The Chemistry Maths Book

Erich Steiner

University of Exeter

Second Edition 2008

Solutions

Chapter 5. Integration

Section 5.2

Evaluate the indefinite integrals:

By Table 5.11,

1. $\int 2\,dx = 2x + C$

2. $\int x^3\,dx = \frac{1}{4}x^4 + C$

3. $\int x^{2/3}\,dx = \frac{x^{(2/3)+1}}{(2/3)+1} + C = \frac{3}{5}x^{5/3} + C$

4. $\int \frac{dx}{x^3} = \int x^{-3}dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$

5. $\int x^{-1/3}\,dx = \frac{x^{-(1/3)+1}}{-(1/3)+1} + C = \frac{3}{2}x^{2/3} + C$

6. $\int \sin 4x\,dx = -\frac{1}{4}\cos 4x + C$

7. $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

8. $\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$

9. $\int \frac{dx}{x-1} = \ln(x-1) + C = \ln A(x-1)$, with $\ln A = C$.

10. $\int \frac{dx}{3-x} = -\ln(3-x) + C = \ln\frac{A}{3-x}$, from type 5 in Table 5.1 with $a = -1, b = 3$.

Evaluate the indefinite integrals subject to the given conditions:

11. $y = \int x^2\,dx$; $y = 0$ when $x = 3$

We have $y = \int x^2\,dx = x^3/3 + C$

Then $y = 0$ when $x = 3 \;\to\; 0 = 3^3/3 + C \;\to\; C = -9$

Therefore $y = x^3/3 - 9$

12. $y = \int \cos 4x \, dx$; $y = 0$ when $x = \pi/4$

We have $$y = \int \cos 4x \, dx = \frac{1}{4}\sin 4x + C$$

$$y = 0 \text{ when } x = \pi/4 \;\rightarrow\; 0 = \frac{1}{4}\sin \pi + C = 0 + C \;\rightarrow\; C = 0$$

Therefore $$y = \frac{1}{4}\sin 4x$$

13. $I = \int (5x^4 + 2x + 3)\, dx$; $I = 4$ when $x = 2$

$$I = \int (5x^4 + 2x + 3)\, dx = 5 \times \frac{x^5}{5} + 2 \times \frac{x^2}{2} + 3x + C$$
$$= x^5 + x^2 + 3x + C$$

$$I = 4 \text{ when } x = 2 \;\rightarrow\; 4 = 32 + 4 + 6 + C = 42 + C \;\rightarrow\; C = -38$$

Therefore $$I = x^5 + x^2 + 3x - 38$$

14. $I = \int \frac{3x^2 + 2x + 1}{x^2}\, dx$; $I = 3$ when $x = 1$

$$I = \int \frac{3x^2 + 2x + 1}{x^2}\, dx = 3\int dx + 2\int \frac{1}{x} dx + \int \frac{1}{x^2} dx$$
$$= 3x + 2\ln x - \frac{1}{x} + C$$

$$I = 3 \text{ when } x = 1 \;\rightarrow\; 3 = 3 + 0 - 1 + C \;\rightarrow\; C = 1$$

Therefore $$I = 3x + 2\ln x - \frac{1}{x} + 1$$

15. $I = \int \left(-4 + 4\cos 2x - \frac{1}{2}e^{2x}\right) dx$; $I = 0$ when $x = 0$

$$I = \int \left(-4 + 4\cos 2x - \frac{1}{2}e^{2x}\right) dx = -4x + 2\sin 2x - \frac{1}{4}e^{2x} + C$$

$$I = 0 \text{ when } x = 0 \;\rightarrow\; 0 = 0 + 0 - \frac{1}{4} + C \;\rightarrow\; C = \frac{1}{4}$$

Therefore $$I = -4x + 2\sin 2x - \frac{1}{4}e^{2x} + \frac{1}{4}$$

Section 5.3.

Evaluate the definite integrals:

16. $\int_{-1}^{+1}(2x^2+3x+4)\,dx=\left[\frac{2x^3}{3}+\frac{3x^2}{2}+4x\right]_{-1}^{+1}$

$$=\left(\frac{2}{3}+\frac{3}{2}+4\right)-\left(-\frac{2}{3}+\frac{3}{2}-4\right)=\frac{28}{3}$$

17. $\int_3^5 dx=\left[x\right]_3^5=5-3=2$

18. $\int_1^2\frac{du}{u^3}=\left[-\frac{1}{2u^2}\right]_1^2=\left(-\frac{1}{8}\right)-\left(-\frac{1}{2}\right)=-\frac{1}{8}+\frac{1}{2}=\frac{3}{8}$

19. $\int_2^3\frac{dv}{v+2}=\left[\ln(v+2)\right]_2^3=\ln 5-\ln 4=\ln 5/4$

20. $\int_1^5 e^{-3t}\,dt=\left[-\frac{1}{3}e^{-3t}\right]_1^5=-\frac{1}{3}e^{-15}+\frac{1}{3}e^{-3}=\frac{1}{3}\left[e^{-3}-e^{-15}\right]$

21. $\int_0^{\pi/2}\cos\theta\,d\theta=\left[\sin\theta\right]_0^{\pi/2}=\sin\frac{\pi}{2}-\sin 0=1-0=1$

22. $\int_0^{\pi}\cos 3\theta\,d\theta=\left[\frac{1}{3}\sin 3\theta\right]_0^{\pi}=0-0=0$

23. $\int_{-\pi/4}^{0}\sin 2x\,dx=\left[-\frac{1}{2}\cos 2x\right]_{-\pi/4}^{0}$

$$=\left(-\frac{1}{2}\cos 0\right)-\left(-\frac{1}{2}\cos(-\pi/2\right)=-\frac{1}{2}+0=-\frac{1}{2}$$

24. (i) Show that the rate equation of a first-order decomposition reaction $\frac{dx}{dt} = -kx$ can be written in the logarithmic form $\frac{d\ln x}{dt} = -k$.

(ii) Integrate this equation with respect to t over the range 0 to t, and show that

$$\ln\left[\frac{x(t)}{x(0)}\right] = -kt \quad \text{and} \quad x(t) = x(0)e^{-kt}$$

(i) By the chain rule, $\frac{d\ln x}{dt} = \frac{1}{x}\frac{dx}{dt}$.

Therefore $\frac{dx}{dt} = -kx \;\to\; \frac{1}{x}\frac{dx}{dt} = -k$

$\to\; \frac{d\ln x}{dt} = -k$

(ii) We have $\int_0^t \frac{d\ln x}{dt}dt = -k\int_0^t dt \;\to\; \big[\ln x\big]_0^t = -k\big[t\big]_0^t$

$\to\; \ln x(t) - \ln x(0) = -kt$

Therefore $\ln\left[\frac{x(t)}{x(0)}\right] = -kt$

Then, because $e^{\ln x} = x$,

$$\frac{x(t)}{x(0)} = e^{-kt} \;\to\; x(t) = x(0)e^{-kt}$$

25. The Clausius-Clapeyron equation for liquid-vapour equilibrium is

$$\frac{d\ln p}{dT} = \frac{\Delta H_{\text{vap}}}{RT^2} .$$

If the enthalpy of vaporization, ΔH_{vap} , is constant in the temperature range T_1 to T_2 show, by integrating both sides of the equation with respect to T, that

$$\ln\left(\frac{p_2}{p_1}\right) = \frac{\Delta H_{\text{vap}}}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right),$$

where $p_1 = p(T_1)$ and $p_2 = p(T_2)$.

We have $\int_{T_1}^{T_2} \frac{d\ln p}{dT}dT = \frac{\Delta H_{\text{vap}}}{R}\int_{T_1}^{T_2}\frac{1}{T^2}dT \;\to\; \big[\ln p\big]_{T_1}^{T_2} = \frac{\Delta H_{\text{vap}}}{R}\left[-\frac{1}{T}\right]_{T_1}^{T_2}$

Therefore $\ln p_2 - \ln p_1 = \frac{\Delta H_{\text{vap}}}{R}\left[-\frac{1}{T_2} + \frac{1}{T_1}\right] \;\to\; \ln\frac{p_2}{p_1} = \frac{\Delta H_{\text{vap}}}{R}\left[\frac{1}{T_1} - \frac{1}{T_2}\right]$

Find the average values in the given intervals:

26. $2x^2+3x+4;\ -1 \le x \le +1$

We have $$\int_{-1}^{+1}(2x^2+3x+4)\,dx = \left[\frac{2x^3}{3}+\frac{3x^2}{2}+4x\right]_{-1}^{+1} = \frac{28}{3} \quad \text{(Exercise 16)}$$

and $$\int_{-1}^{+1} dx = 2$$

Therefore $\overline{2x^2+3x+4} = (28/3)\big/2 \ = 14/3$

27. $\cos 3\theta;\ \ 0 \le \theta \le \pi/2$

We have $$\int_0^{\pi/2}\cos 3\theta\,d\theta = \left[\frac{1}{3}\sin 3\theta\right]_0^{\pi/2} = -\frac{1}{3},\quad \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

Therefore $\overline{\cos 3\theta} = (-1/3)\big/(\pi/2) \ = -2/3\pi$

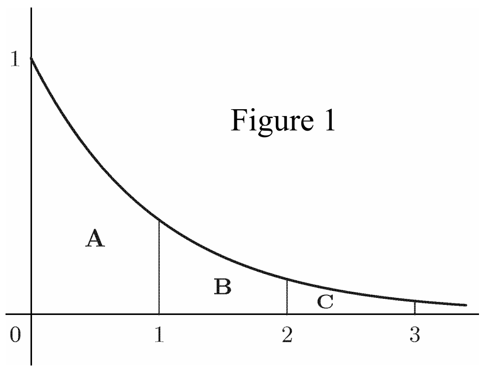
28. $1;\ \ 3 \le x \le 5$

We have $\bar{1} = \left[\int_a^b dx \Big/ \int_a^b dx\right] = 1$ for all intervals

Demonstrate and sketch a graph to interpret:

29. $$\int_0^3 e^{-x}\,dx = \int_0^1 e^{-x}\,dx + \int_1^2 e^{-x}\,dx + \int_2^3 e^{-x}\,dx$$

$= \text{A}+\text{B}+\text{C}$ in Figure 1



We have $$\int_a^b e^{-x}\,dx = \left[-e^{-x}\right]_a^b = e^{-a}-e^{-b}$$

Therefore

$$\int_0^1 e^{-x}\,dx + \int_1^2 e^{-x}\,dx + \int_2^3 e^{-x}\,dx \quad = (e^0 - \cancel{e^{-1}}) + (\cancel{e^{-1}} - \cancel{e^{-2}}) + (\cancel{e^{-2}} - e^{-3})$$

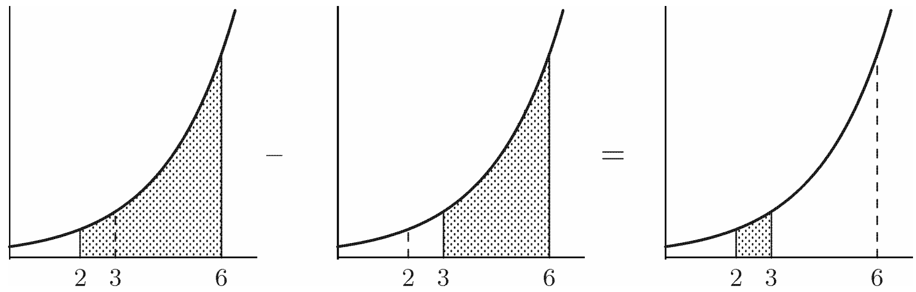
$$= 1 - e^{-3} = \int_0^3 e^{-x}\,dx$$

30. $\int_2^3 e^x\,dx = \int_2^6 e^x\,dx - \int_3^6 e^x\,dx$

We have $\int_a^b e^x\,dx = \left[e^x\right]_a^b = e^b - e^a$

Therefore $\int_2^6 e^x\,dx - \int_3^6 e^x\,dx = \left[\cancel{e^6} - e^2\right] - \left[\cancel{e^6} - e^3\right] = e^3 - e^2$

Figure 2



31. **(i)** Show that $\int_0^{\pi/2} \cos x\,dx = -\int_{\pi/2}^{\pi} \cos x\,dx$.

(ii) Calculate $\int_{-\pi}^{0} \cos x\,dx$, $\int_{-\pi}^{\pi/2} \cos x\,dx$, $\int_{-\pi}^{\pi} \cos x\,dx$.

(iii) Sketch a graph to interpret these results.

We have $\int_a^b \cos x\,dx = \sin b - \sin a$. Therefore

(i) $\int_0^{\pi/2} \cos x\,dx = \sin \pi/2 - \sin 0 = 1 - 0 = 1$ $\quad A_3$ in Figure 3

$\int_{\pi/2}^{\pi} \cos x\,dx = \sin \pi - \sin \pi/2 = 0 - 1 = -1$ $\quad A_4$

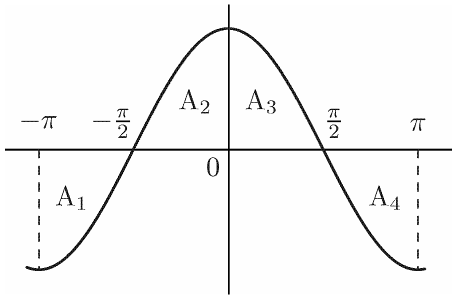
$= -\int_0^{\pi/2} \cos x\,dx$

(ii) $\int_{-\pi}^{0} \cos x\,dx = \sin 0 - \sin(-\pi) = 0 - 0 = 0$ $\quad A_1 + A_2$ in Figure3

$\int_{-\pi}^{\pi/2} \cos x\,dx = \sin \pi/2 - \sin(-\pi) = 1 - 0 = 1$ $\quad A_1 + A_2 + A_3$

$\int_{-\pi}^{\pi} \cos x\,dx = \sin \pi - \sin(-\pi) = 0 - 0 = 0$ $\quad A_1 + A_2 + A_3 + A_4$

(iii) Figure 3



Evaluate and sketch a graph to interpret:

32. $\int_{-1}^{+3} f(x)\,dx$ where $f(x) = \begin{cases} x^2+2 & \text{if } x<1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

We have

$$\int_{-1}^{+3} f(x)\,dx = \int_{-1}^{+1} (x^2+2)\,dx + \int_{1}^{3} x^2\,dx$$

$$= \left[\frac{1}{3}x^3 + 2x\right]_{-1}^{+1} + \left[\frac{1}{3}x^3\right]_{1}^{3} = \frac{14}{3} + \frac{26}{3} = \frac{40}{3}$$

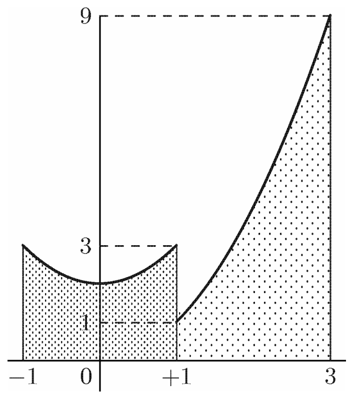


Figure 4

33. $\int_{-1}^{+1} f(x)\,dx$ where $f(x) = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x \leq 0 \end{cases}$

$$\int_{-1}^{+1} f(x)\,dx = \int_{-1}^{0} (-x)\,dx + \int_{0}^{+1} x\,dx$$

$$= \left[-x^2/2\right]_{-1}^{0} + \left[+x^2/2\right]_{0}^{1} = 1/2 + 1/2 = 1$$

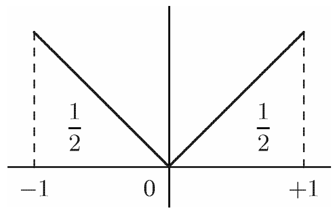


Figure 5

34. $\int_{-a}^{+a} f(x)\,dx$ where $f(x) = \begin{cases} e^{-x} & \text{if } x>0 \\ -e^{+x} & \text{if } x \leq 0 \end{cases}$

$$\int_{-a}^{+a} f(x)\,dx = \int_{-a}^{0} (-e^{x})\,dx + \int_{0}^{a} e^{-x}\,dx$$

$$= \left[-e^{x}\right]_{-a}^{0} + \left[-e^{-x}\right]_{0}^{a}$$

$$= (\cancel{-1} + \cancel{e^{-a}}) + (\cancel{-e^{-a}} + \cancel{1})$$

$$= 0$$

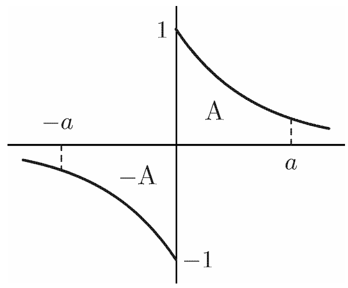


Figure 6

35. **(i)** Show that $\frac{d}{dx}(x\ln x - x) = \ln x$, **(ii)** evaluate $\int_0^1 \ln x\, dx$.

(i) $\frac{d}{dx}x\ln x = x\times\frac{1}{x} + 1\times\ln x = 1 + \ln x$ (product rule)

Therefore $\frac{d}{dx}(x\ln x - x) = (1+\ln x) - 1 = \ln x$

(ii)
$$\int_0^1 \ln x\, dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \ln x\, dx$$
$$= \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{d}{dx}(x\ln x - x)\, dx \qquad \text{(by (i))}$$
$$= \lim_{\varepsilon\to 0}\Big[x\ln x - x\Big]_\varepsilon^1 = \Big[-1\Big] - \lim_{\varepsilon\to 0}\Big[\varepsilon\ln\varepsilon - \varepsilon\Big]$$
$$= -1$$

Evaluate:

36.
$$\int_0^\infty e^{-3t}\, dt = \lim_{b\to\infty}\int_0^b e^{-3t}\, dt = \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3t}\right]_0^b$$
$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3b}\right] + \frac{1}{3} = \frac{1}{3}$$

37.
$$\int_0^\infty e^{-x/2}\, dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\, dx = \lim_{b\to\infty}\Big[-2e^{-x/2}\Big]_0^b = 2$$

38. By partial fractions $\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$.

Therefore
$$\int_2^\infty \frac{dx}{x(x-1)} = \int_2^\infty\left(\frac{1}{x-1} - \frac{1}{x}\right)dx$$
$$= \lim_{b\to\infty}\left[\ln\frac{x-1}{x}\right]_2^b = \lim_{b\to\infty}\left[\ln\frac{b-1}{b}\right] - \ln\frac{1}{2}$$
$$= \ln 1 - \ln\frac{1}{2} = \ln 2$$

39. By partial fractions $\dfrac{1}{x(x-1)^2}=\dfrac{1}{x}-\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}$.

Therefore $$\int_2^\infty \frac{dx}{x(x-1)^2}=\int_2^\infty\left[\frac{1}{x}-\frac{1}{x-1}\right]dx+\int_2^\infty\frac{dx}{(x-1)^2}$$

$$=-\ln 2 \quad \text{(by Exercise 38)}$$

$$+\left[-\frac{1}{x-1}\right]_2^\infty$$

$$=-\ln 2+1$$

For each function, state if it is an even function of *x*, an odd function, or neither. If neither, give the even and odd components.

40. $\sin 2x=-\sin(-2x)$; odd

41. $\cos 3x=+\cos(-3x)$; even

42. $\sin x\cos x=-\sin(-x)\cos(-x)$; $\text{odd}\times\text{even}=\text{odd}$

43. $x=-(-x)$; odd

44. $x^4=+(-x)^4$; even

45. $3x^2+2x+1\neq\pm\left[3(-x)^2+2(-x)+1\right]$; neither (except when $x=0$)

The function has even component $3x^2+1=+\left[3(-x)^2+1\right]$

odd component $2x=-2(-x)$

46. $e^{-x}\neq\pm e^{-(-x)}$; neither (except when $x=0$)

The function has even component $\dfrac{1}{2}\left[e^{-x}+e^{x}\right]$

odd component $\dfrac{1}{2}\left[e^{-x}-e^{x}\right]$

47. $(3x^2+2x+1)e^{-x}=\left[(3x^2+1)+(2x)\right]\times\left[\tfrac{1}{2}(e^{-x}+e^x)+\tfrac{1}{2}(e^{-x}-e^x)\right]$

$=\dfrac{1}{2}\left[(3x^2+1)\times(e^{-x}+e^x)+(x)\times(e^{-x}-e^x)\right]$ $\quad(\text{even}\times\text{even})+(\text{odd}\times\text{odd})=\text{even}$

$+\dfrac{1}{2}\left[(3x^2+1)\times(e^{-x}-e^x)+(x)\times(e^{-x}+e^x)\right]$ $\quad(\text{even}\times\text{odd})+(\text{odd}\times\text{even})=\text{odd}$

Section 5.4

48. The equation of an ellipse with centre at the origin is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where, if $a>b$, a is the major axis and b the minor axis (if $a=b$, we have a circle). Use Method 1 in Example 5.11 to find the area of the ellipse.

In the first quadrant (Figure 7),

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\to\quad y=\frac{b}{a}\sqrt{a^2-x^2}$$

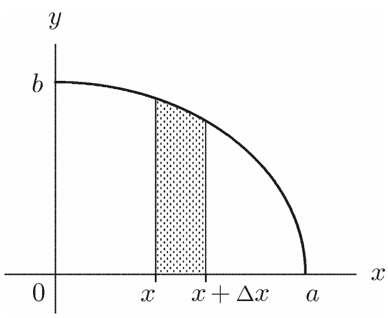


Figure 7

An approximate value of the area of the strip between x and $x+\Delta x$ is

$$\Delta A \approx y\Delta x=\frac{b}{a}\sqrt{a^2-x^2}\,\Delta x$$

and the total area in the first quadrant is

$$A=\int_0^a y\,dx=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\;dx=\frac{b}{a}\times\frac{\pi a^2}{4}=\frac{\pi ab}{4}$$

The total area of the ellipse is therefore $4A=\pi ab$.

49. Find the length of the curve $y=\frac{1}{2}x^{3/2}$ between $x=0$ and $x=1$.

By equation (5.38), the length of the curve is

$$s=\int_0^1\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx=\int_0^1\left[1+\frac{9}{16}x\right]^{1/2}dx$$

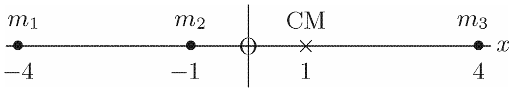
Now $\quad \int(1+ax)^{1/2}dx=\frac{2}{3a}(1+ax)^{3/2}.$

Therefore $\quad s=\left[\frac{2}{3}\times\frac{16}{9}\times\left(1+\frac{16}{9}x\right)^{3/2}\right]_0^1=\frac{32}{27}\left[\frac{125}{64}-1\right]=\frac{61}{54}$

Section 5.6

50. Three masses, $m_1 = 1$, $m_2 = 2$ and $m_3 = 3$, lie on a straight line with m_1 at $x_1 = -4$, m_2 at $x_2 = -1$ and m_3 at $x_3 = +4$ with respect to a point O on the line. Calculate **(i)** the position of the centre of mass, **(ii)** the moment of inertia with respect to O, and **(iii)** the moment of inertia with respect to the centre of mass.

Figure 8



(i) The total mass is $M = 6$, and the centre of mass lies at

$$X = \frac{1}{M}\sum_i m_i x_i = \frac{1}{6}\left[1\times(-4) + 2\times(-1) + 3\times 4\right] = 1$$

(ii) $$I(0) = \sum_i m_i {x_i}^2 = 1\times(-4)^2 + 2\times(-1)^2 + 3\times 4^2 = 66$$

(iii) $$I(X) = \sum_i m_i (x_i - X)^2 = 1\times(-5)^2 + 2\times(-2)^2 + 3\times 3^2 = 60$$

51. The distribution of mass in a straight rod of length l is given by the density function $\rho(x) = x^2;\ 0 \le x \le l$. Find **(i)** the total mass, **(ii)** the mean density, **(iii)** the centre of mass, **(iv)** the moment of inertia with respect to an arbitrary point x_0 on the line, **(v)** the moment of inertia with respect to the centre of mass. **(vi)** Show that the moment of inertia has its smallest value when computed with respect to the centre of mass.

(i) $$M = \int_0^l \rho(x)\,dx = \int_0^l x^2\,dx = l^3/3$$

(ii) $$\bar{\rho} = M/l = l^2/3$$

(iii) $$X = \frac{1}{M}\int_0^l x\rho(x)\,dx = \frac{1}{M}\int_0^l x^3\,dx = \frac{l^4}{4}\Big/\frac{l^3}{3} = 3l/4$$

(iv) $$I(x_0) = \int_0^l (x - x_0)^2 \rho(x)\,dx = \int_0^l (x^2 - 2x_0 x + {x_0}^2)x^2\,dx$$

$$= l^5/5 - x_0\, l^4/2 + {x_0}^2 l^3/3$$

(v) $$I(X) = l^5/5 - X\, l^4/2 + X^2 l^3/3$$

$$= l^5/5 - 3l^5/8 + 16 = l^5/80$$

(vi) $$\frac{d}{dx_0}I(x_0)=\frac{d}{dx_0}\left[l^5/5-x_0\,l^4/2+x_0^{\,2}l^3/3\right]=-l^4/2+2x_0l^3/3$$

$$=0 \text{ when } l/2=2x_0/3,\quad x_0=3l/4$$

The moment of inertia has minimum value when $x_0=3l/4$; that is, when computed with respect to the centre of mass $X=3l/4$.

Section 5.7

52. A body moves in a straight line with velocity $\upsilon=3t^2$ at time *t*. Calculate the distance travelled in time interval **(i)** $t=0\to 1$, **(ii)** $t=1\to 2$, **(iii)** $t=3\to 4$.

We have $$\upsilon=\frac{dx}{dt}=3t^2 \;\to\; x=\int_{t_1}^{t_2}\upsilon\,dt=\left[t_2^{\,3}-t_1^{\,3}\right]$$

Therefore

(i) $t_1=0,\ t_2=1 \quad\to\quad x=1-0=1$

(ii) $t_1=1,\ t_2=2 \quad\to\quad x=8-1=7$

(iii) $t_1=3,\ t_2=4 \quad\to\quad x=64-27=37$

53. A body of mass *m* moves in a straight line (the *x*-direction) under the influence of a force *F* = *kx*. What is the work done on the body between $x=x_{\rm A}$ and $x=x_{\rm B}$?

$$W_{\rm AB}=\int_{\rm A}^{\rm B}F\,dx=k\int_{x_{\rm A}}^{x_{\rm B}}x\,dx=\frac{k}{2}(x_{\rm B}^{\,2}-x_{\rm A}^{\,2})$$

54. A body of mass *m* moves in a straight line (the *x*-direction) under the influence of a force $F = kx$, where *k* is positive (see Exercise 53). **(i)** Find the potential energy $V(x)$ (choose $V(0) = 0$). The body is released from rest at $x = 1$. **(ii)** Find (a) the total energy *E* and (b) the kinetic energy $T(x)$ as functions of *x*. **(iii)** Sketch a graph showing the dependence of $V(x)$, $T(x)$, and *E* on *x*.
(iv) Use the graph to describe the motion of the body. **(v)** What would be the motion if the body were released from rest at (a) $x = -1$, (b) $x = 0$?

(i) By Exercise 53 and Equation (5.56), the work done by the force is

$$W_{\text{AB}} = \frac{k}{2}({x_{\text{B}}}^2 - {x_{\text{A}}}^2) = V(x_{\text{A}}) - V(x_{\text{B}})$$

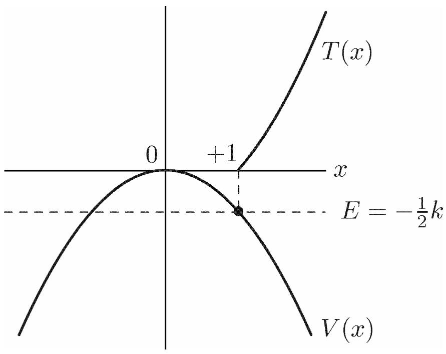
Therefore $\quad V(x) = -\dfrac{1}{2}kx^2$

(ii) The constant total energy is the sum of kinetic and potential energies, $E = T(x) + V(x)$. The body is at rest when $x = 1$**.** Therefore $T(1) = 0$ and

(a) $E = V(1) = -\dfrac{1}{2}k$, (b) $T(x) = E - V(x) = \dfrac{1}{2}k(x^2 - 1)$

(iii)

Figure 9



(iv) The body moves to the right with increasing speed. As in Figure10a it "falls down" the parabolic potential energy surface.

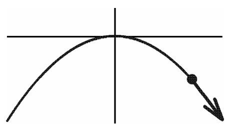


Figure 10a

(v) (a) The body would move to the left with increasing speed:

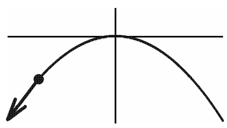


Figure 10b

(b) The body should remain at rest at the top of the potential surface:

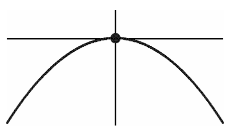


Figure 10c

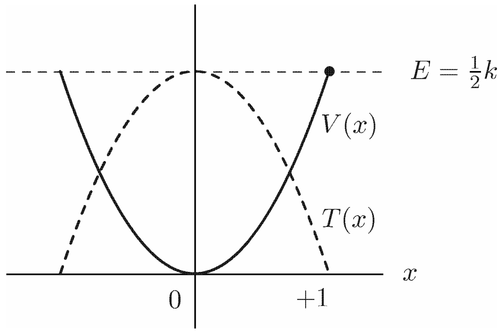
55. Repeat Exercise 54 with $F = -kx$

(i) $$W_{AB} = -\frac{k}{2}({x_B}^2 - {x_A}^2) = V(x_A) - V(x_B)$$

Therefore $$V(x) = \frac{1}{2}kx^2$$

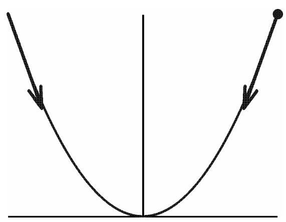
(ii) (a) $E = V(1) = \frac{1}{2}k$ (b) $T(x) = E - V(x) = \frac{1}{2}k(1 - x^2)$

(iii) Figure 11



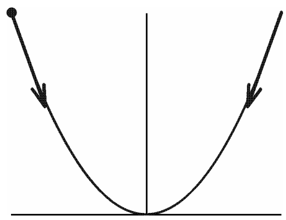
(iv) The body undergoes simple harmonic motion. As in Figure 12a, it moves back and forth between $x = +1$ and $x = -1$ inside the parabolic potential energy surface.

Figure 12a



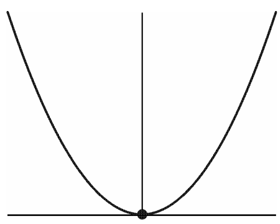
(v) (a) Like case **(iv)**, as in Figure 12b.

Figure 12b



(b) The body would remain at rest at the bottom of the potential surface:

Figure 12c



Section 5.8

56. A slightly imperfect gas obeys the van der Waals equation of state

$$\left(p+\frac{n^2a}{V^2}\right)(V-nb)=nRT$$

Find expressions for the work done by the gas in expanding reversibly from volume V_1 to volume V_2 at **(i)** constant pressure, and **(ii)** constant temperature (assume a and b are constant).

(i) At constant p,

$$W=p\int_{V_1}^{V_2}dV=p(V_2-V_1)$$

(ii) At constant T,

$$p=\frac{nRT}{V-nb}-\frac{n^2a}{V^2}$$

$$W=\int_{V_1}^{V_2}p\,dV=nRT\int_{V_1}^{V_2}\frac{dV}{V-nb}-n^2a\int_{V_1}^{V_2}\frac{dV}{V^2}$$

$$=nRT\ln\left[\frac{V_2-nb}{V_1-nb}\right]+n^2a\left[\frac{1}{V_2}-\frac{1}{V_1}\right]$$