The Chemistry Maths Book

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Second Edition 2008

Solutions

Chapter 4. Differentiation

Section 4.2

1. For $y = x^3$, find **(i)** the change Δy in y that corresponds to change Δx in x, **(ii)** $\Delta y/\Delta x$.

(i) $y + \Delta y = (x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

(ii) $\dfrac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$

2. For $y = x^3$, find $\lim\limits_{\Delta x \to 0}\left(\dfrac{\Delta y}{\Delta x}\right)$

From Exercise 1, $\dfrac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$

Therefore $\lim\limits_{\Delta x \to 0}\dfrac{\Delta y}{\Delta x} = 3x^2$

3. For the Langmuir isotherm

$$\theta = \frac{Kp}{1 + Kp}$$

find **(i)** the change $\Delta\theta$ in θ that corresponds to change Δp in p, **(ii)** $\lim\limits_{\Delta p \to 0}\left(\dfrac{\Delta\theta}{\Delta p}\right)$

(i) $\theta = \dfrac{Kp}{1 + Kp}$, $\theta + \Delta\theta = \dfrac{K(p + \Delta p)}{1 + K(p + \Delta p)}$

Then

$$\Delta\theta = K\left\{\frac{p + \Delta p}{1 + K(p + \Delta p)} - \frac{p}{1 + Kp}\right\} = K\,\frac{\cancel{p} + \Delta p + \cancel{Kp^2} + \cancel{Kp\Delta p} - \cancel{p} - \cancel{Kp^2} - \cancel{Kp\Delta p}}{[1 + K(p + \Delta p)][1 + Kp]}$$

$$= \frac{K\Delta p}{[1 + K(p + \Delta p)][1 + Kp]}$$

(ii) $\dfrac{\Delta\theta}{\Delta p} = \dfrac{K}{[1 + K(p + \Delta p)][1 + Kp]}$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta\theta}{\Delta p}\right) = \frac{K}{(1 + Kp)^2}$$

Section 4.3

Find the discontinuities of the following functions and state which are essential and which removable. Sketch graphs to demonstrate your answers.

4. $\frac{1}{x+1} \to +\infty$ as $x \to -1$ from values $x > -1$ (from the right in Figure 1)

$\to -\infty$ as $x \to -1$ from values $x < -1$ (from the left)

The function has an essential discontinuity at $x = -1$.

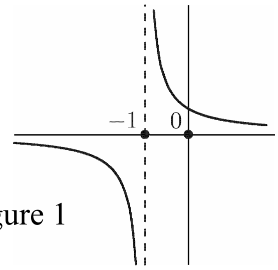


Figure 1

5. $\frac{x^2}{x} = x$ if $x \neq 0$, but is not defined at $x = 0$.

The function has a removable discontinuity at $x = 0$.

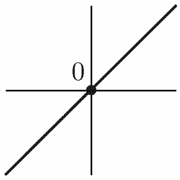


Figure 2

6. $\frac{2x}{x^2 - 3x} = \frac{2x}{x(x-3)}$

The function has a removable discontinuity at $x = 0$ and an essential discontinuity at $x = 3$.

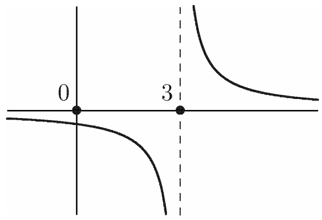


Figure 3

Section 4.4

Find the limits:

7. $\lim_{x\to 0}\left(\frac{x^2}{x}\right)$: $\frac{x^2}{x} = x$ if $x \neq 0$, so that $\lim_{\Delta x\to 0}\left(\frac{x^2}{x}\right) = 0$

8. $\lim_{x\to 0}\left(\frac{x}{x^2}\right)$: $\frac{x}{x^2} = \frac{1}{x}$ if $x \neq 0$, so that $\frac{x}{x^2} \to \pm\infty$ as $x \to 0$

9. $\lim_{x\to 0}\left(\frac{x+1}{x+3}\right)$: $\frac{x+1}{x+3}$ is continuous at $x = 0$, and $\lim_{x\to 0}\left(\frac{x+1}{x+3}\right) = \frac{1}{3}$

10. $\lim_{x\to 1}\left(\frac{x-1}{x^2-1}\right)$: $\frac{x-1}{x^2-1} = \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{1}{x+1}$ if $x \neq 1$.

Therefore $\lim_{x\to 1}\left(\frac{x-1}{x^2-1}\right) = \lim_{\Delta x\to 1}\left(\frac{1}{x+1}\right) = \frac{1}{2}$.

11. $\lim_{x\to\infty}\left(\frac{x+1}{x+3}\right)$: $\frac{x+1}{x+3}\to\frac{x}{x}$ as $x\to\infty$.

Therefore $\lim_{x\to\infty}\left(\frac{x+1}{x+3}\right)=1.$

12. $\lim_{x\to\infty}\left(\frac{x-1}{x^2-1}\right)$**:** $\frac{x-1}{x^2-1}\to\frac{x}{x^2}=\frac{1}{x}$ as $x\to\infty$.

Therefore $\lim_{x\to\infty}\left(\frac{x-1}{x^2-1}\right)=0$

13. $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1}\right)$**:** $\frac{x^2-1}{x-1}\to\frac{x^2}{x}=x$ as $x\to\infty$.

Therefore $\frac{x^2-1}{x+1}\to\infty$ as $x\to\infty$

14. $\lim_{x\to 0}\left[\left(4x^2-\frac{1}{x^2}\right)+\left(2x-\frac{1}{x}\right)^2\right]$**:**

We have $\left(4x^2-\frac{1}{x^2}\right)+\left(2x-\frac{1}{x}\right)^2=4x^2-\frac{1}{x^2}+4x^2-4+\frac{1}{x^2}$

$=8x^2-4$ when $x\neq 0$

Therefore $\lim_{x\to 0}\left[\left(4x^2-\frac{1}{x^2}\right)+\left(2x-\frac{1}{x}\right)^2\right]=-4$

15. $\lim_{x\to 0}\left(\frac{e^{2x}-1}{x}\right)$**:**

We have $\frac{e^{2x}-1}{x}=\frac{(1+2x+2x^2+\cdots)-1}{x}=\frac{2x+2x^2+\cdots}{x}$

$=2+2x+\cdots$ when $x\neq 0$

Therefore $\lim_{x\to 0}\left(\frac{e^{2x}-1}{x}\right)=2$

16. $\lim_{x\to 0}(\ln x-\ln 2x)$**:**

We have $\ln x-\ln 2x=\ln\frac{x}{2x}=\ln\frac{1}{2}$ when $x\neq 0$

Therefore $\lim_{x\to 0}(\ln x-\ln 2x)=-\ln 2$

17. $\lim_{x\to\infty}\left[\ln(x-4)-\ln(3x+2)\right]$**:**

We have $\ln(x-4)-\ln(3x+2)=\ln\frac{x-4}{3x+2}\to\ln\frac{x}{3x}=\ln\frac{1}{3}$ as $x\to\infty$

Therefore $\lim_{x\to\infty}\left[\ln(x-4)-\ln(3x+2)\right]=-\ln 3$

Section 4.5

Differentiate from first principles:

18 $2x^2+3x+4$: Let $y=2x^2+3x+4$

Then $$y+\Delta y = 2(x+\Delta x)^2+3(x+\Delta x)+4$$
$$= (2x^2+3x+4)+(4x+3)\Delta x+2(\Delta x)^2$$

$$\Delta y = (4x+3)\Delta x + 2(\Delta x)^2$$

Therefore $$\frac{\Delta y}{\Delta x} = 4x+3+2\Delta x, \quad \frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = 4x+3$$

19. $y=x^4$ **:** $$y+\Delta y = (x+\Delta x)^4$$
$$= x^4+4x^3\Delta x+6x^2(\Delta x)^2+4x(\Delta x)^3+(\Delta x)^4$$

$$\Delta y = 4x^3\Delta x+6x^2(\Delta x)^2+4x(\Delta x)^3+(\Delta x)^4$$

Therefore $$\frac{\Delta y}{\Delta x} = 4x^3+6x^2(\Delta x)+4x(\Delta x)^2+(\Delta x)^3, \quad \frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = 4x^3$$

20. $y=\frac{2}{x^2}$: $$y+\Delta y = \frac{2}{(x+\Delta x)^2}$$

$$\Delta y = 2\left[\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}\right] = \frac{-4x\Delta x-2(\Delta x)^2}{x^2(x+\Delta x)^2}$$

Therefore $$\frac{\Delta y}{\Delta x} = \frac{-4x-2(\Delta x)}{x^2(x+\Delta x)^2}, \quad \frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{-4x}{x^4} = -\frac{4}{x^3}$$

21. $y=x^{3/2}$: $$y+\Delta y = (x+\Delta x)^{3/2}$$

$$\Delta y = (x+\Delta x)^{3/2}-x^{3/2}$$
$$= \frac{\left[(x+\Delta x)^{3/2}-x^{3/2}\right]\times\left[(x+\Delta x)^{3/2}+x^{3/2}\right]}{(x+\Delta x)^{3/2}+x^{3/2}}$$
$$= \frac{(x+\Delta x)^3-x^3}{(x+\Delta x)^{3/2}+x^{3/2}} = \frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{(x+\Delta x)^{3/2}+x^{3/2}}$$

Therefore $$\frac{\Delta y}{\Delta x} = \frac{3x^2+3x\Delta x+(\Delta x)^2}{(x+\Delta x)^{3/2}+x^{3/2}}, \quad \frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2}$$

22. $y = e^{-x}$ **:** $y + \Delta y = e^{-(x+\Delta x)} = e^{-x} \times e^{-\Delta x} = e^{-x}(1 - \Delta x + \frac{(\Delta x)^2}{2} - \cdots)$

$$\Delta y = e^{-x}(-\Delta x + \frac{(\Delta x)^2}{2} - \cdots)$$

Therefore $\frac{\Delta y}{\Delta x} = -e^{-x} + \frac{\Delta x}{2} e^{-x} + \cdots, \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -e^{-x}$

Section 4.6

Differentiate by rule:

23. x^3 : $\frac{d}{dx} x^3 = 3 \times x^{3-1} = 3x^2$

24. $x^{5/4}$ **:** $\frac{d}{dx} x^{5/4} = \frac{5}{4} x^{(5/4)-1} = \frac{5}{4} x^{1/4}$

25. $x^{1/3}$: $\frac{d}{dx} x^{1/3} = \frac{1}{3} x^{(1/3)-1} = \frac{1}{3} x^{-2/3}$

26. $1/x^3$: $\frac{d}{dx} 1/x^3 = \frac{d}{dx} x^{-3} = -3x^{-4} = -3/x^4$

27. $\frac{d}{dx}(1 - 2x + 3x^2 - 4x^3 + 5\sin x - 6\cos x + 7e^x - 8\ln x)$

$$= \frac{d}{dx}(1) - 2\frac{d}{dx}(x) + 3\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x^3) + 5\frac{d}{dx}(\sin x) - 6\frac{d}{dx}(\cos x) + 7\frac{d}{dx}(e^x) - 8\frac{d}{dx}(\ln x)$$

$$= 0 - 2 \times 1 + 3 \times (2x) - 4 \times (3x^2) + 5 \times \cos x - 6 \times (-\sin x) + 7 \times e^x - 8 \times \frac{1}{x}$$

$$= -2 + 6x - 12x^2 + 5\cos x + 6\sin x + 7e^x - 8/x$$

28. The virial equation of state of a gas at low pressure is

$$pV = nRT\left(1 - \frac{nB}{V}\right).$$

Find $\frac{dp}{dV}$ at constant T and n (assume B is also constant).

$$pV = nRT\left(1 - \frac{nB}{V}\right) \to p = nRT\left(\frac{1}{V} - \frac{nB}{V^2}\right)$$

Then $\frac{dp}{dV} = nRT\left[\frac{d}{dV}\left(\frac{1}{V}\right) - nB\frac{d}{dV}\left(\frac{1}{V^2}\right)\right] = nRT\left[-\frac{1}{V^2} + 2nB\frac{1}{V^3}\right]$

$$= \frac{nRT}{V^3}\left[-V + 2nB\right]$$

Products and quotients

Differentiate

29. $(1-4x^2)\cos x$ **:**

Put $\quad u\times v=(1-4x^2)\times\cos x$

Then $\quad \dfrac{d}{dx}uv=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$

$$=(1-4x^2)\times\frac{d}{dx}\cos x+(\cos x)\times\frac{d}{dx}(1-4x^2)$$

$$=(1-4x^2)\times(-\sin x)+(\cos x)\times(-8x)$$

$$=-(1-4x^2)\sin x-8x\cos x$$

30. $(2+3x)e^x$ **:**

$$\frac{d}{dx}(2+3x)e^x=(2+3x)\times\frac{d}{dx}e^x+e^x\times\frac{d}{dx}(2+3x)$$

$$=(2+3x)\times e^x+e^x\times 3$$

$$=(5+3x)e^x$$

31. $e^x\cos x$ **:**

$$\frac{d}{dx}e^x\cos x=e^x\frac{d}{dx}\cos x+\cos x\frac{d}{dx}e^x=e^x\times(-\sin x)+(\cos x)\times e^x$$

$$=e^x(\cos x-\sin x)$$

32. $x\ln x$ **:**

$$\frac{d}{dx}x\ln x=x\frac{d}{dx}\ln x+\ln x\frac{d}{dx}x=x\times\frac{1}{x}+\ln x\times 1$$

$$=1+\ln x$$

33. $(1+2x+3x^2)\big/(3+x^3)$ **:**

Put $\quad u/v=(1+2x+3x^2)\big/(3+x^3)$

Then $\quad \dfrac{d}{dx}u/v=\left(v\dfrac{du}{dx}-u\dfrac{dv}{dx}\right)\Big/v^2$

$$=\left((3+x^3)\frac{d}{dx}(1+2x+3x^2)-(1+2x+3x^2)\frac{d}{dx}(3+x^3)\right)\Big/(3+x^3)^2$$

$$=\frac{(3+x^3)\times(2+6x)-(1+2x+3x^2)(3x^2)}{(3+x^3)^2}$$

$$=\frac{6+18x-3x^2-4x^3-3x^4}{(3+x^3)^2}$$

34. $(1-4x^2)\big/\sin x$ **:**

$$\frac{d}{dx}(1-4x^2)\big/\sin x = \left(\sin x\frac{d}{dx}(1-4x^2) - (1-4x^2)\frac{d}{dx}\sin x\right)\bigg/\sin^2 x$$

$$= \frac{(\sin x)\times(-8x) - (1-4x^2)\times(\cos x)}{\sin^2 x}$$

$$= -\left[\frac{8x\sin x + (1-4x^2)\cos x}{\sin^2 x}\right]$$

35. $\cos x\big/\sin x$ **:**

$$\frac{d}{dx}\cos x\big/\sin x = \left(\sin x\frac{d}{dx}\cos x - \cos x\frac{d}{dx}\sin x\right)\bigg/\sin^2 x$$

$$= \frac{(\sin x)\times(-\sin x) - (\cos x)\times(\cos x)}{\sin^2 x}$$

$$= -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

36. $(\ln x)\big/x$ **:**

$$\frac{d}{dx}(\ln x)\big/x = \left(x\frac{d}{dx}\ln x - \ln x\frac{d}{dx}x\right)\bigg/x^2 = \left(x\times\frac{1}{x} - (\ln x)\times 1\right)\bigg/x^2$$

$$= \left(1-\ln x\right)\big/x^2$$

Chain rule

Differentiate

37. $(1+x)^5$ **:**

Write $y = u^5$, where $u = (1+x)$

Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx} = (5u^4)\times(1)$

$$= 5(1+x)^4$$

38. $\sqrt{2+x^2}$ **:**

$y = u^{1/2}$, where $u = (2+x^2)$

$$\frac{dy}{dx} = \frac{1}{2}u^{-1/2}\times(2x) = x(2+x^2)^{-1/2} = \frac{x}{\sqrt{2+x^2}}$$

39. $\dfrac{1}{3-x^2}$ **:**

$y = u^{-1}$, where $u = 3-x^2$

$$\frac{dy}{dx} = -u^{-2}\times(-2x) = 2x(3-x^2)^{-2} = \frac{2x}{(3-x^2)^2}$$

40. $\dfrac{3}{(2x^2-3x-1)^{1/2}}$ **:**

$$y = 3u^{-1/2},\ \text{where}\ u = (2x^2-3x-1)$$

$$\frac{dy}{dx} = -\frac{3}{2}u^{-3/2} \times (4x-3) = -\frac{3}{2}(4x-3)\big/(2x^2-3x-1)^{3/2}$$

41. $\sin 4x$ **:**

$$y = \sin u,\ \text{where}\ u = 4x$$

$$\frac{dy}{dx} = \cos u \times (4) = 4\cos 4x$$

42. e^{-2x} **:**

$$y = e^u,\ \text{where}\ u = -2x$$

$$\frac{dy}{dx} = e^u \times (-2) = -2e^{-2x}$$

43. e^{2x^2-3x+1} **:**

$$y = e^u,\ \text{where}\ u = 2x^2-3x+1$$

$$\frac{dy}{dx} = e^u \times (4x-3) = (4x-3)e^{2x^2-3x+1}$$

44. $\ln(2x^2-3x+1)$ **:**

$$y = \ln u,\ \text{where}\ u = 2x^2-3x+1$$

$$\frac{dy}{dx} = \frac{1}{u} \times (4x-3) = \frac{(4x-3)}{2x^2-3x+1}$$

45. $\cos(2x^2-3x+1)$ **:**

$$y = \cos u,\ \text{where}\ u = 2x^2-3x+1$$

$$\frac{dy}{dx} = -\sin u \times (4x-3) = -(4x-3)\sin(2x^2-3x+1)$$

46. $e^{\sin x}$ **:**

$$y = e^u,\ \text{where}\ u = \sin x$$

$$\frac{dy}{dx} = e^u \times \cos x = e^{\sin x}\cos x$$

47. $\ln(\cos x)$ **:**

$$y = \ln u,\ \text{where}\ u = \cos x$$

$$\frac{dy}{dx} = \frac{1}{u} \times (-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$$

48. $e^{-\cos(3x^2+2)}$ **:**

Let $y = e^u$, where $u = -\cos(3x^2+2)$

$= -\cos v$, where $v = 3x^2+2$

We have $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and $\frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx}$.

Therefore $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$

$= (e^u) \times (\sin v) \times (6x)$

$= 6x \sin(3x^2+2) e^{-\cos(3x^2+2)}$

49. $\ln\left(\frac{2+x}{3-x}\right)$ **:**

$y = \ln(2+x) - \ln(3-x) = \ln u - \ln v$, where $u = 2+x$ and $v = 3-x$

$\frac{dy}{dx} = \frac{1}{u} \times 1 - \frac{1}{v} \times (-1)$

$= \frac{1}{2+x} + \frac{1}{3-x}$

50. $\ln(\sin 2x + \sin^2 x)$ **:**

We have $y = \ln u$, where $u = \sin 2x + \sin^2 x$

and

$u = \sin v + w^2$, where $v = 2x$ and $w = \sin x$

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{du} \times \left[\frac{d}{dx}\sin v + \frac{d}{dx} w^2\right]$

$= \frac{1}{u}\left[\cos v \times 2 + 2w \times \cos x\right]$

$= \frac{2\cos 2x + 2\sin x \cos x}{\sin 2x + \sin^2 x} = \frac{2\cos 2x + \sin 2x}{\sin 2x + \sin^2 x}$

51. $3x^2(2+x)^{1/2}$ **:**

$y = 3x^2(2+x)^{1/2} = u \times v$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 3x^2 \times \frac{1}{2}(2+x)^{-1/2} + (2+x)^{1/2} \times 6x$

$= 6x(2+x)^{1/2} + \frac{3x^2}{2}(2+x)^{-1/2}$

52. $\sin x \cos 2x$ **:**

We have $y = \sin x \cos 2x = u \times v$

$$u = \sin x,\ \frac{du}{dx} = \cos x; \quad v = \cos 2x,\ \frac{dv}{dx} = -2\sin 2x$$

Therefore $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = \sin x \times (-2\sin 2x) + \cos 2x \times \cos x$

$$= \cos x \cos 2x - 2\sin x \sin 2x$$

53. $\tan 4x \cos^2 2x$ **:**

We have $y = \tan 4x \cos^2 2x = u \times v$

$$u = \tan 4x,\quad \frac{du}{dx} = 4\sec^2 4x$$

$$v = \cos^2 2x,\ \frac{dv}{dx} = (2\cos 2x) \times (-2\sin 2x) = -2\sin 4x$$

Therefore $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = (\tan 4x) \times (-2\sin 4x) + (\cos^2 2x) \times (4\sec^2 4x)$

$$= 4\cos^2 2x \sec^2 4x - 2\tan 4x \sin 4x$$

54. $x^2 e^{2x^2+3}$ **:**

We have $y = x^2 e^{2x^2+3} = u \times v$

$$u = x^2,\ \frac{du}{dx} = 2x; \quad v = e^{2x^2+3},\ \frac{dv}{dx} = 4xe^{2x^2+3}$$

Therefore $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = x^2 \times 4xe^{2x^2+3} + e^{2x^2+3} \times 2x$

$$= 2x(1+2x^2)e^{2x^2+3}$$

55. $\dfrac{3x^2}{(2+x^2)^{1/2}}$ **:**

We have $y = 3x^2(2+x^2)^{-1/2} = u \times v$

$$u = 3x^2,\ \frac{du}{dx} = 6x; \quad v = (2+x^2)^{-1/2},\ \frac{dv}{dx} = -x(2+x^2)^{-3/2}$$

Therefore $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx} = 3x^2 \times (-x)(2+x^2)^{-3/2} + 6x \times (2+x^2)^{-1/2}$

$$= \frac{-3x^3}{(2+x^2)^{3/2}} + \frac{6x}{(2+x^2)^{1/2}} = \frac{3x(x^2+4)}{(2+x^2)^{3/2}}$$

Inverse functions

56. If $x = 2y^2 - 3y + 1$, find $\frac{dy}{dx}$.

We have $x = 2y^2 - 3y + 1, \quad \frac{dx}{dy} = 4y - 3$

Therefore $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} = \frac{1}{4y-3}$

Find $\frac{dV}{dp}$ at constant T and n for the following equations of state (assume that B, a and b are constants).

57. $pV = nRT\left(1 + \frac{nB}{V}\right)$**:**

$$pV = nRT\left(1 + \frac{nB}{V}\right) \to p = nRT\left(\frac{1}{V} + \frac{nB}{V^2}\right)$$

$$\frac{dp}{dV} = nRT\left(-\frac{1}{V^2} - 2\frac{nB}{V^3}\right) = -\frac{nRT}{V^3}(V + 2nB)$$

$$\frac{dV}{dp} = \frac{-V^3}{nR(V + 2nB)T}$$

58. $p(V - nb) - nRT = 0$:

$$p(V - nb) - nRT = 0 \to p = \frac{nRT}{V - nb}$$

$$\frac{dp}{dV} = -\frac{nRT}{(V - nb)^2} = -\frac{p}{V - nb}$$

$$\frac{dV}{dp} = -\frac{V - nb}{p}$$

59. $\left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$ **:**

$$\left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \to p = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}$$

$$\frac{dp}{dV} = -\frac{nRT}{(V - nb)^2} + \frac{2n^2 a}{V^3}$$

$$\frac{dV}{dp} = \left[\frac{2n^2 a}{V^3} - \frac{nRT}{(V - nb)^2}\right]^{-1}$$

Differentiate

60. $\sin^{-1} 2x$ **:**

We have $\quad y = \sin^{-1} 2x \to x = \frac{1}{2}\sin y$

Therefore $\quad \frac{dx}{dy} = \frac{1}{2}\cos y \to \frac{dy}{dx} = \frac{2}{\cos y} = \frac{2}{\sqrt{1-4x^2}}$

(or by formula from Table 2.5 with $a = 1/2$)

61. $\tan^{-1} x^2$ **:**

We have $\quad y = \tan^{-1} x^2$

$$= \tan^{-1} u, \text{ where } u = x^2 \to \frac{du}{dx} = 2x$$

Therefore $\quad \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = 2x\frac{dy}{du}$

Then, by formula from Table 4.5,

$$\frac{dy}{dx} = 2x \times \frac{1}{1+u^2} = \frac{2x}{1+x^4}$$

62. $\cos^{-1}\left(\frac{1-x}{1+x}\right)$**:**

We have $\quad y = \cos^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1} u, \text{ where } u = \frac{1-x}{1+x} \to \frac{du}{dx} = \frac{-2}{(1+x)^2}$

Then, by formula from Table 4.5,

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \left(\frac{-1}{\sqrt{1-u^2}}\right)\times\left(\frac{-2}{(1+x)^2}\right)$$

Now $\quad 1-u^2 = 1-\left(\frac{1-x}{1+x}\right)^2 = \frac{4x}{(1+x)^2}$

Therefore $\quad \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \left(-\frac{(1+x)}{2\sqrt{x}}\right)\times\left(\frac{-2}{(1+x)^2}\right)$

$$= \frac{1}{\sqrt{x}(1+x)}$$

63. $\sinh^{-1} 2x$ **:**

By formula from Table 4.6 with $a = 1/2$,

$$\frac{d}{dx}\sinh^{-1} 2x = \frac{1}{\sqrt{x^2 + (1/2)^2}} = \frac{2}{\sqrt{4x^2+1}}$$

64. $\tanh^{-1} x^2$ **:**

By formula from Table 4.6,

$$y = \tanh^{-1} x^2 = \tanh^{-1} u, \text{ where } u = x^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1-u^2} \times 2x = \frac{2x}{1-x^4}$$

Section 4.7

Find $\frac{dy}{dx}$:

65. $x^2 + y^2 = 4$:

$$\frac{d}{dx}\left[x^2 + y^2 = 4\right] \to 2x + 2y\frac{dy}{dx} = 0$$

Therefore $\frac{dy}{dx} = -\frac{x}{y}$

66. $y^3 + 3x + x^2 - 1 = 0$:

$$\frac{d}{dx}\left[y^3 + 3x + x^2 - 1 = 0\right] \to 3y^2\frac{dy}{dx} + 3 + 2x = 0$$

Therefore $\frac{dy}{dx} = -\frac{3+2x}{3y^2}$

67. $x = y \ln xy$:

$$\frac{d}{dx}\left[x = y\ln xy\right] \to 1 = y \times \frac{d}{dx}\ln xy + \ln xy \times \frac{dy}{dx}$$

$$= y \times \left[\frac{1}{x} + \frac{1}{y}\frac{dy}{dx}\right] + \ln xy \times \frac{dy}{dx} = \frac{y}{x} + (1 + \ln xy)\frac{dy}{dx}$$

Therefore $\frac{dy}{dx} = \frac{x-y}{x(1+\ln xy)}$

68. $y^2 + \frac{2}{y} - x^2y^2 + 3x + 2 = 0$:

$$\frac{d}{dx}\left[y^2 + \frac{2}{y} - x^2y^2 + 3x + 2 = 0\right] \to 2y\frac{dy}{dx} - \frac{2}{y^2}\frac{dy}{dx} - 2xy^2 - 2x^2y\frac{dy}{dx} + 3 = 0$$

$$\to \left[2y - \frac{2}{y^2} - 2x^2y\right]\frac{dy}{dx} = 2xy^2 - 3$$

Therefore $\frac{dy}{dx} = \frac{2xy^2 - 3}{2y - 2/y^2 - 2x^2y}$

Section 4.8

Differentiate:

69. $\left(\dfrac{3-x}{4+x}\right)^{1/3}$:

We have $y=\left(\dfrac{3-x}{4+x}\right)^{1/3} \rightarrow \ln y=\dfrac{1}{3}\left[\ln(3-x)-\ln(4+x)\right]$

Therefore $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{1}{3}\left[-\dfrac{1}{3-x}-\dfrac{1}{4+x}\right]=-\dfrac{7}{3(3-x)(4+x)}$

$$\frac{dy}{dx}=-\frac{7}{3(3-x)(4+x)}\left(\frac{3-x}{4+x}\right)^{1/3}$$

70. $\dfrac{(1+x^2)(x-1)^{1/2}}{(2x+1)(3x^2+2x-1)^{1/3}}$:

We have $y=\dfrac{(1+x^2)(x-1)^{1/2}}{(2x+1)(3x^2+2x-1)^{1/3}}$

$$\rightarrow\ \ln y=\ln(1+x^2)+\frac{1}{2}\ln(x-1)-\ln(2x+1)-\frac{1}{3}\ln(3x^2+2x-1)$$

Therefore $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{2x}{1+x^2}+\dfrac{1}{2(x-1)}-\dfrac{2}{2x+1}-\dfrac{6x+2}{3(3x^2+2x-1)}$

$$\frac{dy}{dx}=\left[\frac{2x}{1+x^2}+\frac{1}{2(x-1)}-\frac{2}{2x+1}-\frac{6x+2}{3(3x^2+2x-1)}\right]\times\frac{(1+x^2)(x-1)^{1/2}}{(2x+1)(3x^2+2x-1)^{1/3}}$$

71. $\sin^{1/2}x\cos^3(x^2+1)\tan^{1/3}2x$:

We have $y=\sin^{1/2}x\cos^3(x^2+1)\tan^{1/3}2x$

$$\rightarrow \ln y=\frac{1}{2}\ln(\sin x)+3\ln(\cos(x^2+1))+\frac{1}{3}\ln(\tan 2x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\frac{1}{\sin x}\times\cos x+\frac{3}{\cos(x^2+1)}\times(-\sin(x^2+1))\times 2x+\frac{1}{3}\frac{1}{\tan 2x}\times\sec^2 2x\times 2$$

$$=\left[\frac{1}{2}\cot x-6x\tan(x^2+1)+\frac{4}{3\sin 4x}\right]$$

Therefore $\dfrac{dy}{dx}=\left[\dfrac{1}{2}\cot x-6x\tan(x^2+1)+\dfrac{4}{3\sin 4x}\right]\times\sin^{1/2}x\cos^3(x^2+1)\tan^{1/3}2x$

72. Show that the equations

$$\frac{d\ln p}{dT}=\frac{\Delta H_{\text{vap}}}{RT^2} \quad \text{and} \quad \frac{dp}{dT}=\frac{p\Delta H_{\text{vap}}}{RT^2}$$

are equivalent expressions of the Clausius-Clapeyron equation.

We have $$\frac{d\ln p}{dT}=\frac{1}{p}\frac{dp}{dT}$$

Therefore $$\frac{1}{p}\frac{dp}{dT}=\frac{\Delta H_{\text{vap}}}{RT^2} \quad \text{and} \quad \frac{dp}{dT}=\frac{p\Delta H_{\text{vap}}}{RT^2}$$

73. The decomposition of dinitrogen pentoxide in tetrachloromethane at $T = 45\ °\text{C}$ has stoichiometry:

$$N_2O_5 \rightarrow 2NO_2 + \tfrac{1}{2}O_2$$

and obeys first-order kinetics. From the volumes of oxygen liberated after various times *t*, the following concentrations of N_2O_5 were obtained:

$x=[N_2O_5]/\text{mol dm}^{-3}$	2.33	1.91	1.36	1.11	0.72	0.55
t/s	0	319	867	1196	1877	2315

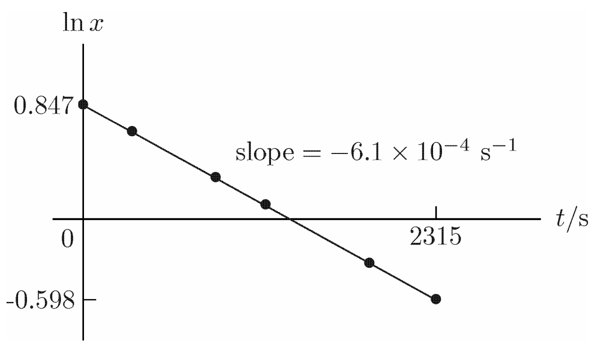
Plot a graph of $\ln x$ against t/s and determine the rate constant.

We have

$\ln x$	0.846	0.647	0.307	0.104	–0.329	–0.598
t/s	0	319	867	1196	1877	2315

In first-order kinetics, a plot of $\ln x$ against t gives a straight line with slope $d(\ln x)/dt = -k$. The tabulated values for the decomposition of N_2O_5 are plotted in Figure 4.

Figure 4



In the present case, the straight line fit obtained by the method of least squares (Section 21.10) is almost identical to that obtained by drawing a straight line through the end points. The slope corresponds to rate constant $k = 6.1\times10^{-4}\ \text{s}^{-1}$

Section 4.9

74. Find all the nonzero derivatives of the function $y = 3x^5 + 4x^4 - 3x^3 + x^2 - 2x + 1$.

$$y = 3x^5 + 4x^4 - 3x^3 + x^2 - 2x + 1$$

$$y' = 3\times 5x^4 + 4\times 4x^3 - 3\times 3x^2 + 2x - 2$$

$$= 15x^4 + 16x^3 - 9x^2 + 2x - 2$$

$$y'' = 15\times 4x^3 + 16\times 3x^2 - 9\times 2x + 2$$

$$= 60x^3 + 48x^2 - 18x + 2$$

$$y''' = 60\times 3x^2 + 48\times 2x - 18$$

$$= 180x^2 + 96x - 18$$

$$y^{(4)} = 360x + 96$$

$$y^{(5)} = 360$$

75. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ for the function $y = \ln x$.

$$y = \ln x$$

$$\frac{dy}{dx} = 1/x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^{-1}) = -x^{-2} = -1/x^2$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(-x^{-2}) = 2x^{-3} = 2/x^3$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}(2x^{-3}) = -6x^{-4} = -6/x^4$$

76. Find a general formula for the *n*th derivative of e^{3x}.

We have $\frac{d}{dx}e^{3x} = 3e^{3x}$, $\frac{d^2}{dx^2}e^{3x} = 3^2 e^{3x}$, $\frac{d^3}{dx^3}e^{3x} = 3^3 e^{3x}$, and so on

Therefore $\frac{d^{(n)}}{dx^{(n)}}e^{3x} = 3^n e^{3x}$

77. Find a general formula for the *n*th derivative of $\cos 2x$.

$$y = \cos 2x$$
$$y' = -2\sin 2x$$
$$y'' = -2^2 \cos 2x$$
$$y^{(3)} = +2^3 \sin 2x$$
$$y^{(4)} = +2^4 \cos 2x$$
$$y^{(5)} = -2^5 \sin 2x \quad \text{and so on}$$

Then $y^{(n)} = (-1)^{n/2} 2^n \cos 2x$ when n is an even integer (or zero),

$y^{(n)} = (-1)^{(n+1)/2} 2^n \sin 2x$ when n is an odd integer.

Section 4.10

Find the maximum and minimum values and the points of inflection of the following functions. In each case, sketch the graph and show the positions of these points.

78. $y = x^2 - 3x + 2$:

We have $y = x^2 - 3x + 2,\quad \dfrac{dy}{dx} = 2x - 3$
$= 0$ when $x = 3/2$

The quadratic has a single stationary value at

$$x = \frac{3}{2}, \text{ when } y = \left(\frac{3}{2}\right)^2 - 3\times\frac{3}{2} + 3 = -\frac{1}{4}$$

The second derivative $\dfrac{d^2y}{dx^2} = 2$ is positive, so that the stationary value is a (local) minimum value.

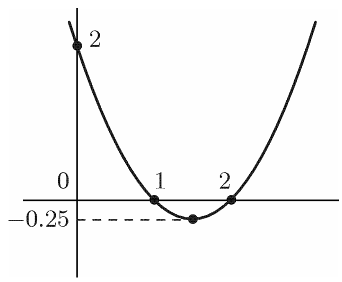


Figure 5

The sketch of the graph should look like Figure 5.

79. $y = x^3 - 7x^2 + 15x - 9$ **:**

$$y = x^3 - 7x^2 + 15x - 9$$

$$\frac{dy}{dx} = 3x^2 - 14x + 15 = (x-3)(3x-5)$$
$$= 0 \text{ when } x = 3 \text{ and } x = 5/3$$

$$\frac{d^2y}{dx^2} = 6x - 14 \begin{cases} > 0 & \text{when } x = 3, \quad \text{a minimum point} \\ < 0 & \text{when } x = 5/3, \text{ a maximum point} \\ = 0 & \text{when } x = 7/3, \text{ a point of inflection} \end{cases}$$

The graph of the function has

minimum point at $x = 3$, when $y = 0$

maximum point at $x = 5/3$, when $y = 32/27$

point of inflection at $x = 7/3$, when $y = 16/27$

The sketch of the graph should look like Figure 6.

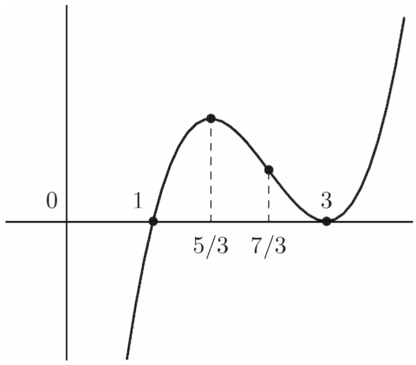


Figure 6

80. $y = 4x^3 + 6x^2 + 3$

$$y = 4x^3 + 6x^2 + 3$$

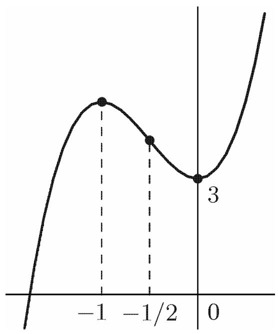
$$\frac{dy}{dx} = 12x^2 + 12x = 12x(x+1)$$

$$= 0 \text{ when } x = 0 \text{ and } x = -1$$

$$\frac{d^2y}{dx^2} = 24x + 12 \begin{cases} > 0 & \text{when } x = 0,\ y = 3 & \text{a minimum point} \\ < 0 & \text{when } x = -1,\ y = 5 & \text{a maximum point} \\ = 0 & \text{when } x = -1/2,\ y = 4 & \text{a point of inflection} \end{cases}$$

The sketch of the graph should look like Figure 7.

Figure 7



81. $y = xe^{-x}$ (see Figure 3.20)

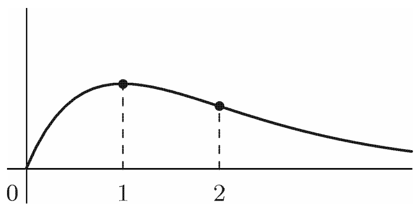
$$y = xe^{-x}$$

$$\frac{dy}{dx} = (1-x)e^{-x} = 0 \text{ when } x = 1$$

$$\frac{d^2y}{dx^2} = -(2-x)e^{-x} \begin{cases} < 0 & \text{when } x = 1,\ y = e^{-1} & \text{a maximum point} \\ = 0 & \text{when } x = 2,\ y = 2e^{-2} & \text{a point of inflection} \end{cases}$$

The sketch of the graph should look like Figure 8.

Figure 8



82. Confirm that the cubic $y = x^3 - 7x^2 + 16x - 10$, discussed in Example 2.23, has local maximum and minimum values at $x = 2$ and $x = 8/3$.

$$y = x^3 - 7x^2 + 16x - 10$$

$$\frac{dy}{dx} = 3x^2 - 14x + 16 = (3x-8)(x-2)$$

$$= 0 \text{ when } x = 8/3 \text{ and } x = 2$$

$$\frac{d^2y}{dx^2} = 6x - 14 \begin{cases} = +2 > 0 & \text{when } x = 8/3 & \text{a minimum point} \\ = -2 < 0 & \text{when } x = 2 & \text{a maximum point} \end{cases}$$

83. Find the maximum and minimum values and the points of inflection of $y = 2x^5 - 5x^4 + 3$. Sketch a graph to show the positions of these points.

$$y = 2x^5 - 5x^4 + 3$$

$$\frac{dy}{dx} = 10x^4 - 20x^3 = 10x^3(x-2)$$

$$= 0 \text{ when } x = 0, \text{ a triple root}$$

$$\text{and } x = 2$$

$$\frac{d^2y}{dx^2} = 40x^3 - 60x^2 = 20x^2(2x-3) \begin{cases} > 0 & \text{when } x = 2,\ y = -13 & \text{a minimum point} \\ = 0 & \text{when } x = 0,\ y = 3 & ? \\ = 0 & \text{when } x = 3/2,\ y = -57/8 & \text{a point of inflection} \end{cases}$$

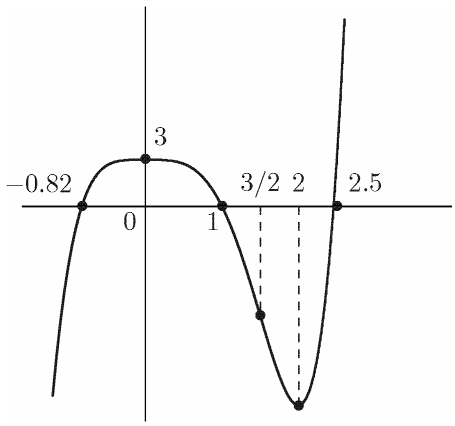
The nature of the point $(0, 3)$ is determined by the first non-zero higher derivative:

$$\frac{d^3y}{dx^3} = 120x^2 - 120x = 120x(x-1) = 0 \text{ when } x = 0$$

$$\frac{d^4y}{dx^4} = 120(2x-1) < 0 \text{ when } x = 0, \quad \text{a maximum point}$$

The sketch of the graph should look like Figure 9.

Figure 9



84. The Lennard-Jones potential for the interaction of two molecules separated by distance R is

$$U(R)=\frac{A}{R^{12}}-\frac{B}{R^{6}}$$

where A and B are constants. The equilibrium separation R_e is that value of R at which $U(R)$ is a minimum and the binding energy is $D_e=-U(R_e)$. Express **(i)** A and B in terms of R_e and D_e, **(ii)** $U(R)$ in terms of R, R_e and D_e.

To obtain the minimum value of the function:

Let $$U(R)=\frac{A}{R^{12}}-\frac{B}{R^{6}}=\frac{A}{x^{2}}-\frac{B}{x},\ \text{where } x=R^{6}$$

Then $$\frac{dU}{dR}=\frac{dU}{dx}\times\frac{dx}{dR}=\left[-\frac{2A}{x^{3}}+\frac{B}{x^{2}}\right]\times 6R^{5}=-\frac{6}{R^{7}}\left[\frac{2A}{R^{6}}-B\right]$$

$$=0 \text{ when } R^{6}=2A/B$$

Therefore $$R_e^{\,6}=2A/B,\quad D_e=-U(R_e)=\frac{A}{R_e^{\,12}}-\frac{B}{R_e^{\,6}}=\frac{A}{R_e^{\,12}}.$$

Then **(i)** $$D_e=\frac{A}{R_e^{\,12}}\quad\to A=D_eR_e^{\,12}$$

$$R_e^{\,6}=2A/B\ \to B=2D_eR_e^{\,6}$$

(ii) $$U(R)=\frac{A}{R^{12}}-\frac{B}{R^{6}}=D_e\left[\left(\frac{R_e}{R}\right)^{12}-2\left(\frac{R_e}{R}\right)^{6}\right]$$

85. The probability that a molecule of mass m in a gas at temperature T has speed v is given by the Maxwell-Boltzmann distribution

$$f(v)=4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}v^{2}e^{-mv^{2}/2kT}$$

where k is Boltzmann's constant. Find the most probable speed (for which $f(v)$ is a maximum).

Let $$f(v)=4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}y(v),\ \text{where } y(v)=v^{2}e^{-av^{2}},\ a=m/2kT$$

Then, for a maximum,

$$\frac{d}{dv}y(v)=(2v)\times(e^{-av^{2}})+(v^{2})\times(-2ave^{-av^{2}})$$

$$=2ve^{-av^{2}}(1-av^{2})=0 \text{ when } v=1/\sqrt{a}$$

and the most probable speed is $v=\sqrt{\dfrac{2kT}{m}}$

86. The concentration of species B in the rate process $A \xrightarrow{k_1} B \xrightarrow{k_2} C$, consisting of two consecutive irreversible first-order reactions, is given by (when $k_1 \neq k_2$)

$$[B] = [A]_0 \frac{k_1}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t})$$

(i) Find the time *t*, in terms of the rate constants k_1 and k_2, at which B has its maximum concentration, and **(ii)** show that the maximum concentration is

$$[B]_{max} = [A]_0 \left(\frac{k_1}{k_2}\right)^{k_2/(k_2-k_1)}$$

(i) Let $$[B] = [A]_0 \frac{k_1}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t})$$

$$= [A]_0 \frac{k_1}{k_2 - k_1} b(t), \text{ where } b(t) = e^{-k_1 t} - e^{-k_2 t}$$

Then, for a maximum,

$$\frac{db}{dt} = -k_1 e^{-k_1 t} + k_2 e^{-k_2 t} = 0 \text{ when } k_1 e^{-k_1 t} = k_2 e^{-k_2 t}$$

Therefore $$\frac{k_2}{k_1} = e^{(k_2 - k_1)t} \rightarrow \ln\frac{k_2}{k_1} = (k_2 - k_1)t$$

$$\rightarrow t = \frac{1}{k_2 - k_1}\ln\frac{k_2}{k_1} \qquad \text{(Equation 1)}$$

(ii) At the maximum, $k_1 e^{-k_1 t} = k_2 e^{-k_2 t}$.

Therefore $$[B] = [A]_0 \frac{k_1}{k_2 - k_1}(e^{-k_1 t} - e^{-k_2 t}) = [A]_0 \frac{1}{k_2 - k_1}(k_2 e^{-k_2 t} - k_1 e^{-k_2 t})$$

$$= [A]_0 e^{-k_2 t}$$

Now, by Equation 1,

$$-k_2 t = \frac{-k_2}{k_2 - k_1}\ln\frac{k_2}{k_1} = \ln\left(\frac{k_1}{k_2}\right)^{k_2/(k_2-k_1)}$$

Therefore $e^{-k_2 t} = \left(\frac{k_1}{k_2}\right)^{k_2/(k_2-k_1)}$ and, at maximum concentration of B,

$$[B]_{max} = [A]_0 e^{-k_2 t} = [A]_0 \left(\frac{k_1}{k_2}\right)^{k_2/(k_2-k_1)}$$

Section 4.11

87. A particle moving along a straight line travels the distance $s = 2t^2 - 3t$ in time *t*. **(i)** Find the velocity v and acceleration *a* at time *t*. **(ii)** Sketch graphs of *s* and v as functions of *t* in the interval $t = 0 \to 2$, **(iii)** find the stationary values, and describe the motion of the particle.

(i) $s = 2t^2 - 3t, \quad v = \dfrac{ds}{dt} = 4t - 3, \quad a = \dfrac{dv}{dt} = 4$

(ii) The sketch of the graphs should look like Figure 10.

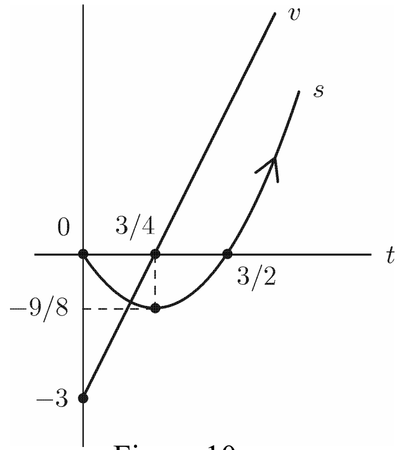


Figure 10

(iii) $v = \dfrac{ds}{dt} = 4t - 3$

$= 0$ when $t = 3/4, \quad s = -9/8$

The particle moves from $s = 0$ at time $t = 0$ with velocity $v = -3$, in the negative *s*-direction (downwards in Figure 10). The acceleration is constant, $a = +4$, in the positive *s*-direction. The particle slows down, and turns at $t = 3/4$ and $s = -9/8$, when $v = 0$ It then moves in the positive *s*-direction (upwards) with increasing speed.

88. A particle moving on the circumference of a circle of radius $r = 2$ travels distance $s = t^3 - 2t^2 - 4t$ in time *t*. **(i)** Express the distance travelled in terms of the angle θ subtended at the centre of the circle, **(ii)** find the angular velocity ω and acceleration $\dot{\omega}$ around the centre of the circle, **(iii)** Sketch graphs of θ, ω and $\dot{\omega}$ as functions of *t* in the interval $t = 0 \to 4$, **(iv)** find the stationary values, and describe the motion of the particle.

(i) $s = r\theta \to \theta = \dfrac{s}{r} = \dfrac{1}{2}(t^3 - 2t^2 - 4t)$

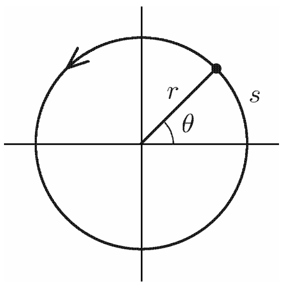


Figure 11

(ii) $\omega = \dfrac{d\theta}{dt} = \dfrac{1}{2}(3t^2 - 4t - 4)$

$\dot{\omega} = \dfrac{d\omega}{dt} = \dfrac{1}{2}(6t - 4) = 3t - 2$

(iii) The sketch of the graphs should look like Figure 12.

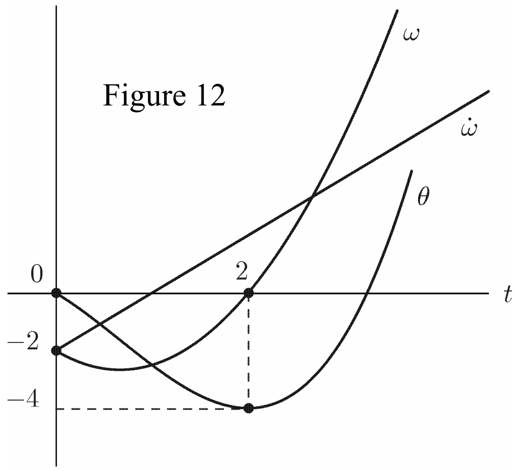


Figure 12

(iv) $\omega = \frac{1}{2}(3t^2 - 4t - 4)$

$$= \frac{1}{2}(3t+2)(t-2) = 0 \text{ when } t = 2$$

The particle moves from $\theta = 0$ at time $t = 0$ in a clockwise direction around the circle with decreasing speed, comes to rest at $t = 2$ when $\theta = -4$, then moves in an anticlockwise direction with increasing speed.

Section 4.12

Find the differential dy:

89. $y = 2x$:

$$y = \frac{dy}{dx}dx = 2\,dx$$

90. $y = 3x^2 + 2x + 1$:

$$dy = \frac{dy}{dx}dx = (6x+2)dx$$

91. $y = \sin x$

$$dy = \frac{dy}{dx}dx = \cos x\,dx$$

92. The volume of a sphere of radius r is $V = 4\pi r^3/3$. Derive the differential dV from first principles. Give a geometric interpretation of the result.

We have $$V + \Delta V = \frac{4\pi}{3}(r + \Delta r)^3 = \frac{4\pi}{3}\left[r^3 + 3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3\right]$$

$$\Delta V = \frac{4\pi}{3}\left[3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3\right] \to \frac{\Delta V}{\Delta r} = \frac{4\pi}{3}\left[3r^2 + 3r\Delta r + (\Delta r)^2\right]$$

Then $$\frac{dV}{dr} = \lim_{\Delta r \to 0}\left(\frac{\Delta V}{\Delta r}\right) = 4\pi r^2$$

and $$dV = 4\pi r^2 dr$$

= surface area of sphere × differential radius

= volume of a spherical shell of radius r and thickness dr