(-x)^3$

$= 1 - 3x + 3x^2 - x^3$

30. $(1+3x)^4 = \binom{4}{0} + \binom{4}{1}(3x) + \binom{4}{2}(3x)^2 + \binom{4}{3}(3x)^3 + \binom{4}{4}(3x)^4$

$= 1 + 12x + 54x^2 + 108x^3 + 81x^4$

31. $(1-4x)^5 = 1 + \binom{5}{1}(-4x) + \binom{5}{2}(-4x)^2 + \binom{5}{3}(-4x)^3 + \binom{5}{4}(-4x)^4 + (-4x)^5$

$= 1 - 20x + 160x^2 - 640x^3 + 1280x^4 - 1024x^5$

32. $(3-2x)^4 = (3)^4(-2x)^0 + 4\times(3)^3(-2x)^1 + 6\times(3)^2(-2x)^2 + 4\times(3)^1(-2x)^3 + (3)^0(-2x)^4$

$= 81 - 216x + 216x^2 - 96x^3 + 16x^4$

33. $(3+x)^6 = 3^6 + 6\times 3^5\times x + 15\times 3^4\times x^2 + 20\times 3^3\times x^3 + 15\times 3^2\times x^4 + 6\times 3\times x^5 + x^6$

$= 729 + 1458x + 1215x^2 + 540x^3 + 135x^4 + 18x^5 + x^6$

34. (i) Calculate the distinct trinomial coefficients $\dfrac{4!}{n_1!n_2!n_3!}$.

(ii) Use the coefficients to expand $(a+b+c)^4$.

(i) The possible values of (n_1, n_2, n_3) for which $n_1 + n_2 + n_3 = 4$ are

(4, 0, 0), (0, 4, 0), (0, 0, 4),
(3, 1, 0), (3, 0, 1), (1, 3, 0), (1,0, 3), (0, 3, 1), (0, 1, 3),
(2, 2, 0), (2, 0, 2), (0, 2, 2),
(2, 1, 1), (1, 2, 1), (1, 1, 2)

The corresponding distinct coefficients are

$$\frac{4!}{4!0!0!} = 1, \quad \frac{4!}{3!1!0!} = 4, \quad \frac{4!}{2!2!0!} = 6, \quad \frac{4!}{2!1!1!} = 12$$

(ii)
$$(a+b+c)^4 = \left(\frac{4!}{4!0!0!}\right)(a^4+b^4+b^4) + \left(\frac{4!}{3!1!0!}\right)(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$$
$$+\left(\frac{4!}{2!2!0!}\right)(a^2b^2+a^2c^2+b^2c^2) + \left(\frac{4!}{2!1!1!}\right)(a^2bc+ab^2c+abc^2)$$
$$= (a^4+b^4+b^4) + 4(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$$
$$+6(a^2b^2+a^2c^2+b^2c^2) + 12(a^2bc+ab^2c+abc^2)$$

35. (i) Calculate the distinct coefficients $\dfrac{3!}{n_1!n_2!n_3!n_4!}$.

(ii) Use the coefficients to expand $(a+b+c+d)^3$.

(i)
$$\frac{3!}{3!0!0!0!} = 1, \quad \frac{3!}{2!1!0!0!} = 3, \quad \frac{3!}{1!1!1!0!} = 6$$

(ii)
$$(a+b+c+d)^3 = (a^3+b^3+c^3+d^3)$$
$$+3(a^2b+a^2c+a^2d+b^2a+b^2c+b^2d+c^2a+c^2b+c^2d+d^2a+d^2b+d^2c)$$
$$+6(abc+abd+acd+bcd)$$

36. Find $\displaystyle\sum_{n=1}^{10}\frac{1}{n(n+1)}$.

By Example 7.6, $\displaystyle\sum_{n=1}^{N}\frac{1}{n(n+1)} = \frac{N}{N+1}$

$$= \frac{10}{11} \text{ when } N = 10$$

37. **(i)** Verify that $\dfrac{1}{r(r+2)} = \dfrac{1}{2}\left(\dfrac{1}{r} - \dfrac{1}{r+2}\right)$, then

(ii) find the sum of the series $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)}$.

(i) $\dfrac{1}{2}\left(\dfrac{1}{r} - \dfrac{1}{r+2}\right) = \dfrac{1}{2}\left(\dfrac{(r+2)}{r(r+2)} - \dfrac{r}{r(r+2)}\right) = \dfrac{1}{r(r+2)}$

(ii) We have $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\left(\sum_{r=1}^{n} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r+2}\right)$

Now $\displaystyle\sum_{r=1}^{n} \frac{1}{r+2} = \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1} + \frac{1}{n+2} = \sum_{r=3}^{n+2} \frac{1}{r}$

Therefore $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\left(\sum_{r=1}^{n} \frac{1}{r} - \sum_{r=3}^{n+2} \frac{1}{r}\right) = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$

$$= \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$

38. **(i)** Express $\dfrac{1}{r(r+1)(r+2)}$ in partial fractions, then

(ii) show that $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$

(i) $$\frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left[\frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}\right]$$

(ii) $$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left[\sum_{r=1}^{n} \frac{1}{r} - 2\sum_{r=1}^{n} \frac{1}{r+1} + \sum_{r=1}^{n} \frac{1}{r+2}\right]$$

$$= \frac{1}{2}\left[\sum_{r=1}^{n} \frac{1}{r} - 2\sum_{r=2}^{n+1} \frac{1}{r} + \sum_{r=3}^{n+2} \frac{1}{r}\right]$$

$$= \frac{1}{2}\left[\cancel{\sum_{r=1}^{n} \frac{1}{r}} - 2\left(\cancel{\sum_{r=1}^{n} \frac{1}{r}} - 1 + \frac{1}{n+1}\right) + \left(\cancel{\sum_{r=1}^{n} \frac{1}{r}} - 1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right] = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

39. **(i)** Verify that $(1+r)^3 - r^3 = 3r^2 + 3r + 1$, then

(ii) show that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

(i) $(1+r)^3 - r^3 = (r^3 + 3r^2 + 3r + 1) - r^3 = 3r^2 + 3r + 1$

(ii) We have $$\sum_{r=1}^{n}(1+r)^3 - \sum_{r=1}^{n} r^3 = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \qquad \text{(A)}$$

and $$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \quad \sum_{r=1}^{n} 1 = n$$

On the left of the equal sign in (A),

$$\sum_{r=1}^{n}(1+r)^3 - \sum_{r=1}^{n} r^3 = \sum_{r=2}^{n+1} r^3 - \sum_{r=1}^{n} r^3$$
$$= \left[\sum_{r=1}^{n} r^3 - 1 + (n+1)^3\right] - \left[\sum_{r=1}^{n} r^3\right] = (n+1)^3 - 1$$
$$= n^3 + 3n^2 + 3n$$

Equation (A) is then

$$n^3 + 3n^2 + 3n = 3\sum_{r=1}^{n} r^2 + \frac{3}{2}n(n+1) + n$$

and $$\sum_{r=1}^{n} r^2 = \frac{1}{3}\left[n^3 + 3n^2 + 3n - \frac{3}{2}n(n+1) - n\right] = \frac{1}{6}(2n^3 + 3n^2 + n)$$
$$= \frac{1}{6}n(2n+1)(n+1)$$

40. **(i)** Expand $(1+r)^6 - r^6$, then

(ii) use the series in Table 7.1 to find the sum of the series $\sum_{r=1}^{n} r^5$.

(i) $(1+r)^6 - r^6 = 1 + 6r + 15r^2 + 20r^3 + 15r^4 + 6r^5$

(ii) We have $$\sum_{r=1}^{n}(1+r)^6 - \sum_{r=1}^{n} r^6 = \sum_{r=1}^{n} 1 + 6\sum_{r=1}^{n} r + 15\sum_{r=1}^{n} r^2 + 20\sum_{r=1}^{n} r^3 + 15\sum_{r=1}^{n} r^4 + 6\sum_{r=1}^{n} r^5 \qquad \text{(B)}$$

On the left of the equal sign in equation (B),

$$\sum_{r=1}^{n}(1+r)^6 - \sum_{r=1}^{n} r^6 = (1+n)^6 - 1 = 6n + 15n^2 + 20n^3 + 15n^4 + 6n^5 + n^6$$

On the right of the equal sign in equation (B), by Table 7.1,

$$\sum_{r=1}^{n} 1 + 6\sum_{r=1}^{n} r + 15\sum_{r=1}^{n} r^2 + 20\sum_{r=1}^{n} r^3 + 15\sum_{r=1}^{n} r^4 + 6\sum_{r=1}^{n} r^5$$

$$= n + 6\left[\frac{1}{2}n(n+1)\right] + 15\left[\frac{1}{6}n(n+1)(2n+1)\right] + 20\left[\frac{1}{4}n^2(n+1)^2\right]$$

$$+15\left[\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n\right] + 6\sum_{r=1}^{n} r^5$$

$$= 6n + \frac{31}{2}n^2 + 20n^3 + \frac{25}{2}n^4 + 3n^5 + 6\sum_{r=1}^{n} r^5$$

Hence $$\sum_{r=1}^{n} r^5 = \frac{n^2}{12}\left[2n^4 + 6n^3 + 5n^2 - 1\right]$$

Section 7.4

(i) Expand in powers of *x* to terms in x^6 . **(ii)** Find the values of *x* for which the series converge:

41. (i) $\frac{1}{1-3x} = 1 + (3x) + (3x)^2 + (3x)^3 + (3x)^4 + (3x)^5 + (3x)^6$ **(ii)** $|3x| < 1 \;\to\; |x| < 1/3$

$= 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 + 729x^6$

42. (i) $\frac{1}{1+5x^2} = 1 + (-5x^2) + (-5x^2)^2 + (-5x^2)^3$ **(ii)** $|-5x^2| < 1 \;\to\; |x| < 1/\sqrt{5}$

$= 1 - 5x^2 + 25x^4 - 125x^6$

43. (i) $\frac{1}{2+x} = \frac{1}{2(1+x/2)} = \frac{1}{2}\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} - \frac{x^5}{32} + \frac{x^6}{64}\right]$ **(ii)** $|x/2| < 1 \;\to\; |x| < 2$

44. (i) Use the geometric series to express the number $1/(10^6 - 1)$ as a decimal fraction. **(ii)** Show that the decimal representation of $1/7$ can be written as $142857/(10^6 - 1)$ (see Section 1.4)

(i) $$\frac{1}{10^6 - 1} = \frac{10^{-6}}{1 - 10^{-6}} = 10^{-6}\left[1 + 10^{-6} + 10^{-12} + 10^{-18} + \cdots\right]$$

$$= 10^{-6} + 10^{-12} + 10^{-18} + 10^{-24} + \cdots$$

$$= 0.000001 + 0.000000\,000001 + 0.000000\,000000\,000001 + \cdots$$

$$= 0.000001\,000001\,000001\,\ldots$$

(ii) $$\frac{142857}{10^6 - 1} = 142857 \times 0.000001\,000001\,000001\,\ldots$$

$$= 0.142857\,142857\,142857\,\ldots = \frac{1}{7}$$

45. The vibrational partition function of a harmonic oscillator is given by the series $q_{\mathrm{v}} = \sum_{n=0}^{\infty} e^{-n\theta_{\mathrm{v}}/T}$ where $\theta_{\mathrm{v}} = h\nu_{\mathrm{e}}/k$ is the vibrational temperature. Confirm that the series is a convergent geometric series, and find its sum.

We have $\quad q_{\mathrm{v}} = \sum_{n=0}^{\infty} e^{-n\theta_{\mathrm{v}}/T} = \sum_{n=0}^{\infty} \left(e^{-\theta_{\mathrm{v}}/T}\right)^n = \sum_{n=0}^{\infty} x^n$

Now $\quad \theta_{\mathrm{v}} > 0$ and $T > 0$, so that $\theta_{\mathrm{v}}/T > 0$.

Therefore $\quad x = e^{-\theta_{\mathrm{v}}/T} < 1$

and the geometric series is convergent, with sum

$$q_{\mathrm{v}} = (1-x)^{-1} = \left[1 - e^{-\theta_{\mathrm{v}}/T}\right]^{-1}$$

Section 7.5

Examine the following series for convergence by

Comparison test

46. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

We have $\ln n < n$, $\frac{1}{\ln n} > n$, and each term of the series is greater than the corresponding term of the harmonic series. Therefore $\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$, and the series diverges.

47. $\sum_{r=1}^{\infty} \frac{\ln r}{r^3}$

We have $\ln r < r$, $\frac{\ln r}{r^3} < \frac{1}{r^2}$. Therefore $\sum_{r=1}^{\infty} \frac{\ln r}{r^3} < \sum_{r=1}^{\infty} \frac{1}{r^2}$**,** and the series converges.

D'Alembert ratio test

48. $\sum_{s=0}^{\infty} \frac{s^a}{(s+1)!}$

We have $u_s = \frac{s^a}{(s+1)!}$, $\frac{u_{s+1}}{u_s} = \left(\frac{s+1}{s}\right) \times \left(\frac{1}{s+2}\right) \to 0$ as $s \to \infty$**,** and the series converges for all values of a.

49. $\sum_{r=1}^{\infty} \frac{1}{r^a}$

We have $u_r = \frac{1}{r^a}$, $\frac{u_{r+1}}{u_r} = \left(\frac{r}{r+1}\right)^a \to 1$ as $r \to \infty$**,** and D'Alembert's ratio test fails (see Exercise 50).

Cauchy integral test:

50. $\sum_{r=1}^{\infty} \frac{1}{r^a}$

For $a = 1$, the harmonic series diverges.

For $a \neq 1$, $\int_1^{\infty} \frac{1}{r^a} dr = \lim_{b\to\infty} \left[-\frac{1}{(a-1)r^{a-1}} \right]_1^b \begin{cases} = \frac{1}{a-1} & \text{if } a > 1 \\ \text{divergees if } a < 1 \end{cases}$

The series therefore converges if $a > 1$, diverges if $a \leq 1$.

51. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Let $x = \ln n$.

Then $\int_2^{\infty} \frac{1}{n \ln n} dn = \int_{\ln 2}^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} \left[\ln x \right]_{\ln 2}^{b}$, and the series diverges.

Section 7.6

Find the radius of convergence of each of the following series:

52. $\sum_{m=0}^{\infty} \frac{x^m}{4^m}$

Let $c_m = \frac{1}{4^m}$, $c_{m+1} = \frac{1}{4^{m+1}}$.

Then $R = \frac{c_m}{c_{m+1}} = 4$, and the series converges when $|x| < 4$

53. $\sum_{r=0}^{\infty} (-1)^r x^{2r}$

Let $c_r = (-1)^r$, $c_{r+1} = (-1)^{r+1}$.

Then $R = \left|\frac{c_r}{c_{r+1}}\right| = 1$, and the series converges when $x^2 < 1$, $|x| < 1$

54. $\sum_{n=1}^{\infty} nx^n$

Let $c_n = n, \quad c_{n+1} = n+1$.

Then $R = \left|\dfrac{c_n}{c_{n+1}}\right| = 1$, and the series converges when $|x| < 1$. In fact, $\sum_{n=1}^{\infty} nx^n = \dfrac{1}{1-x}\sum_{n=1}^{\infty} x^n$.

55. $\sum_{n=1}^{\infty} \dfrac{x^n}{n^2}$

Let $c_n = \dfrac{1}{n^2}, \quad c_{n+1} = \dfrac{1}{(n+1)^2}$.

Then $R = \lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^2 = 1$, and the series converges when $|x| < 1$.

56. $\sum_{m=1}^{\infty} m^m x^m$

Let $c_m = m^m, \quad c_{m+1} = (m+1)^{m+1}$

Then $\dfrac{c_m}{c_{m+1}} = \dfrac{m^m}{(m+1)^{m+1}} = \left(\dfrac{m}{m+1}\right)^m \times \left(\dfrac{1}{m+1}\right) \to 0$ as $m \to \infty$.

Therefore $R = 0$, and the series never converges.

57. $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{3^n}$

Let $c_n = \left(-\dfrac{1}{3}\right)^n, \quad c_{n+1} = \left(-\dfrac{1}{3}\right)^{n+1}$.

Then $R = \left|\dfrac{c_n}{c_{n+1}}\right| = 3$, and the series converges when $x^2 < 3, \quad |x| < \sqrt{3}$.

Write down the first 5 terms of the MacLaurin series of the following functions:

58. $(1+x)^{1/3} = 1 + \dfrac{1}{3}x + \dfrac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}x^2 + \dfrac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3 + \dfrac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}x^4$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4$$

59. $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8$

60. $(1-x)^{-1/2} = 1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3+\frac{\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}(-x)^4$

$$= 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4$$

61. $\frac{1}{3+x} = \frac{1}{3}\left(1+\frac{x}{3}\right)^{-1} = \frac{1}{3}\left[1-\frac{x}{3}+\left(\frac{x}{3}\right)^2-\left(\frac{x}{3}\right)^3+\left(\frac{x}{3}\right)^4\right]$

$$= \frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}-\frac{x^3}{81}+\frac{x^4}{243}$$

62. $\sin 2x^2 = (2x^2)-\frac{(2x^2)^3}{3!}+\frac{(2x^2)^5}{5!}-\frac{(2x^2)^7}{7!}+\frac{(2x^2)^9}{9!}$

$$= 2x^2-\frac{4}{3}x^6+\frac{4}{15}x^{10}-\frac{8}{315}x^{14}+\frac{4}{2835}x^{18}$$

63. $\frac{\ln(1-2x)+2x}{x^2} = \frac{1}{x^2}\left\{\left[\cancel{(-2x)}-\frac{(-2x)^2}{2}+\frac{(-2x)^3}{3}-\frac{(-2x)^4}{4}+\frac{(-2x)^5}{5}-\frac{(-2x)^6}{6}\right]+\cancel{2x}\right\}$

$$= -\left[2+\frac{8}{3}x+4x^2+\frac{32}{5}x^3+\frac{32}{3}x^4\right]$$

64. $e^{-3x} = 1+(-3x)+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}+\frac{(-3x)^4}{4!}$

$$= 1-3x+\frac{9}{2}x^2-\frac{9}{2}x^3+\frac{27}{8}x^4$$

65. $\frac{e^{x^2}-1}{x} = \frac{1}{x}\left\{\left[\cancel{1}+x^2+\frac{x^4}{2}+\frac{x^6}{3!}+\frac{x^8}{4!}+\frac{x^{10}}{5!}\right]-\cancel{1}\right\}$

$$= x+\frac{x^3}{2}+\frac{x^5}{6}+\frac{x^7}{24}+\frac{x^9}{120}$$

66. A body with rest mass m_0 and speed v has relativistic energy $E = mc^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}}$ and kinetic energy $T = E-m_0c^2$. Express T as a power series in v and show that the series reduces to the nonrelativistic kinetic energy in the limit $v/c \to 0$.

$$T = E-m_0c^2 = m_0c^2\left[(1-v^2/c^2)^{-1/2}-1\right]$$

$$= m_0c^2\left\{\left[\cancel{1}+\frac{v^2}{2c^2}+\frac{3v^4}{8c^4}+\cdots\right]-\cancel{1}\right\}$$

$$= \frac{1}{2}m_0v^2\left[1+\frac{3v^2}{8c^2}+\cdots\right] \to \frac{1}{2}m_0v^2 \text{ as } \frac{v}{c}\to 0$$

67. The equation of state of a gas can be expressed in terms of the series

$$pV = nRT\sum_{i=0}^{\infty} B_i(T)\left(\frac{n}{V}\right)^i$$

where the B_i are called virial coefficients. Find the first three coefficients for

(i) the van der Waals equation, $\left(p+\frac{n^2a}{V^2}\right)(V-nb) = nRT$

(ii) the Dieterici equation, $p(V-nb) = nRTe^{-an/RTV}$

(i) We have $\left(p+\frac{n^2a}{V^2}\right)(V-nb) = nRT \;\rightarrow\; p = \frac{nRT}{V-nb} - \frac{n^2a}{V^2}$

Then $$p = \frac{nRT}{V}\left[1-\frac{nb}{V}\right]^{-1} - \frac{n^2a}{V^2}$$

$$= \frac{nRT}{V}\left[1+\frac{nb}{V}+\frac{n^2b^2}{V^2}+\cdots\right] - \frac{n^2a}{V^2}$$

and $$pV = nRT\left[1+\frac{n}{V}\left(b-\frac{a}{RT}\right)+\frac{n^2b^2}{V^2}+\cdots\right]$$

Therefore $B_0 = 1,\quad B_1 = b-\frac{a}{RT},\quad B_2 = b^2$

(ii) We have $p(V-nb) = nRTe^{-an/RTV} \;\rightarrow\; p = \frac{nRT}{V-nb}e^{-an/RTV}$

Then $$pV = nRT\left[1-\frac{nb}{V}\right]^{-1} e^{-an/RTV}$$

$$= nRT\left[1+b\left(\frac{n}{V}\right)+b^2\left(\frac{n}{V}\right)^2+\cdots\right]\times\left[1-\frac{a}{RT}\left(\frac{n}{V}\right)+\frac{a^2}{2R^2T^2}\left(\frac{n}{V}\right)^2+\cdots\right]$$

$$= nRT\left[1+\left(b-\frac{a}{RT}\right)\left(\frac{n}{V}\right)+\left(\frac{a^2}{2R^2T^2}+b^2-\frac{ab}{RT}\right)\left(\frac{n}{V}\right)^2+\cdots\right]$$

Therefore $B_0 = 1,\quad B_1 = b-\frac{a}{RT},\quad B_2 = b^2+\frac{a^2}{2R^2T^2}-\frac{ab}{RT}$

(i) Expand each of the following functions as a Taylor series about the given point, and

(ii) find the values of *x* for which the series converges:

68. (i) Expand $1/x$ about the point $x = 1$:

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} f^{(n)}(1)$$

We have $f(x) = \frac{1}{x} \quad f'(x) = -\frac{1}{x^2} \quad f''(x) = \frac{2}{x^3} \quad f'''(x) = -\frac{3!}{x^4} \quad \ldots$

$f(1) = 1 \quad f'(1) = -1 \quad f''(1) = 2! \quad f'''(1) = -3! \ \ldots$

Therefore $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \ldots$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

(ii) The series converges for $|1-x| < 1,\ 0 < x < 2$

69. (i) Expand e^x about the point $x = 2$:

We have $f^{(n)}(x) = e^x, \quad f^{(n)}(2) = e^2, \quad$ all n

Therefore $e^x = e^2 + (x-2)e^2 + \frac{(x-2)^2}{2!}e^2 + \frac{(x-2)^3}{3!}e^2 + \cdots = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$

(ii) The series converges for all values of *x*.

70. (i) Expand $\sin x$ about the point $x = \pi/2$:

We have $f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f''(x) = -\cos x,\ f^{(4)}(x) = \sin x \ \ldots$

$f(\pi/2) = 1, \quad f'(\pi/2) = 0, \quad f''(\pi/2) = -1, \quad f'''(\pi/2) = 0, \quad f^{(4)}(\pi/2) = 1 \quad \ldots$

Therefore $\sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$$

(ii) The series converges for all values of *x*.

71. Expand $\ln x$ about the point $x = 2$:

(i) We have

$$f(x) = \ln x \quad f'(x) = 1/x \quad f''(x) = -1/x^2 \quad f''(x) = 2!/x^3 \quad f^{(4)}(x) = -3!/x^4 \quad \ldots$$

$$f(2) = \ln 2 \quad f'(2) = 1/2 \quad f''(2) = -1/4 \quad f'''(2) = 2!/8 \quad f^{(4)}(2) = -3!/16 \quad \ldots$$

Therefore $$\ln x = \ln 2 + \left(\frac{x-2}{2}\right) - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 - \frac{1}{4}\left(\frac{x-2}{2}\right)^4 + \cdots$$

$$= \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{x-2}{2}\right)^n$$

(ii) The series converges for $-1 < \dfrac{x-2}{2} \le 1, \quad 0 < x \le 4$.

Section 7.7

72. **(i)** Find the MacLaurin expansion of the function $(8+x)^{1/3}$ up to terms in x^4. **(ii)** Use this expansion to find an approximate value of $\sqrt[3]{9}$. **(iii)** Use this value and Taylor's theorem for the remainder to compute upper and lower bounds to the value of $\sqrt[3]{9}$.

(i) $f(x) = (8+x)^{1/3} = 2\left(1 + \dfrac{x}{8}\right)^{1/3}$

$$= 2\left[1 + \left(\frac{1}{3}\right)\times\left(\frac{x}{8}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\times\left(\frac{x}{8}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\times\left(\frac{x}{8}\right)^3 \right.$$

$$\left. + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}\times\left(\frac{x}{8}\right)^4 + \cdots \right]$$

$$= 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{20736}x^3 - \frac{5}{248832}x^4 + \cdots$$

(ii) $x = 1$: $\quad 9^{1/3} \approx 2 + \dfrac{1}{12} - \dfrac{1}{288} + \dfrac{5}{20736} - \dfrac{5}{248832} = 2.08008214$

(iii) By Taylor's theorem, the remainder term in the MacLaurin expansion, with $a = 0$ in equation (7.26) is

$$R_n = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(b)$$

where *b* is some point in the interval 0 to *x*. In the present case, $n = 4$, $x = 1$, and

$$R_4 = \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{5!}(8+b)^{-14/3} = \frac{22}{729}\times\frac{1}{(8+b)^{14/3}}$$

The largest and smallest values of R_4 are

$$R_{\max} \approx 1.84\times 10^{-6} \text{ for } b=0$$

$$R_{\min} = \frac{22}{729}\times\frac{1}{A^{14}} \approx \frac{0.03017833}{A^{14}} \text{ for } b=1$$

where $A = 9^{1/3}$ is the quantity that is being estimated. Therefore

$$2.08008214 + \frac{0.03017833}{A^{14}} < A < 2.08008214 + 0.00000184 = 2.08008398$$

Taking the upper bound as the value of A in the lower bound, an estimate of the lower bound is

$$2.08008214 + 0.00000106 = 2.08008320$$

Therefore, to 8 significant figures,

$$2.0800832 < 9^{1/3} < 2.0900840$$

Find the limits:

73. $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}$

We have $\dfrac{e^x-1}{x} = \dfrac{\left[\cancel{1} + x + x^2/2 + \cdots\right] - \cancel{1}}{x} = 1 + \dfrac{x}{2} + \cdots$

Therefore $\lim\limits_{x\to 0}\dfrac{e^x-1}{x} = 1$

74. $\lim\limits_{x\to 0}\dfrac{\tan x - \sin x}{x^3}$

We have $\dfrac{\tan x - \sin x}{x^3} = \dfrac{\left[\cancel{x} + x^3/3 + 2x^5/15 + \cdots\right] - \left[\cancel{x} - x^3/3! + x^5/5! + \cdots\right]}{x^3}$

$= \dfrac{\left[x^3/2 + x^5/8 + \cdots\right]}{x^3}$

Therefore $\lim\limits_{x\to 0}\dfrac{\tan x - \sin x}{x^3} = \dfrac{1}{2}$

75. $\lim_{x\to 0}\dfrac{e^x+e^{-x}-2}{\cos x-1}$

We have $\dfrac{e^x+e^{-x}-2}{\cos x-1}=\dfrac{\left[\cancel{1}+\cancel{x}+x^2/2+\cancel{x^3/6}+x^4/24+\cdots\right]+\left[\cancel{1}-\cancel{x}+x^2/2-\cancel{x^3/6}+x^4/24+\cdots\right]-\cancel{2}}{\left[\cancel{1}-x^2/2+x^4/24+\cdots\right]-\cancel{1}}$

$$=\frac{x^2+x^4/12+\cdots}{-x^2/2+x^4/24+\cdots}$$

Therefore $\lim_{x\to 0}\dfrac{e^x+e^{-x}-2}{\cos x-1}=-2$

76. $\lim_{x\to 1}\dfrac{\ln x}{x^2-1}$

We have $\dfrac{\ln x}{x^2-1}=\dfrac{\ln[1+(x-1)]}{x^2-1}=\dfrac{(x-1)-(x-1)^2/2+\cdots}{(x-1)(x+1)}=\dfrac{1}{x+1}\left[1-(x-1)/2+\cdots\right]$

Therefore $\lim_{x\to 1}\dfrac{\ln x}{x^2-1}=\dfrac{1}{2}$

77. The energy density of black-body radiation at temperature T is given by the Planck formula

$$\rho(\lambda)=\frac{8\pi hc}{\lambda^5}\left[e^{hc/\lambda kT}-1\right]^{-1}$$

where λ is the wavelength. Show that the formula reduces to the classical Rayleigh-Jeans law $\rho=8\pi kT/\lambda^4$

(i) for long wavelengths ($\lambda\to\infty$),

(ii) if Planck's constant is set to zero ($h\to 0$).

We have $\rho(\lambda)=\dfrac{8\pi hc}{\lambda^5}\left[e^{hc/\lambda kT}-1\right]^{-1}=\dfrac{8\pi hc}{\lambda^5}\left[\cancel{1}+\dfrac{hc}{\lambda kT}+\dfrac{1}{2}\left(\dfrac{hc}{\lambda kT}\right)^2+\cdots-\cancel{1}\right]^{-1}$

$$=\frac{8\pi kT}{\lambda^4}\left[1+\frac{1}{2}\left(\frac{hc}{\lambda kT}\right)+\cdots\right]^{-1}=\frac{8\pi kT}{\lambda^4}\left[1-\frac{1}{2}\left(\frac{hc}{\lambda kT}\right)+\cdots\right]$$

Then **(i)** $\rho(\lambda)=\dfrac{8\pi kT}{\lambda^4}\left[1-\dfrac{1}{2}\left(\dfrac{hc}{\lambda kT}\right)+\cdots\right]\to\dfrac{8\pi kT}{\lambda^4}$ as $\lambda\to\infty$

(ii) $\rho(\lambda)=\dfrac{8\pi kT}{\lambda^4}\left[1-\dfrac{1}{2}\left(\dfrac{hc}{\lambda kT}\right)+\cdots\right]\to\dfrac{8\pi kT}{\lambda^4}$ as $h\to 0$

Section 7.8

78. Find the Cauchy product of the power series expansions of $\sin x$ and $\cos x$, and show that it is equal to $\frac{1}{2}\sin 2x$.

We have
$$\sin x\cos x = \left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right]\times\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right]$$
$$=\left[x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\cdots\right]+\left[-\frac{x^3}{3!}+\frac{x^5}{3!2!}-\frac{x^7}{3!4!}+\cdots\right]+\left[\frac{x^5}{5!}-\frac{x^7}{5!2!}+\cdots\right]+\left[-\frac{x^7}{7!}+\cdots\right]$$
$$=x-\frac{4x^3}{3!}+\frac{16x^5}{5!}-\frac{64x^7}{7!}+\cdots$$

and
$$\frac{1}{2}\sin 2x=\frac{1}{2}\left[(2x)-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\cdots\right]=x-\frac{4x^3}{3!}+\frac{16x^5}{5!}-\frac{64x^7}{7!}+\cdots=\sin x\cos x$$

79. Differentiate the power series expansion of $\sin x$ and show that the result is $\cos x$.

$$\frac{d}{dx}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right]=1-\frac{3x^2}{3!}+\frac{5x^4}{5!}-\frac{7x^6}{7!}+\cdots$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots=\cos x$$

80. Integrate the power series expansion of $\sin x$ and show that the result is $C-\cos x$, where C is a constant.

$$\int\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right]dx=\frac{x^2}{2}-\frac{x^4}{4\times 3!}+\frac{x^6}{6\times 5!}-\frac{x^8}{8\times 7!}+\cdots+A$$
$$=A+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\cdots$$
$$=(1+A)-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\cdots\right]$$
$$=C-\cos x$$