The Chemistry Maths Book

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Solutions

Chapter 6. Methods of Integration

Section 6.2

Evaluate the indefinite integrals:

1. $\int \sin^2 3x\,dx = \frac{1}{2}\int (1-\cos 6x)\,dx = \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right] + C$

$= \frac{1}{12}\left[6x - \sin 6x\right] + C$

2. $\int \sin 3x \cos 3x\,dx = \frac{1}{2}\int \sin 6x\,dx = -\frac{1}{12}\cos 6x + C$

3. $\int \sin 3x \cos 2x\,dx = \frac{1}{2}\int \left[\sin x + \sin 5x\right]dx = \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right] + C$

$= -\frac{1}{10}\left[5\cos x + \cos 5x\right] + C$

4. $\int \sin x \cos 3x\,dx = \frac{1}{2}\int \left[\sin 4x - \sin 2x\right]dx = \frac{1}{2}\left[-\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x\right] + C$

$= \frac{1}{8}\left[2\cos 2x - \cos 4x\right] + C$

5. $\int \sin 3x \sin x\,dx = \frac{1}{2}\int \left[\cos 2x - \cos 4x\right]dx = \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right] + C$

$= \frac{1}{8}\left[2\sin 2x - \sin 4x\right] + C$

6. $\int \cos 5x \cos 2x\,dx = \frac{1}{2}\int \left[\cos 7x + \cos 3x\right]dx = \frac{1}{2}\left[\frac{1}{7}\sin 7x + \frac{1}{3}\sin 3x\right] + C$

$= \frac{1}{42}\left[3\sin 7x + 7\sin 3x\right] + C$

Evaluate the definite integrals:

7. $\int_0^{\pi/2} \cos^2 3x\,dx = \frac{1}{2}\int_0^{\pi/2}\left[\cos 6x + 1\right]dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + x\right]_0^{\pi/2}$

$= \frac{1}{2}\left\{\left[\frac{1}{6}\sin 3\pi + \frac{\pi}{2}\right] - 0\right\} = \frac{\pi}{4}$

8. $$\int_0^{\pi/2} \sin 2x \cos 2x\, dx = \frac{1}{2}\int_0^{\pi/2} \sin 4x\, dx = -\frac{1}{8}\Big[\cos 4x\Big]_0^{\pi/2}$$
$$= -\frac{1}{8}\Big[\cos 2\pi\text{-}\cos 0\Big] = 0$$

9. $$\int_0^{\pi} \sin x \cos 2x\, dx = \frac{1}{2}\int_0^{\pi}\Big[\sin 3x - \sin x\Big]dx = -\frac{1}{2}\left[\frac{1}{3}\cos 3x - \cos x\right]_0^{\pi}$$
$$= -\frac{1}{2}\left\{\left[-\frac{1}{3}+1\right] - \left[\frac{1}{3}-1\right]\right\} = -\frac{2}{3}$$

10. The wave functions for a particle in a box of length l are

$$\psi_n(x) = \sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right), \qquad n = 1, 2, 3, \ldots$$

Show that the functions satisfy the orthonormality conditions

$$\int_0^l \psi_n\psi_m\, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

We have $$\int_0^l \psi_n\psi_m\, dx = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\sin\frac{\text{m}\pi x}{l}dx$$
$$= \frac{1}{l}\int_0^l \left[\cos(m-n)\frac{\pi x}{l} - \cos(m+n)\frac{\pi x}{l}\right]dx$$

For $m = n$: $$\int_0^l \psi_n{}^2\, dx = \frac{1}{l}\int_0^l\left[1 - \cos\frac{2n\pi x}{l}\right]dx = \frac{1}{l}\left[x - \frac{l}{2n\pi x}\sin\frac{2n\pi x}{l}\right]_0^l$$
$$= \frac{1}{l}\times l = 1$$

For $m \neq n$: $$\int_0^l \psi_n\psi_m\, dx = \frac{1}{l}\int_0^l \left[\cos(m-n)\frac{\pi x}{l} - \cos(m+n)\frac{\pi x}{l}\right]dx$$
$$= \frac{1}{l}\left[\frac{l}{(m-n)\pi}\sin(m-n)\frac{\pi x}{l} - \frac{l}{(m+n)\pi}\sin(m+n)\frac{\pi x}{l}\right]_0^l$$
$$= 0 \quad \text{because } \sin p\pi = 0 \text{ if integer } p \neq 0$$

Section 6.3

Evaluate the indefinite integrals (use the substitutions in parentheses, when given):

11. $\int (3x+1)^5\, dx \qquad (u = 3x+1)$

We have $du = \dfrac{du}{dx}dx = 3dx,\ \ dx = \dfrac{1}{3}du$

Therefore $\displaystyle\int (3x+1)^5\, dx = \frac{1}{3}\int u^5\, du = \frac{u^6}{18} + C$

$$= \frac{(3x+1)^6}{18} + C$$

12. $\int (2x-1)^{1/2}\, dx$

Let $u = 2x - 1,\ \ du = 2\,dx$

Then $\displaystyle\int (2x-1)^{1/2}\, dx = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{2}\times\frac{2}{3}u^{3/2} + C$

$$= \frac{1}{3}(2x-1)^{3/2} + C$$

13. $\int (3x^2+2x+5)^3(3x+1)\, dx \qquad (u = 3x^2+2x+5)$

We have $du = (6x+2)\,dx$

Therefore $\displaystyle\int (3x^2+2x+5)^3(3x+1)\, dx = \frac{1}{2}\int u^3\, du = \frac{1}{8}u^4 + C$

$$= \frac{1}{8}(3x^2+2x+5)^4 + C$$

14. $\int (2x^3+3x-1)^{1/3}(2x^2+1)\, dx$

Let $u = 2x^3 + 3x - 1,\ \ du = (6x^2+3)\,dx$

Then $\displaystyle\int (2x^3+3x-1)^{1/3}(2x^2+1)\, dx = \frac{1}{3}\int u^{1/3}\, du = \frac{1}{3}\times\frac{3}{4}u^{4/3} + C$

$$= \frac{1}{4}(2x^3+3x-1)^{4/3} + C$$

15. $\int (3x^2+2)e^{-(x^3+2x)}\,dx \qquad (u = x^3+2x)$

We have $du = (3x^2+2)\,dx$

Therefore
$$\int (3x^2+2)e^{-(x^3+2x)}\,dx = \int e^{-u}\,du = -e^{-u} + C$$
$$= -e^{-(x^3+2x)} + C$$

16. $\int (1-x)e^{4x-2x^2}\,dx$

Let $u = 4x-2x^2,\ du = (4-4x)\,dx$

Then
$$\int (1-x)e^{4x-2x^2}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C$$
$$= \frac{1}{4}e^{4x-2x^2} + C$$

17. $\int x\sqrt{4-x^2}\,dx \qquad (u = 4-x^2)$

We have $du = -2x\,dx$

Therefore
$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{2}\times\frac{2}{3}u^{3/2} + C$$
$$= -\frac{1}{3}(4-x^2)^{3/2} + C$$

18. $\int \cos x\, e^{2\sin x}\,dx$

Let $u = 2\sin x,\ du = 2\cos x\,dx$

Then
$$\int \cos x\, e^{2\sin x}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$$
$$= \frac{1}{2}e^{2\sin x} + C$$

19. $\int e^x(1+e^x)^{1/2}\,dx \quad (u = 1+e^x)$

We have $du = e^x\,dx$

Therefore
$$\int e^x(1+e^x)^{1/2}\,dx = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{3}(1+e^x)^{3/2} + C$$

20. $\int x\cos(3x^2-1)\,dx$

Let $u = 3x^2 - 1,\;\; du = 6x\,dx$

Then $\int x\cos(3x^2-1)\,dx = \frac{1}{6}\int \cos u\,du = \frac{1}{6}\sin u + C$

$$= \frac{1}{6}\sin(3x^2-1) + C$$

21. $\int \frac{2x+1}{x^2+x+2}\,dx \quad (u = x^2+x+2)$

We have $du = (2x+1)\,dx$

Therefore $\int \frac{2x+1}{x^2+x+2}\,dx = \int \frac{1}{u}\,du = \ln u + C$

$$= \ln(x^2+x+2) + C$$

22. $\int \frac{3x^2-x}{2x^3-x^2+3}\,dx$

Let $u = 2x^3 - x^2 + 3,\;\; du = (6x^2-2x)\,dx$

Then $\int \frac{3x^2-x}{2x^3-x^2+3}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u + C$

$$= \frac{1}{2}\ln(2x^3-x^2+3) + C$$

23. $\int \frac{\cos x}{1-\sin x}\,dx \quad (u = 1-\sin x)$

We have $du = -\cos x\,dx$

Therefore $\int \frac{\cos x}{1-\sin x}\,dx = -\int \frac{1}{u}\,du = -\ln u + C$

$$= -\ln(1-\sin x) + C$$

24. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x,\;\; du = -\sin x\,dx$

Then $\int \tan x\,dx = -\int \frac{1}{u}\,du = -\ln u + C$

$$= -\ln(\cos x) + C$$

25. $\int \frac{x}{\sqrt{4-x^2}}\,dx$

Let $u = 4 - x^2,\ du = -2x\,dx$

Then $$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{u^{1/2}}\,du = -\frac{1}{2}\times 2u^{1/2} + C$$

$$= -\sqrt{4-x^2} + C$$

26. $\int \frac{\tan x}{\ln(\cos x)}\,dx$

Let $u = \ln(\cos x),\ du = -\frac{\sin x}{\cos}\,dx = -\tan x\,dx$

Then $$\int \frac{\tan x}{\ln(\cos x)}\,dx = -\int \frac{1}{u}\,du = -\ln u + C$$

$$= -\ln\left[\ln(\cos x)\right] + C$$

27. $\int \sin^3 x \cos x\,dx \quad (u = \sin x)$

We have $du = \cos x\,dx$

Therefore $$\int \sin^3 x \cos x\,dx = \int u^3\,du = \frac{1}{4}u^4 + C$$

$$= \frac{1}{4}\sin^4 x + C$$

28. $\int \ln(\cos x) \sin x\,dx$

Let $u = \cos x,\ du = -\sin x\,dx$

Then $$\int \ln(\cos x)\sin x\,dx = -\int \ln u\,du = -\left[u\ln u - u\right] + C$$

$$= \cos x\left[1 - \ln(\cos x)\right] + C$$

29. $\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ **(**Integral 4 in Table 6.3)

30. $\displaystyle\int \frac{x^2 dx}{\sqrt{1-x^2}} \quad (x = \sin\theta)$

We have $\quad dx = \cos\theta\, d\theta, \quad \sqrt{1-x^2} = \cos\theta$

Therefore
$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = \int \sin^2\theta\, d\theta = \frac{1}{2}\int \big[1-\cos 2\theta\big] d\theta$$
$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C = \frac{1}{2}\big[\theta - \sin\theta\cos\theta\big] + C$$
$$= \frac{1}{2}\left[\sin^{-1} x - x\sqrt{1-x^2}\,\right] + C$$

31. $\displaystyle\int \frac{\sqrt{x}}{1+x}\, dx \quad (u = \sqrt{x})$

We have $\quad du = \dfrac{1}{2\sqrt{x}} dx, \quad dx = 2u\, du$

Therefore
$$\int \frac{\sqrt{x}}{1+x}\, dx = 2\int \frac{u^2}{1+u^2}\, du$$
$$= 2\int \left[\frac{1+u^2}{1+u^2} - \frac{1}{1+u^2}\right] du = 2\int \left[1 - \frac{1}{1+u^2}\right] du$$
$$= 2\left[u - \tan^{-1} u\,\right] + C$$
$$= 2\left[\sqrt{x} - \tan^{-1}\sqrt{x}\,\right] + C$$

32. (i) Use the substitution $x = a\sinh u$ to show that $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$.

(ii) Use the substitution $u = x + \sqrt{x^2+a^2}$ to show that $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x + \sqrt{x^2+a^2}\,\right] + C$.

(i) We have $\quad x = a\sinh u, \quad dx = a\cosh u\, du$

and $\quad \sqrt{x^2+a^2} = a\sqrt{1+\sinh^2 u} = a\cosh u$

Therefore
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int du = u + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

(ii) We have $u = x + \sqrt{x^2 + a^2}$, $du = \left[1 + \dfrac{x}{\sqrt{x^2 + a^2}}\right] dx = \dfrac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} dx$

Therefore $du = \dfrac{u}{\sqrt{x^2 + a^2}} dx$

and
$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{1}{u} du = \ln u + C$$
$$= \ln\left[x + \sqrt{x^2 + a^2}\right] + C$$

Evaluate the definite integrals:

33. $\displaystyle\int_1^2 \frac{x\,dx}{3x^2 - 2}$

Let $u = 3x^2 - 2$, $du = 6x\,dx$

We have $u = 1$ when $x = 1$, $u = 10$ when $x = 2$

Therefore
$$\int_1^2 \frac{x\,dx}{3x^2 - 2} = \frac{1}{6}\int_1^{10} \frac{du}{u} = \frac{1}{6}\Big[\ln u\Big]_1^{10}$$
$$= \frac{1}{6}\ln 10$$

34. $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x} + \pi)}{\sqrt{x}} dx$

Let $u = \sqrt{x} + \pi$, $du = \dfrac{1}{2\sqrt{x}} dx$

We have $u = \pi$ when $x = 0$, $u = 2\pi$ when $x = \pi^2$

Therefore
$$\int_0^{\pi^2} \frac{\sin(\sqrt{x} + \pi)}{\sqrt{x}} dx = 2\int_\pi^{2\pi} \sin u\,du = -2\Big[\cos u\Big]_\pi^{2\pi}$$
$$= -2\Big[\cos 2\pi - \cos\pi\Big]$$
$$= -4$$

35. $\int_0^{\pi/2} \sqrt{\sin\theta}\cos\theta\, d\theta$

Let $u = \sin\theta,\ du = \cos\theta\, d\theta$

We have $u = 0$ when $\theta = 0$, $u = 1$ when $\theta = \pi/2$

Therefore $\int_0^{\pi/2} \sqrt{\sin\theta}\cos\theta\, d\theta = \int_0^1 u^{1/2}\, du = \left[\frac{2}{3}u^{3/2}\right]_0^1$

$= 2/3$

36. $\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$ (Integral 1 in Table 6.3)

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0) = \pi/4 - 0 = \pi/4$

37. $\int_0^{\infty} xe^{-x^2}\, dx$

Let $u = x^2,\ du = 2x\, dx$

Then $\int_0^{\infty} xe^{-x^2}\, dx = \frac{1}{2}\int_0^{\infty} e^{-u}\, du = \frac{1}{2}\left[-e^{-u}\right]_0^{\infty} = \frac{1}{2}$

38. Line shapes in magnetic resonance spectroscopy are often described by the Lorentz function

$g(\omega) = \frac{1}{\pi}\frac{T}{1+T^2(\omega-\omega_0)^2}$. Find $\int_{\omega_0}^{\infty} g(\omega)\, d\omega.$

We have $\int_{\omega_0}^{\infty} g(\omega)\, d\omega = \frac{1}{\pi}\int_{\omega_0}^{\infty} \frac{T}{1+T^2(\omega-\omega_0)^2}\, d\omega$

Let $u = T(\omega-\omega_0),\ du = T\, d\omega$. Then $u = 0$ when $\omega = \omega_0$

and $\int_{\omega_0}^{\infty} g(\omega)\, d\omega = \frac{1}{\pi}\int_0^{\infty} \frac{du}{1+u^2} = \frac{1}{\pi}\left[\tan^{-1} u\right]_0^{\infty}$

Now $\lim_{u\to\infty} \tan^{-1} u = \frac{\pi}{2},\ \tan^{-1} 0 = 0$

Therefore $\int_{\omega_0}^{\infty} g(\omega)\, d\omega = \frac{1}{2}$

39. An approximate expression for the rotational partition function of a linear rotor is

$$q_r = \int_0^\infty (2J+1)e^{-J(J+1)\theta_R/T}\, dJ$$

where $\theta_R = \hbar^2/2Ik$ is the rotational temperature, I is the moment of inertia, and k is Boltzmann's constant. Evaluate the integral.

Let $u = J(J+1),\ du = (2J+1)\,dJ$

Then $$q_r = \int_0^\infty (2J+1)e^{-J(J+1)\theta_R/T}\, dJ = \int_0^\infty e^{-u\theta_R/T}\,du = \left[-\frac{T}{\theta_R}e^{-u\theta_R/T}\right]_0^\infty$$

$$= \frac{T}{\theta_R}$$

Section 6.4

Evaluate the integrals:

40. $\int x \sin x\, dx$

Let $u = x,\ \dfrac{dv}{dx} = \sin x$ in equation (6.14) for integration by parts.

Then $$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

41. $\int x^3 \sin x\, dx$

We integrate by parts three times:

$u = x^3,\ \dfrac{dv}{dx} = \sin x \quad \to \quad \displaystyle\int x^3 \sin x\,dx = -x^3\cos x + 3\int x^2\cos x\,dx$

$u = x^2,\ \dfrac{dv}{dx} = \cos x \quad \to \quad \displaystyle = -x^3\cos x + 3\left\{x^2\sin x - 2\int x\sin x\,dx\right\}$

$u = x,\ \dfrac{dv}{dx} = \sin x, \quad \to \quad \displaystyle = -x^3\cos x + 3\left\{x^2\sin x - 2\left[-x\cos x + \int \cos x\,dx\right]\right\}$

$$= -x^3\cos x + 3\left\{x^2\sin x - 2\left[-x\cos x + \sin x\right]\right\} + C$$

$$= 3(x^2-2)\sin x - x(x^2-6)\cos x + C$$

42. $\int (x+1)^2 \cos 2x \, dx$

By parts twice:

$$u=(x+1)^2,\ \frac{dv}{dx}=\cos 2x \rightarrow \int (x+1)^2 \cos 2x\, dx$$

$$=\frac{1}{2}(x+1)^2 \sin 2x - \int (x+1)\sin 2x\, dx$$

$$u=(x+1),\ \frac{dv}{dx}=\sin 2x \quad \rightarrow \quad =\frac{1}{2}(x+1)^2 \sin 2x - \left\{-\frac{1}{2}(x+1)\cos 2x + \frac{1}{2}\int \cos 2x\, dx\right\}$$

$$=\frac{1}{2}(x+1)^2 \sin 2x + \frac{1}{2}(x+1)\cos 2x - \frac{1}{4}\sin 2x + C$$

43. $\int x^2 e^{2x}\, dx$

By parts twice: $u=x^2,\ \frac{dv}{dx}=e^{2x}$, then $u=x,\ \frac{dv}{dx}=e^{2x}$.

$$\int x^2 e^{2x}\, dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\, dx$$

$$=\frac{1}{2}x^2 e^{2x} - \left\{\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x}\, dx\right\}$$

$$=\frac{1}{2}x^2 - \left\{\frac{1}{2}x e^{2x} - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right]\right\} + C$$

$$=\frac{1}{4}\left[2x^2 - 2x + 1\right]e^{2x} + C$$

44. $\int_0^1 x e^x\, dx = \left[x e^x\right]_0^1 - \int_0^1 e^x\, dx = \left[x e^x\right]_0^1 - \left[e^x\right]_0^1 = \left[e^1 - 0\right] - \left[e^1 - 1\right] = 1$

45. $\int_0^\infty x^2 e^{-2x}\, dx = \frac{2!}{2^3} = \frac{1}{4}$ (by formula)

46. $\int x \ln x\, dx$

By parts with $u=\ln x,\ \frac{dv}{dx}=x$

Then $\int x \ln x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

$$=\frac{x^2}{4}\left[2\ln x - 1\right] + C$$

47. $\displaystyle\int \frac{\ln x}{x^2}\,dx$

By parts with $u = \ln x$

Then $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + C$

$$= -\frac{1}{x}(\ln x + 1) + C$$

48. $\displaystyle\int_0^1 x^2 \ln x\,dx = \left[\frac{x^3}{3}\ln x\right]_0^1 - \frac{1}{3}\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\ln x\right]_0^1 - \frac{1}{3}\left[\frac{x^3}{3}\right]_0^1$

$$= -\frac{1}{9}$$

49. $\displaystyle\int e^{-x}\sin 2x\,dx$

Twice by parts with $u = e^{-x}$

We have $\displaystyle\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\,dx$

$$= -e^{-x}\sin 2x + 2\left\{-e^{-x}\cos 2x - 2\int e^{-x}\sin 2x\,dx\right\}$$

$$= -e^{-x}\left[\sin 2x + 2\cos 2x\right] - 4\int e^{-x}\sin 2x\,dx$$

The integral occurs on both sides of the equal sign.

Then $\displaystyle\int e^{-x}\sin 2x\,dx = -\frac{1}{5}e^{-x}\left[\sin 2x + 2\cos 2x\right] + C$

50. $\displaystyle\int e^{ax}\cos bx\,dx$

Twice by parts with $u = e^{ax}$

$$\int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$$

$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left\{\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx\right\}$$

$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$$

Therefore $\displaystyle\int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}\left[a\cos bx + b\sin bx\right]e^{ax} + C$

51. $\int_0^{\pi/2} e^{-2x} \cos 3x \, dx$

By Exercise 50 with $a = -2,\ b = 3$

$$\int_0^{\pi/2} e^{-2x} \cos 3x \, dx = \left[\frac{1}{2^2 + 3^2} \left[-2\cos 3x + 3\sin 3x \right] e^{-2x} \right]_0^{\pi/2}$$
$$= \frac{1}{13}\left[(-3e^{-\pi}) + (2) \right]$$
$$= \frac{1}{13}\left[2 - 3e^{-\pi} \right]$$

Section 6.5

52. Determine a reduction formula for $\int \sin^n x \, dx$, where n is a positive integer.

Write $$I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$$

Then, by parts

$$I_n = \int \sin^{n-1} x \sin x \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

Therefore, solving for I_n,

$$I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \left(\frac{n-1}{n}\right) I_{n-2}$$

53. Show that, for integers $m \ge 0$ and $n \ge 1$,

$$\int \sin^m \theta \cos^n \theta \, d\theta = \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+n} + \frac{n-1}{m+n}\int \sin^m \theta \cos^{n-2}\theta \, d\theta$$

Write $\quad I = \int \sin^m \theta \cos^n \theta \, d\theta = \int \left[\sin^m \theta \cos\theta\right] \cos^{n-1}\theta \, d\theta$

Then by parts, with

$$u = \cos^{n-1}\theta, \ \frac{du}{d\theta} = -(n-1)\cos^{n-2}\theta \sin\theta$$

and $\quad \frac{dv}{d\theta} = \sin^m \theta \cos\theta, \ v = \frac{\sin^{m+1}\theta}{m+1}$

we have

$$\begin{aligned}
I &= \int \left[\sin^m \theta \cos\theta\right] \cos^{n-1}\theta \, d\theta \\
&= \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+1} + \left(\frac{n-1}{m+1}\right)\int \sin^{m+2}\theta \cos^{n-2}\theta \, d\theta \\
&= \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+1} + \left(\frac{n-1}{m+1}\right)\int \sin^{m}\theta (1-\cos^2\theta) \cos^{n-2}\theta \, d\theta \\
&= \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+1} + \left(\frac{n-1}{m+1}\right)\int \sin^{m}\theta \cos^{n-2}\theta \, d\theta - \left(\frac{n-1}{m+1}\right)\int \sin^{m}\theta \cos^{n}\theta \, d\theta \\
&= \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+1} + \left(\frac{n-1}{m+1}\right)\int \sin^{m}\theta \cos^{n-2}\theta \, d\theta - \left(\frac{n-1}{m+1}\right) I
\end{aligned}$$

Therefore, solving for I,

$$I = \frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+n} + \left(\frac{n-1}{m+n}\right)\int \sin^{m}\theta \cos^{n-2}\theta \, d\theta$$

54. Use the results of Exercises **52** and **53** to evaluate $\int \sin^5 x \cos^4 x \, dx$.

By Exercise 53,

$$\begin{aligned}
\int \sin^5 x \cos^4 x \, dx &= \frac{\sin^6 x \cos^3 x}{9} + \frac{1}{3}\int \sin^5 x \cos^2 x \, dx \\
&= \frac{\sin^6 x \cos^3 x}{9} + \frac{1}{3}\left\{\frac{\sin^6 x \cos x}{7} + \frac{1}{7}\int \sin^5 x \, dx\right\} \\
&= \frac{\sin^6 x \cos^3 x}{9} + \frac{\sin^6 x \cos x}{21} + \frac{1}{21}\int \sin^5 x \, dx
\end{aligned}$$

By Exercise 52,

$$\int \sin^5 x\,dx = -\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}\int \sin^3 x\,dx$$
$$= -\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}\left\{-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\,dx\right\}$$
$$= -\frac{1}{5}\sin^4 x\cos x - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x + C$$

Therefore
$$\int \sin^5 x\cos^4 x\,dx = \frac{\sin^6 x\cos^3 x}{9} + \frac{\sin^6 x\cos x}{21}$$
$$+\frac{1}{21}\left\{-\frac{1}{5}\sin^4 x\cos x - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x\right\} + C$$
$$= \frac{\sin^6 x\cos^3 x}{9} + \frac{\sin^6 x\cos x}{21}$$
$$-\frac{1}{105}\sin^4 x\cos x - \frac{4}{315}\sin^2 x\cos x - \frac{8}{315}\cos x + C$$

55. Show that, for integers $m \ge 0$ and $n > 1$,

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m\theta\cos^{n-2}\theta\,d\theta$$

By Exercise 53,

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \left[\frac{\sin^{m+1}\theta\cos^{n-1}\theta}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m\theta\cos^{n-2}\theta\,d\theta$$

At $\theta = \pi/2$, $\sin^{m+1}\theta\cos^{n-1}\theta = \cos^{n-1}\pi/2 = 0$ if integer $n > 1$

At $\theta = 0$, $\sin^{m+1}\theta\cos^{n-1}\theta = \sin^{m+1} 0 = 0$ if integer $\mathrm{m} \ge 0$

Therefore, when integers $m \ge 0$ and $n > 1$,

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m\theta\cos^{n-2}\theta\,d\theta$$

Evaluate

56. $\int_0^{\pi/2} \sin^5 x\cos^5 x\,dx$

By Exercise 55,

$$\int_0^{\pi/2} \sin^5 x\cos^5 x\,dx = \frac{4}{10}\int_0^{\pi/2} \sin^5 x\cos^3 x\,dx = \frac{4\times 2}{10\times 8}\int_0^{\pi/2} \sin^5 x\cos x\,dx$$
$$= \frac{4\times 2}{10\times 8}\left[\frac{\sin^6 x}{6}\right]_0^{\pi/2} = \frac{4\times 2}{10\times 8\times 6} = \frac{1}{60}$$

57. $\int_0^\infty r\,e^{-2r^2}\,dr$

Put $u = 2r^2,\ du = 4r\,dr$

Then $\int_0^\infty r\,e^{-2r^2}\,dr = \frac{1}{4}\int_0^\infty e^{-u}\,du = \frac{1}{4}$ (I_1 of Example 6.16)

58. $\int_0^\infty r^2\,e^{-2r^2}\,dr = I_2$ in Example 16.6.

Then $I_2 = \frac{1}{4}I_0 = \frac{1}{8}\sqrt{\frac{\pi}{2}}$

59. $\int_0^\infty r^3\,e^{-2r^2}\,dr = I_3$ in Example 6.16.

Then $I_3 = \frac{1}{2}I_1 = \frac{1}{8}$

60. The probability that a molecule of mass m in a gas at temperature T has speed v is given by the Maxwell-Boltzmann distribution

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

where k is Boltzmann's constant. Find the average speed $\overline{v} = \int_0^\infty v\,f(v)\,dv$.

Write $f(v) = Av^2e^{-Bv^2}$, where $A = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}$, $B = m/2kT$

Then $\overline{v} = \int_0^\infty v\,f(v)\,dv = A\int_0^\infty v^3e^{-Bv^2}dv = \frac{A}{2B^2}$ (Exercise 59 with $B = 2$)

$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\Big/ 2(m/2kT)^2$$

$$= \left(\frac{8kT}{\pi m}\right)^{1/2}$$

61. For the Maxwell-Boltzmann distribution in Exercise 60, find the root mean square speed $\sqrt{\overline{v^2}}$, where $\overline{v^2} = \int_0^\infty v^2 f(v)\,dv$.

As in Exercise 60, let

$$f(v) = Av^2 e^{-Bv^2}, \quad \text{where } A = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}, \; B = m/2kT$$

Then $$\overline{v^2} = \int_0^\infty v^2 f(v)\,dv = A\int_0^\infty v^4 e^{-Bv^2}\,dv$$

Now, in Example 6.16 with $a = B$,

$$I_4 = \int_0^\infty v^4 e^{-Bv^2}\,dv = \frac{3}{2B} I_2 = \frac{3}{2B}\times\frac{1}{2B} I_0 = \frac{3}{8B^2}\sqrt{\frac{\pi}{B}}$$

Therefore $$\overline{v^2} = \int_0^\infty v^2 f(v)\,dv = \frac{3A}{8B^2}\sqrt{\frac{\pi}{B}} = \frac{3kT}{m}$$

and $$\sqrt{\overline{v^2}} = \sqrt{\frac{3kT}{m}}$$

62. Line shapes in spectroscopy are sometimes analysed in terms of second moments. The second moment of a signal centred at angular frequency ω_0 is $\int_{\omega_0}^\infty (\omega - \omega_0)^2 g(\omega)\,d\omega$ where $g(\omega)$ is a shape function for the signal. Evaluate the integral for the gaussian curve

$$g(\omega) = \sqrt{\frac{2}{\pi}}\,T \exp\left[-\frac{1}{2}T^2(\omega-\omega_0)^2\right]$$

We have $$I = \int_{\omega_0}^\infty (\omega-\omega_0)^2 g(\omega)\,d\omega = \sqrt{\frac{2}{\pi}}\,T\int_{\omega_0}^\infty e^{-T^2(\omega-\omega_0)^2/2}(\omega-\omega_0)^2\,d\omega$$

Let $$A = \sqrt{2/\pi}\,T, \; B = T^2/2, \; x = \omega - \omega_0$$

Then $$I = A\int_0^\infty e^{-Bx^2} x^2 dx$$

Now, by Example 6.16 with $a = B$,

$$I_2 = \int_0^\infty x^2 e^{-Bx^2}\,dv = \frac{1}{2B} I_0 = \frac{1}{4B}\sqrt{\frac{\pi}{B}}$$

Therefore $$I = A\int_0^\infty e^{-Bx^2} x^2 dx = \frac{A}{4B}\sqrt{\frac{\pi}{B}} = \frac{1}{T^2}$$

Section 6.6

Evaluate the indefinite integrals:

63. $\int \frac{dx}{(2x-1)(x+3)} = \frac{1}{7}\int \left[\frac{2}{2x-1} - \frac{1}{x+3}\right] dx = \frac{1}{7}\left[\ln(2x-1) - \ln(x+3)\right] + C$

$= \frac{1}{7}\ln\frac{2x-1}{x+3} + C$

64. $\int \frac{(x+2)}{(x+3)(x+4)}\, dx = \int \left[\frac{2}{x+4} - \frac{1}{x+3}\right] dx = \left[2\ln(x+4) - \ln(x+3)\right] + C$

$= \ln\frac{(x+4)^2}{x+3} + C$

65. $\int \frac{(x^2 - 3x + 3)}{(x+1)(x+2)(x+3)}\, dx = \frac{1}{2}\int \left[\frac{7}{x+1} - \frac{26}{x+2} + \frac{21}{x+3}\right] dx$

$= \frac{1}{2}\left[7\ln(x+1) - 26\ln(x+2) + 21\ln(x+3)\right] + C$

$= \frac{1}{2}\ln\frac{(x+1)^7 (x+3)^{21}}{(x+2)^{26}} + C$

66. $\int \frac{x+2}{x^2 + 4x + 5}\, dx = \frac{1}{2}\int \frac{2x+4}{x^2 + 4x + 5}\, dx$

$= \frac{1}{2}\ln(x^2 + 4x + 5) + C$

67. $\int \frac{x}{(x^2 + 3)(x^2 + 4)}\, dx = \frac{1}{2}\int \left[\frac{2x}{x^2+3} - \frac{2x}{x^2+4}\right] dx$

$= \frac{1}{2}\left[\ln(x^2+3) - \ln(x^2+4)\right] + C$

$= \frac{1}{2}\ln\frac{x^2+3}{x^2+4} + C$

68. $\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1}$

$= \tan^{-1}(x+2) + C$

69. $\displaystyle\int \frac{dx}{(x^2+4x+5)^2}$

We have $\displaystyle\int \frac{dx}{(x^2+4x+5)^2} = \int \frac{dx}{((x+2)^2+1)^2}$

By (6.26), with $n=2,\ a=1,\ u=\tan\theta = x+2$,

$$\int \frac{dx}{(x^2+4x+5)^2} = \int \frac{dx}{(u^2+1)^2} = \int \cos^2\theta\, d\theta$$
$$= \frac{1}{2}(\sin\theta\cos\theta + \theta) + C$$

Then as in Example 6.20, $\theta = \tan^{-1} u,\ \sin\theta = u\big/\sqrt{u^2+1},\quad \cos\theta = 1\big/\sqrt{u^2+1}$,

and $$\int \frac{dx}{(x^2+4x+5)^2} = \frac{1}{2}\left[\frac{u}{u^2+1} + \tan^{-1} u\right]$$
$$= \frac{1}{2}\left[\frac{x+2}{x^2+4x+5} + \tan^{-1}(x+2)\right] + C$$

70. $\displaystyle\int \frac{x}{x^2+4x+5}\,dx$

We have $\displaystyle\int \frac{x}{x^2+4x+5}\,dx = \int \frac{x+2}{x^2+4x+5}\,dx - 2\int \frac{1}{x^2+4x+5}\,dx$

Therefore, by Exercises 66 and 68,

$$\int \frac{x}{x^2+4x+5}\,dx = \frac{1}{2}\ln(x^2+4x+5) - 2\tan^{-1}(x+2) + C$$

71. $\displaystyle\int \frac{4x+3}{(x^2+4x+5)^2}\,dx$

We have $$\int \frac{4x+3}{(x^2+4x+5)^2}\,dx = 2\int \frac{2x+4}{(x^2+4x+5)^2}\,dx - 5\int \frac{1}{(x^2+4x+5)^2}\,dx$$
$$= 2A - 5B$$

In A, let $u = x^2+4x+5,\ du = (2x+4)\,dx$

Then $$\int \frac{2x+4}{(x^2+4x+5)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{x^2+4x+5}$$

From Exercise 69,

$$B = \int \frac{dx}{(x^2+4x+5)^2} = \frac{1}{2}\left[\frac{x+2}{x^2+4x+5} + \tan^{-1}(x+2)\right]$$

Therefore $\displaystyle\int \frac{4x+3}{(x^2+4x+5)^2}\,dx = -\frac{2}{x^2+4x+5} - \frac{5}{2}\left[\frac{x+2}{x^2+4x+5} + \tan^{-1}(x+2)\right]$

$$= -\frac{1}{2}\left[\frac{14+5x}{x^2+4x+5} + 5\tan^{-1}(x+2)\right] + C$$

72. If $t = \tan\dfrac{\theta}{2}$, show that $d\theta = \dfrac{2}{1+t^2}\,dt$ (Equation (6.33))

We have $\displaystyle\frac{dt}{d\theta} = \frac{1}{2}\sec^2\theta/2 = \frac{1}{2\cos^2\theta/2} \;\rightarrow\; d\theta = 2\cos^2\theta/2\; dt$

Now $\displaystyle t = \tan\theta/2 \;\rightarrow\; 1+t^2 = 1+\frac{\sin^2\theta/2}{\cos^2\theta/2} = \frac{\cos^2\theta/2+\sin^2\theta/2}{\cos^2\theta/2} = \frac{1}{\cos^2\theta/2}$

Therefore $\displaystyle\cos^2\theta/2 = \frac{1}{1+t^2}$

and $\displaystyle d\theta = \frac{2}{1+t^2}\,dt$

Evaluate by means of the substitution $t = \tan\theta/2$:

73. $\displaystyle\int \frac{1}{\cos\theta}\,d\theta = \int \frac{\cancel{1+t^2}}{1-t^2}\times\frac{2}{\cancel{1+t^2}}\,dt = 2\int\frac{dt}{1-t^2}$

$\displaystyle = \ln\frac{1+t}{1-t} + C$ (Integral 5 in Table 6.3)

$\displaystyle = \ln\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right) + C$

74. $\displaystyle\int \frac{d\theta}{5-3\cos\theta} = \int\left[\frac{1}{5-3(1-t^2)/(1+t^2)}\right]\frac{2}{1+t^2}\,dt$

$\displaystyle = \int \frac{1}{1+4t^2}\,dt = \frac{1}{2}\tan^{-1}(2t) + C$ (Integral 4 in Table 6.3)

$\displaystyle = \frac{1}{2}\tan^{-1}(2\tan\theta/2) + C$

75. $\displaystyle\int \frac{1}{1+\sin\theta+\cos\theta}\,d\theta = \int\left[\frac{1}{1+(2t)/(1+t^2)+(1-t^2)/(1+t^2)}\right]\frac{2}{1+t^2}\,dt$

$\displaystyle = \int\frac{1}{1+t}\,dt = \ln(1+t) + C$

$= \ln(1+\tan\theta/2) + C$

Section 6.7.

76. By differentiation of the integral $\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ with respect to a,

show that $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$

We have

$$\frac{d}{da}e^{-ax^2} = (-x^2)e^{-ax^2}$$

$$\frac{d^2}{da^2}e^{-ax^2} = (-x^2)^2 e^{-ax^2}$$

$$\cdots$$

$$\frac{d^n}{da^n}e^{-ax^2} = (-x^2)^n e^{-ax^2} = (-1)^n x^{2n} e^{-ax^2}$$

and

$$\frac{d}{da}a^{-1/2} = \left(-\frac{1}{2}\right)a^{-3/2}$$

$$\frac{d^2}{da^2}a^{-1/2} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)a^{-5/2}$$

$$\frac{d^3}{da^3}a^{-1/2} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)a^{-7/2}$$

$$\cdots$$

$$\frac{d^n}{da^n}a^{-1/2} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)a^{-(2n+1)/2}$$

$$= (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}a^{-(2n+1)/2} = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n a^n}a^{-1/2}$$

Then

$$\frac{d^n}{dx^n}\int_0^\infty e^{-ax^2}\,dx = \int_0^\infty\left[\frac{d^n}{dx^n}e^{-ax^2}\right]dx = (-1)^n\int_0^\infty x^{2n}e^{-ax^2}\,dx$$

$$\frac{d^n}{dx^n}\frac{1}{2}\sqrt{\frac{\pi}{a}} = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

and

$$\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$