The Chemistry Maths Book

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Second Edition 2008

Solutions

Chapter 3 Transcendental functions

Section 3.2

1. The right-angled triangle ABC has sides $a = 12$ and $b = 5$ (Figure 1). Find *c* and the sin, cos, tan, cosec, sec and cot of the internal angles A and B.

By Pythagoras, $c^2 = a^2 + b^2$.

Therefore $c = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$.

Then $\sin A = \frac{a}{c} = \frac{12}{13}$, $\cos A = \frac{b}{c} = \frac{5}{13}$, $\tan A = \frac{a}{b} = \frac{12}{5}$,

$\operatorname{cosec} A = \frac{c}{a} = \frac{13}{12}$, $\sec A = \frac{c}{b} = \frac{13}{5}$, $\cot A = \frac{b}{a} = \frac{5}{12}$,

$\sin B = \frac{b}{c} = \frac{5}{13}$, $\cos B = \frac{a}{c} = \frac{12}{13}$, $\tan B = \frac{b}{a} = \frac{5}{12}$,

$\operatorname{cosec} B = \frac{13}{5}$, $\sec B = \frac{13}{12}$, $\cot B = \frac{12}{5}$

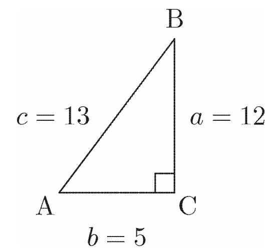


Figure 1

2. For the triangle in Exercise 1, find **(i)** $\sin^2 A + \cos^2 A$, **(ii)** $\sin^2 B + \cos^2 B$.

(i) $\sin^2 A + \cos^2 A = \left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2 = \frac{144+25}{169} = 1$

(ii) $\sin^2 B + \cos^2 B = \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = \frac{25+144}{169} = 1$

3. Express the following angles in radians: **(i)** $5°$, **(ii)** $87°$, **(iii)** $120°$, **(iv)** $260°$, **(v)** $540°$, **(vi)** $720°$

We have $180° = \pi$ (π rad). Therefore

(i) $5° = 5 \times \frac{\pi}{180} = \frac{\pi}{36} \approx 0.0873$ (0.0873 rad)

(ii) $87° = 87 \times \frac{\pi}{180} \approx 1.5184$

(iii) $120° = \frac{2\pi}{3} \approx 2.0944$

(iv) $260° = 260 \times \frac{\pi}{180} = \frac{13\pi}{9} \approx 4.5349$

(v) $540° = 3\pi \approx 9.4248$

(vi) $720° = 4\pi \approx 12.5664$

4. Express the following angles in degrees:

(i) $\pi/10 = 180°/10 = 18°$

(ii) $\pi/4 = 45°$

(iii) $\pi/6 = 30°$

(iv) $\pi/3 = 60°$

(v) $3\pi/8 = 3\times 180°/8 = 67.5°$

(vi) $7\pi/8 = 157.5°$

5. For a circle of radius $r = 4$, find

(i) the angle subtended at the centre of the circle by arc of length 6:

$$s = r\theta,\ \ \theta = s/r = 6/4 = 3/2 \text{ rad} \approx 85.9437°$$

(ii) the length of arc that subtends angle $\pi/10$ at the centre of the circle:

$$s = r\theta = 4\times\pi/10 = 2\pi/5 \approx 1.2566$$

(iii) the length of arc that subtends angle $\pi/2$ at the centre of the circle:

$$s = r\theta = 4\times\pi/2 = 2\pi \approx 6.2832$$

(iv) the circumference of the circle:

$$s = 2\pi r = 8\pi \approx 25.1327$$

6. Use Table 3.2 to find the sine, cosine and tangent of **(i)** $3\pi/4$ **, (ii)** $5\pi/4$ **, (iii)** $7\pi/4$ **.**

From Table 3.2, $\sin\pi/4 = \cos\pi/4 = 1/\sqrt{2},\ \ \tan\pi/4 = 1$.

Then

(i) $\sin 3\pi/4 = +1/\sqrt{2},\ \ \cos 3\pi/4 = -1/\sqrt{2},\ \ \tan 3\pi/4 = -1$

(ii) $\sin 5\pi/4 = -1/\sqrt{2},\ \ \cos 5\pi/4 = -1/\sqrt{2},\ \ \tan 5\pi/4 = +1$

(iii) $\sin 7\pi/4 = -1/\sqrt{2},\ \ \cos 7\pi/4 = +1/\sqrt{2},\ \ \tan 7\pi/4 = -1$

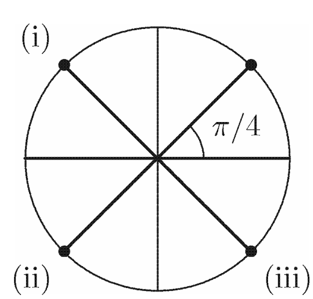


Figure 2

7. By considering the limit $\theta \to 0$ of an internal angle of a right-angled triangle (Figure 3), show that **(i)** $\sin 0 = 0$ **,** **(ii)** $\cos 0 = 1$ **.**

In Figure 3,

as $\theta \to 0$, $a \to 0$ and $b \to c$.

Therefore **(i)** $\sin\theta = a/c \to 0$

(ii) $\cos\theta = b/c \to 1$

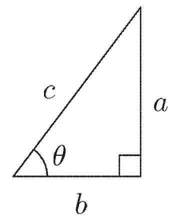


Figure 3

8. Use the properties of the right-angled isosceles triangle to verify the values of the trigonometric functions for $\theta = \pi/4$ in Table 3.2.

By Pythagoras, $c = \sqrt{2}a$.

Therefore

$\sin \pi/4 = \cos \pi/4 = a/c = 1/\sqrt{2}$

$\tan \pi/4 = a/a = 1$

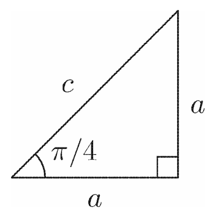


Figure 4

9. Sketch a diagram to show that

(i) $\sin(\pi - \theta) = \sin\theta$ **(ii)** $\cos(\pi - \theta) = -\cos\theta$ **(iii)** $\sin(\pi + \theta) = -\sin\theta$

(iv) $\cos(\pi + \theta) = -\cos\theta$ **(v)** $\sin(\pi/2 - \theta) = \cos\theta$ **(vi)** $\cos(\pi/2 - \theta) = \sin\theta$

From Figure 5, $\sin\theta = a/r$, $\cos\theta = b/r$

Therefore **(i)** $\sin(\pi - \theta) = a/r = \sin\theta$

(ii) $\cos(\pi - \theta) = -b/r = -\cos\theta$

(iii) $\sin(\pi + \theta) = -a/r = -\sin\theta$

(iv) $\cos(\pi + \theta) = -b/r = -\cos\theta$

(v) $\sin(\pi/2 - \theta) = b/r = \cos\theta$

(vi) $\cos(\pi/2 - \theta) = a/r = \sin\theta$

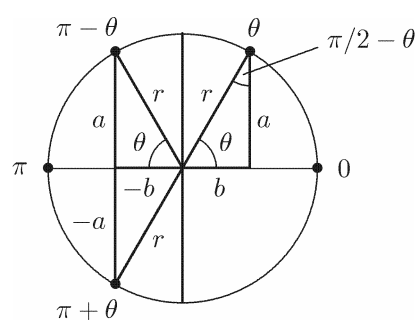


Figure 5

10. Find the period and sketch the graph ($-\pi \le x \le 2\pi$) of **(i)** $\sin 2x$, **(ii)** $\cos 3x$.

(i) If $f(x) = \sin 2x$ then $f(x+a) = \sin(2x+2a) = f(x)$ if $2a = 2\pi$, and the period of $\sin 2x$ is $a = \pi$.

In the graph, $\sin 2x = 0$ when $2x$ is zero or an integer multiple of π ; that is, when

$x = 0,\ \pm\pi/2,\ \pm\pi,\ \pm 3\pi/2,\ \ldots$

and $\sin 2x = \pm 1$ when $2x$ is an odd integer multiple of $\pi/2$; that is, when

$x = \pm\pi/4,\ \pm 3\pi/4,\ \ldots$

The sketch of $\sin 2x$ should look like

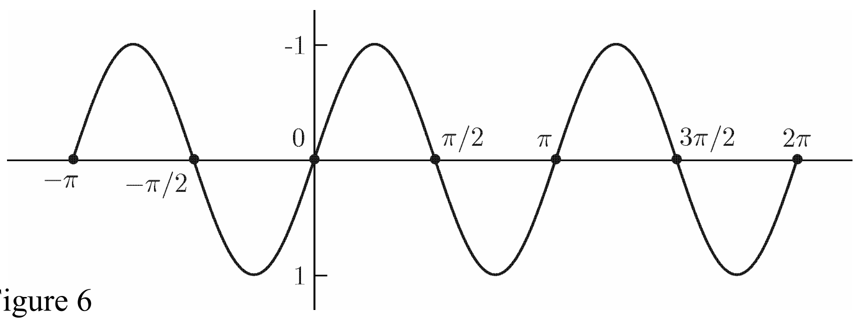


Figure 6

(ii) If $f(x) = \cos 3x$ then $f(x+a) = \cos(3x+3a) = f(x)$ if $3a = 2\pi$, and the period of $\cos 3x$ is $a = 2\pi/3$

In the graph, $\cos 3x = 0$ when $3x$ is an odd integer multiple of $\pi/2$; that is, when

$x = \pm\pi/6,\ \pm 3\pi/6 = \pm\pi/2,\ \pm 5\pi/6, \ldots$

and $\cos 3x = \pm 1$ when $3x$ is zero or an integer multiple of π ; that is, when

$x = 0,\ \pm\pi/3,\ \pm 2\pi/3,\ \ldots$

The sketch of $\cos 3x$ should look like

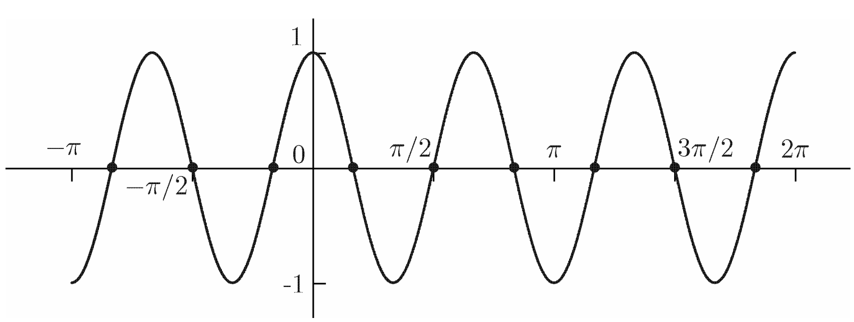


Figure 7

11. Sketch the graph of the harmonic wave $\phi(x,t) = \sin 2\pi(x-t)$ as a function of x ($-1 \le x \le 2$) for values of time, **(i)** $t = 0$, **(ii)** $t = 1/4$, **(iii)** $t = 1/2$.

In the graphs, $\phi(x,t) = \sin 2\pi(x-t) = 0$ when $2\pi(x-t)$ is zero or an integer multiple of π; that is, when

$$x = t + n/2 \text{ for } n = 0, \ \pm 1, \ \pm 2, \ \ldots$$

(i) $\phi(x,0) = 0$ when $x = 0, \ \pm 1/2, \ \pm 1, \ \pm 3/2, \ \ldots$

The function is represented by the solid line in Figure 8 below.

(ii) $\phi(x,1/4) = 0$ when $x = \pm 1/4, \ \pm 3/4, \ \pm 5/4, \ \ldots$

The function is represented by the long dashes in Figure 8.

(iii) $\phi(x,1/2) = 0$ when $x = 0, \ \pm 1/2, \ \pm 1, \ \pm 3/2, \ \ldots$

The function is represented by the short dashes Figure 8.

The sketches of $\phi(x,t) = \sin 2\pi(x-t)$ should look like

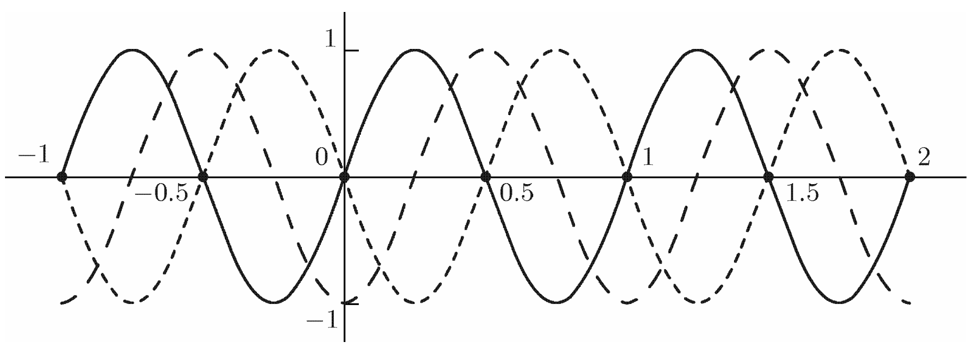


Figure 8

Section 3.3

12. Find the principal values of:

(i) $\theta = \sin^{-1}(1/2)$: The principal value of the inverse sine lies in quadrant I or IV. But $\sin\theta = 1/2 > 0$. Therefore $\theta = \pi/6$, in the first quadrant.

(ii) $\theta = \sin^{-1}(1)$: $\theta = \pi/2$, on the I/II border.

(iii) $\theta = \cos^{-1}(1/2)$: The principal value of the inverse cosine lies in quadrant I or II. But $\cos\theta = 1/2 > 0$. Therefore $\theta = \pi/3$, in the first quadrant.

(iv) $\theta = \cos^{-1}(-1)$: $\theta = \pi$, on the II/III border.

13. The Bragg equation for the reflection of radiation of wavelength λ from the planes of a crystal is $n\lambda = 2d\sin\theta$ where d is the separation of the planes, θ is the angle of incidence of the radiation, and n is an integer. Calculate the angles θ at which X-rays of wavelength 1.5×10^{-10} m are reflected by planes separated by 3.0×10^{-10} m.

We have, $d = 3.0\times10^{-10}$ m and $\lambda = 1.5\times10^{-10}$ m, so that $\dfrac{\lambda}{2d} = \dfrac{1.5\times10^{-10}}{2\times3\times10^{-10}} = \dfrac{1}{4}$.

Therefore $\theta = \sin^{-1}\left(\dfrac{n\lambda}{2d}\right) = \sin^{-1}\left(\dfrac{n}{4}\right)$:

$n = 1$ $\quad \sin^{-1}(1/4) \approx 14.48°$

$n = 2$ $\quad \sin^{-1}(1/2) = 30°$

$n = 3$ $\quad \sin^{-1}(3/4) \approx 48.59°$

$n = 4$ $\quad \sin^{-1}(1) = 90°$

Section 3.4

14. Given the side $a = 1$ and angles $\mathrm{A} = \pi/4$ and $\mathrm{B} = \pi/3$ of a triangle ABC (Figure 9), find the third angle and the other two sides.

The third angle is $\mathrm{C} = \pi - \mathrm{A} - \mathrm{B} = \pi - \pi/4 - \pi/3 = 5\pi/12 = 75°$

By the sine rule, $\dfrac{\sin \mathrm{A}}{a} = \dfrac{\sin \mathrm{B}}{b} = \dfrac{\sin \mathrm{C}}{c}$.

Therefore $$b = a\,\frac{\sin \mathrm{B}}{\sin \mathrm{A}} = \frac{\sqrt{3}/2}{1/\sqrt{2}} = \sqrt{3/2}$$

$$c = a\,\frac{\sin \mathrm{C}}{\sin \mathrm{A}} = \sqrt{2}\sin 75° \approx 1.3660$$

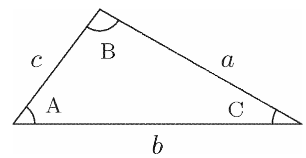


Figure 9

15. Given the sides $a = 2$, $b = 2.5$ and $c = 3$ of a triangle ABC (Figure 9), find the angles.

By the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos \mathrm{A}$.

Therefore $\cos \mathrm{A} = (b^2 + c^2 - a^2)/2bc = 0.75$

and $\mathrm{A} = \cos^{-1}(0.75) \approx 41.41°$.

Similarly $\cos \mathrm{B} = (a^2 + c^2 - b^2)/2ac = 0.5625$

and $\mathrm{B} = \cos^{-1}(0.5625) \approx 55.77°$.

Then $\mathrm{C} = 180° - \mathrm{A} - \mathrm{B} \approx 82.82°$.

16. Given the sides $a = 3,\ b = 4,$ and included angle $\mathrm{C} = \pi/4$ of triangle ABC (Figure 9), find the third side and the other two angles.

By the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos\mathrm{C} = 25 - 24/\sqrt{2} \approx 8.0294$.

Therefore $c \approx 2.8336$.

By the cosine rule, $\cos\mathrm{A} = \dfrac{b^2 + c^2 - a^2}{2bc} \approx 0.6630$ and $\cos\mathrm{B} = \dfrac{a^2 + c^2 - b^2}{2ac} \approx 0.0605$.

Therefore, taking principal values,

$$\mathrm{A} \approx \cos^{-1}(0.6630) \approx 48.47°$$

$$\mathrm{B} \approx \cos^{-1}(0.0605) \approx 86.53°$$

Check: $\mathrm{A} + \mathrm{B} + \mathrm{C} = 45° + 48.47° + 86.53° = 180°$

17. Given the sides $a = \sqrt{2}, b = 3,$ and included angle $\mathrm{C} = \pi/4$ of the triangle ABC (Figure 9), find the third side and the other two angles.

By the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos\mathrm{C} = 2 + 9 - 2 \times \sqrt{2} \times 3 \times 1/\sqrt{2} = 5$.

Therefore, $c = \sqrt{5} \approx 2.2361$.

By the cosine rule, $\cos\mathrm{A} = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{2}{\sqrt{5}} \approx 0.8944$ and $\cos\mathrm{B} = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{-1}{\sqrt{10}} \approx -0.3162$.

Therefore, taking principal values,

$$\mathrm{A} \approx \cos^{-1}(2/\sqrt{5}) \approx 26.57°$$

$$\mathrm{B} \approx \cos^{-1}(-1/\sqrt{10}) \approx 108.43°$$

Check: $\mathrm{A} + \mathrm{B} + \mathrm{C} = 45° + 26.57° + 108.43° = 180°$

18. Express in terms of in terms of the sines and cosines of 2θ and 5θ:

(i) $\sin 7\theta = \sin(5\theta + 2\theta) = \sin 5\theta \cos 2\theta + \cos 5\theta \sin 2\theta$

(ii) $\sin 3\theta = \sin(5\theta - 2\theta) = \sin 5\theta \cos 2\theta - \cos 5\theta \sin 2\theta$

(iii) $\cos 7\theta = \cos(5\theta + 2\theta) = \cos 5\theta \cos 2\theta - \sin 5\theta \sin 2\theta$

(iv) $\cos 3\theta = \cos(5\theta - 2\theta) = \cos 5\theta \cos 2\theta + \sin 5\theta \sin 2\theta$

19. Express **(i)** $\sin 3\theta$ in terms of $\sin\theta$, **(ii)** $\cos 3\theta$ in terms of $\cos\theta$.

(i)
$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\ &= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta \\ &= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\ &= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

(ii)
$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

20. Express $\cos 4x$ in terms of **(i)** $\sin 2x$ and $\cos 2x$, **(ii)** $\sin 2x$ only, **(iii)** $\cos 2x$ only, **(iv)** $\sin x$ only, **(v)** $\cos x$ only.

(i) $\cos 4x = \cos^2 2x - \sin^2 2x$

(ii) $\cos 4x = 1 - 2\sin^2 2x$

(iii) $\cos 4x = 2\cos^2 2x - 1$

(iv)
$$\begin{aligned}\cos 4x &= 1 - 2\sin^2 2x \\ &= 1 - 2(2\sin x\cos x)^2 = 1 - 8\sin^2 x\cos^2 x \\ &= 1 - 8\sin^2 x(1 - \sin^2 x) \\ &= 1 - 8\sin^2 x + 8\sin^4 x\end{aligned}$$

(v) $\cos 4x = 2\cos^2 2x - 1$

$= 2(2\cos^2 x - 1)^2 - 1$

$= 8\cos^4 x - 8\cos^2 x + 2 - 1$

$= 1 - 8\cos^2 x + 8\cos^4 x$

21. Given $\sin 10° = 0.1736, \sin 30° = 1/2, \sin 50° = 0.7660$, find $\cos 20°$ (without using a calculator).

We have $\sin 50° = \sin(30° + 20°) = \sin 30° \cos 20° + \cos 30° \sin 20°$

$\sin 10° = \sin(30° - 20°) = \sin 30° \cos 20° - \cos 30° \sin 20°$

Then $\sin 50° + \sin 10° = 2\sin 30° \cos 20° = 2 \times \frac{1}{2} \times \cos 20°$

$= \cos 20°$

and $\cos 20° = 0.7660 + 0.1736 = 0.9396$

22. Express in terms of the sines of $8x$ and $2x$: **(i)** $\sin 5x \cos 3x$, **(ii)** $\cos 5x \sin 3x$

By equation (3.21),

$$\sin 8x = \sin(5x + 3x) = \sin 5x \cos 3x + \cos 5x \sin 3x$$

$$\sin 2x = \sin(5x - 3x) = \sin 5x \cos 3x - \cos 5x \sin 3x$$

Therefore **(i)** $\sin 5x \cos 3x = \frac{1}{2}\left[\sin 8x + \sin 2x\right]$

(ii) $\cos 5x \sin 3x = \frac{1}{2}\left[\sin 8x - \sin 2x\right]$

23. Express in terms of the cosines of $8x$ and $2x$: **(i)** $\sin 5x \sin 3x$, **(ii)** $\cos 5x \cos 3x$.

By equation (3.22),

$$\cos 8x = \cos(5x + 3x) = \cos 5x \cos 3x - \sin 5x \sin 3x$$

$$\cos 2x = \cos(5x - 3x) = \cos 5x \cos 3x + \sin 5x \sin 3x$$

Therefore **(i)** $\sin 5x \sin 3x = \frac{1}{2}\left[\cos 2x - \cos 8x\right]$

(ii) $\cos 5x \cos 3x = \frac{1}{2}\left[\cos 2x + \cos 8x\right]$

24. Express **(i)** $\sin(\pi \pm \theta)$ and **(ii)** $\cos(\pi \pm \theta)$ in terms of $\sin\theta$ and $\cos\theta$.

(i) We have $\sin(\pi \pm \theta) = \sin\pi\,\cos\theta \pm \cos\pi\,\sin\theta = \mp\sin\theta$

Therefore $\sin(\pi + \theta) = -\sin\theta$
$\sin(\pi - \theta) = +\sin\theta$

(ii) We have $\cos(\pi \pm \theta) = \cos\pi\,\cos\theta \mp \sin\pi\,\sin\theta = -\cos\theta$

Therefore $\cos(\pi + \theta) = \cos(\pi - \theta) = -\cos\theta$

25. The function $\psi(x,t) = \sin\pi x \cos 2\pi t$ represents a standing wave. Find the values of time t for which ψ has **(i)** maximum amplitude, **(ii)** zero amplitude. **(iii)** Sketch the wave function between $x = 0$ and $x = 3$ at (a) $t = 0$, (b) $t = 1/8$.

(i) $|\cos 2\pi t| = 1$ when $2\pi t = n\pi$ for integer values of n.

Then $t = n/2,\ n = 0, 1, 2, \ldots$

(ii) $\cos 2\pi t = 0$ when $2\pi t = n\pi/2$ for odd integer values of n:

Then $t = (2n+1)/4,\ n = 0, 1, 2, \ldots$

(iii) (a) At $t = 0$, $\psi(x,0) = \sin\pi x$ is represented by the solid line in Figure 10.

(b) At $t = 1/8$, $\psi(x,1/8) = \sin\pi x \cos\pi/4 = \dfrac{\sin\pi x}{\sqrt{2}}$ is represented by the dashed line.

The sketches should look like

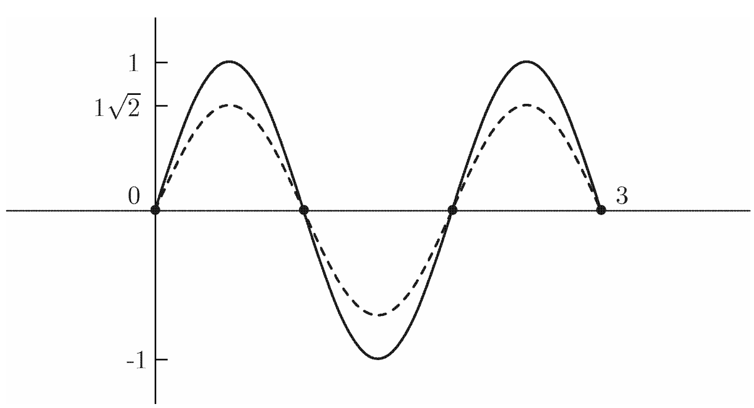


Figure 10

26. The function

$$\phi(x) = a\sin\frac{2\pi x}{\lambda} + b\cos\frac{2\pi x}{\lambda}$$

represents the superposition of two harmonic waves with the same wavelength . Show that ϕ is

(i) also harmonic with the same wavelength, and

(ii) can be written as $\phi(x) = A\sin\left(\frac{2\pi x}{\lambda} + \alpha\right)$ where $A = \sqrt{a^2 + b^2}$ and $\tan\alpha = b/a$.

(i) $$\phi(x+\lambda) = a\sin\left(\frac{2\pi x}{\lambda} + 2\pi\right) + b\cos\left(\frac{2\pi x}{\lambda} + 2\pi\right)$$

$$= a\left[\sin\frac{2\pi x}{\lambda}\cos 2\pi + \cos\frac{2\pi x}{\lambda}\sin 2\pi\right] + b\left[\cos\frac{2\pi x}{\lambda}\cos 2\pi - \sin\frac{2\pi x}{\lambda}\sin 2\pi\right]$$

$$= a\sin\frac{2\pi x}{\lambda} + b\cos\frac{2\pi x}{\lambda} = \phi(x)$$

Therefore $\phi(x)$ is periodic with period λ, and represents a harmonic wave with wavelength λ.

(ii) $$A\sin\left(\frac{2\pi x}{\lambda} + \alpha\right) = A\sin\frac{2\pi x}{\lambda}\cos\alpha + A\cos\frac{2\pi x}{\lambda}\sin\alpha$$

$$= a\sin\frac{2\pi x}{\lambda} + b\cos\frac{2\pi x}{\lambda} \quad \text{if } a = A\cos\alpha \text{ and } b = A\sin\alpha$$

Then $b/a = \tan\alpha$

and $a^2 + b^2 = A^2\left(\cos^2\alpha + \sin^2\alpha\right) = A^2$

so that $\sqrt{a^2 + b^2} = A$

Section 3.5

27. Find the cartesian coordinates of the points whose polar coordinates are

(i) $r = 3, \theta = \pi/3$: $x = r\cos\theta = 3\cos\pi/3 = 3/2$

$y = r\sin\theta = 3\sin\pi/3 = 3\sqrt{3}/2$

(ii) $r = 3, \theta = 5\pi/3$: $x = r\cos\theta = 3\cos 5\pi/3 = 3/2$

$y = r\sin\theta = 3\sin 5\pi/3 = -3\sqrt{3}/2$

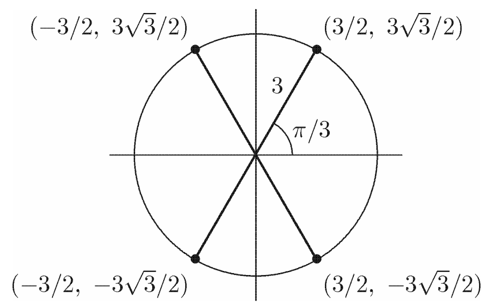


Figure 11

28. Find the cartesian coordinates of the points whose polar coordinates are

(i) $r = 3, \theta = 2\pi/3$: $x = r\cos\theta = 3\cos 2\pi/3 = -3/2$

$y = r\sin\theta = 3\sin 2\pi/3 = 3\sqrt{3}/2$

(ii) $r = 3, \theta = 4\pi/3$: $x = r\cos\theta = 3\cos 2\pi/3 = -3/2$

$y = r\sin\theta = 3\sin 2\pi/3 = -3\sqrt{3}/2$

(see Figure 11)

29. Find the polar coordinates of the points whose cartesian coordinates are

(i) (3, 2) **:** The point lies in the quadrant I, and

$$r = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.7°$$

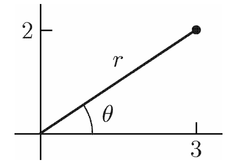


Figure 12

(ii) $(3,-2)$ **:** The point lies in the quadrant IV, and

$$r=\sqrt{13}\,,\;\theta=\tan^{-1}\left(-\frac{2}{3}\right)\approx -33.7°$$

But angle θ is conventionally defined to lie in the range $0\to 2\pi$, with $\tan(2\pi+\theta)=\tan\theta$. In the present case,

$$\theta=\tan^{-1}\left(-\frac{2}{3}\right)+2\pi\approx 326.3$$

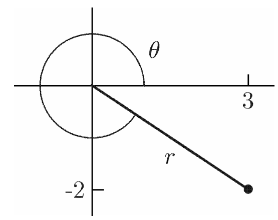


Figure 13

30. Find the polar coordinates of the points whose cartesian coordinates are

(i) $(-3,2)$ **:** The point lies in the quadrant II, and

$$r=\sqrt{3^2+2^2}=\sqrt{13}$$

$$\theta=\tan^{-1}\left(-\frac{2}{3}\right)+\pi\approx 146.3°$$

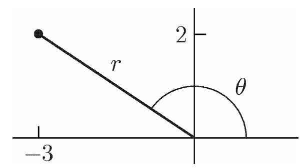


Figure 14

(ii) $(-3,-2)$ **:** The point lies in the quadrant III, and

$$r=\sqrt{13}$$

$$\theta=\tan^{-1}\left(\frac{2}{3}\right)+\pi\approx 213.7°$$

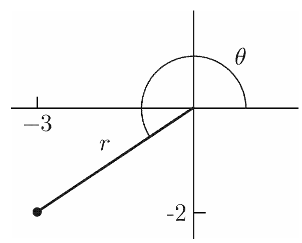


Figure 15

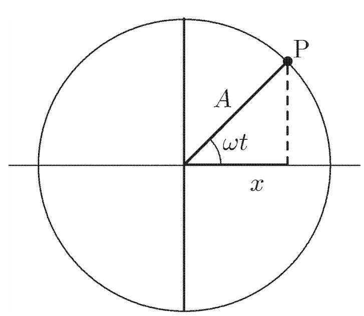
31. A solution of the equation of motion for the harmonic oscillator is given in Example 3.8 as

$$x(t)=A\cos\omega t\,.$$

Show that $x(t)$ can be interpreted as the *x*-coordinate of a point moving with constant angular speed ω in a circle in the *xy*-plane, with centre at the origin and radius *A*.

The point $x(t)=A\cos\omega t$ is interpreted in Figure 16 as the *x*-coordinate of point P on a circle. The angle ωt increases at constant rate with time *t*, and the point moves round the circle at constant speed.

Figure 16



Section 3.6

32. Simplify

(i) $e^2 e^3 = e^{2+3} = e^5$

(ii) $e^3 e^{-3} = e^{3-3} = e^0 = 1$

(iii) $e^3 e^{-4} = e^{3-4} = e^{-1} = 1/e$

(iv) $e^3 / e^2 = e^{3-2} = e^1 = e$

(v) $e^5 / e^{-4} = e^{5+4} = e^9$

33. **(i)** Write down the expansion of $e^{-x/3}$ in powers of *x* to terms in x^5. **(ii)** Use the expansion to calculate an approximate value of $e^{-1/3}$. Determine how many significant figures of this value are correct, and quote your answer to this number of figures.

(i)
$$e^{-x/3} = 1 + \left(-\frac{x}{3}\right) + \frac{1}{2!}\left(-\frac{x}{3}\right)^2 + \frac{1}{3!}\left(-\frac{x}{3}\right)^3 + \frac{1}{4!}\left(-\frac{x}{3}\right)^4 + \frac{1}{5!}\left(-\frac{x}{3}\right)^5 + \cdots$$
$$= 1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{162} + \frac{x^4}{1944} - \frac{x^5}{29160} + \cdots$$

(ii) Holding 6 decimal places throughout,

$$e^{-1/3} = 1 - \frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \frac{1}{1944} - \frac{1}{29160} + \cdots$$
$$\approx 1 - 0.333333 + 0.055555 - 0.006173 + 0.000514 - 0.000034$$
$$\approx 0.716529$$

The terms are decreasing by a factor greater than 10, and the truncated series is accurate to 5 significant figures: $e^{-1/3} \approx 0.71653$

34. **(i)** Write down the expansion of e^{-x^3} in powers of *x* to terms in x^{15}. Use the expansion to calculate an approximate value of e^{-x^3} that is correct to 12 significant figures for the following values of *x*, in each case giving the smallest number of terms required: **(ii)** 10^{-1}, **(iii)** 10^{-2}, **(iv)** 10^{-3}, **(v)** 10^{-4}, **(vi)** 10^{-5}.

(i) $$e^{-x^3} = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \frac{x^{12}}{24} - \frac{x^{15}}{120} + \ldots$$

Then

(ii) $\exp(-10^{-3}) \approx 1-10^{-3}+\frac{1}{2}\times10^{-6}-\frac{1}{6}\times10^{-6}+\frac{1}{24}\times10^{-12}-\frac{1}{120}\times10^{-15}$

$\approx 1-0.001+0.000\,000\,5-0.000\,000\,000\,166\,67+0.000\,000\,000\,000\,04$

$\approx 1.000\,000\,500\,000\,04-0.001\,000\,000\,166\,67$

$\approx 0.999\,000\,499\,833$ (4 terms)

(iii) $\exp(-10^{-6}) \approx 1-10^{-6}+\frac{1}{2}\times10^{-12}-\frac{1}{6}\times10^{-18}$

$\approx 1-0.000\,001+0.000\,000\,000\,000\,50$

$\approx 1.000\,000\,000\,000\,50-0.000\,001\,000\,000\,00$

$\approx 0.999\,999\,000\,001$ (3 terms)

(iv) $\exp(-10^{-9}) \approx 1-10^{-9}+\frac{1}{2}\times10^{-18}$

$\approx 1-0.000\,000\,001$

$\approx 0.999\,999\,999\,000$ (2 terms)

(v) $\exp(-10^{-12}) \approx 1-10^{-12}+$

$\approx 1-0.000\,000\,000\,001$

$\approx 0.999\,999\,999\,999$ (2 terms)

(v) $\exp(-10^{-15}) \approx 1.000\,000\,000\,00$ (1 term)

35. Sketch the graphs of e^{2x} and e^{-2x} for values $-1.5 \le x \le 1.5$.

x	e^{2x}	e^{-2x}
–1.5	0.050	20.09
–1	0.135	7.389
–0.5	0.368	2.718
0	1	1
0.5	2.718	0.368
1.0	7.389	0.135
1.5	20.09	0.050

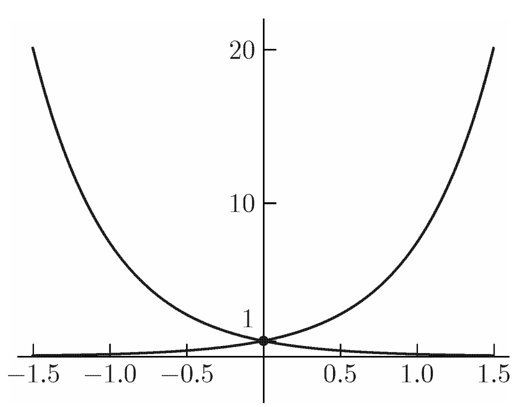


Figure 17

36. For a system composed of N identical molecules, the Boltzmann distribution

$$\frac{n_i}{N} = e^{-\varepsilon_i/kT}$$

gives the average fraction of molecules in the molecular state i with energy ε_i. **(i)** Show that the ratio n_i/n_j of the populations of states i and j depends only on the difference in energy of the two states. **(ii)** What is the ratio for two states with the same energy (degenerate states)?

(i) $\dfrac{n_i}{n_j} = \dfrac{n_i}{N}\Big/\dfrac{n_j}{N} = e^{-\varepsilon_i/kT}\Big/e^{-\varepsilon_j/kT} = e^{-(\varepsilon_i-\varepsilon_j)/kT}$

Therefore n_i/n_j depends only on $\Delta\varepsilon = \varepsilon_i - \varepsilon_j$ (and T)

(ii) $\dfrac{n_i}{n_j} = e^0 = 1$ if $\varepsilon_i = \varepsilon_j$

Section 3.7

37. Simplify:

(i) $\log_{10} 100 = \log_{10} 10^2 = 2$

(ii) $\log_2 16 = \log_2 2^4 = 4$

(iii) $\ln e^{-5} = \log_e e^{-5} = -5$

(iv) $\ln e^{x^2} = x^2$

(v) $\ln e^{-(ax^2+bx+c)} = -(ax^2+bx+c)$

(vi) $\ln e^{-kt} = -kt$

38. Express the following as the log of a single number:

(i) $\ln 2 + \ln 3 = \ln\,(2\times 3) = \ln 6$

(ii) $\ln 2 - \ln 3 = \ln\,(2/3)$

(iii) $5\ln 2 = \ln 2^5 = \ln 32$

(iv) $\ln 3 + \ln 4 - \ln 6 = \ln\dfrac{3\times 4}{6} = \ln 2$

39. Simplify:

(i) $\ln x^3 - \ln x = \ln \dfrac{x^3}{x} = \ln x^2$

(ii) $\ln(2x^3 - 3x^2) + \ln x^{-2} = \ln \dfrac{2x^3 - 3x^2}{x^2} = \ln(2x - 3)$

(iii) $\ln(x^5 - 3x^2) + 2\ln x^{-1} - \ln(x^3 - 3) = \ln \dfrac{x^5 - 3x^2}{x^2(x^3 - 3)} = \ln \dfrac{x^2(x^3 - 3)}{x^2(x^3 - 3)} = \ln 1 = 0$

(iv) $\ln e^x = x$

(v) $\ln e^{x^2+3} - \ln e^3 = \ln \dfrac{e^{x^2+3}}{e^3} = \ln e^{x^2} = x^2$

40. The barometric formula

$$p = p_0 e^{-Mgh/RT}$$

gives the pressure of a gas of molar mass M at altitude h, when p_0 is the pressure at sea level. Express h in terms of the other variables.

$$p = p_0 e^{-Mgh/RT} \;\rightarrow\; p/p_0 = e^{-Mgh/RT} \;\rightarrow\; \ln(p/p_0) = -Mgh/RT$$

Therefore $h = -\dfrac{RT}{Mg}\ln(p/p_0)$

41. The chemical potential of a gas at pressure p and temperature T is

$$\mu = \mu^{\ominus} + RT\ln\left(f/p^{\ominus}\right)$$

where $f = \gamma p$ is the fugacity and γ is the fugacity coefficient. Express p as an explicit function of the other variables.

$$\mu = \mu^{\ominus} + RT\ln\left(f/p^{\ominus}\right) \;\rightarrow\; \frac{\mu - \mu^{\ominus}}{RT} = \ln\left(f/p^{\ominus}\right) \;\rightarrow\; e^{\left(\mu-\mu^{\ominus}\right)/RT} = f/p^{\ominus}$$

Therefore, putting $f = \gamma p$,

$$p = \frac{p^{\ominus}}{\gamma}\exp\left[\left(\mu - \mu^{\ominus}\right)/RT\right]$$

42. In a first-order decomposition reaction, $\text{A} \rightarrow \text{products}$, the amount of substance A at time t is

$$x(t) = x(0)e^{-kt}$$

where $x(0)$ is the initial amount of A, and k is the rate constant. The time taken for the amount of A to fall to half of its initial value is called the half-life, $\tau_{1/2}$, of the reaction. Find the half-life for rate constants: **(i)** $k = 3\ \text{s}^{-1}$, **(ii)** $k = 10^{-5}\ \text{s}^{-1}$.

We have $x(\tau_{1/2}) = x(0)e^{-k\tau_{1/2}} = x(0)/2$.

Therefore $\dfrac{x(\tau_{1/2})}{x(0)} = e^{-k\tau_{1/2}} = \dfrac{1}{2} \rightarrow -k\tau_{1/2} = \ln\dfrac{1}{2} = -\ln 2$

$$\tau_{1/2} = \frac{\ln 2}{k} \approx \frac{0.6931}{k}$$

(i) $k = 3\ \text{s}^{-1}$, $\tau_{1/2} = \dfrac{\ln 2}{3\ \text{s}^{-1}} \approx 0.2310\ \text{s}$

(ii) $k = 10^{-5}\ \text{s}^{-1}$, $\tau_{1/2} = \dfrac{\ln 2}{10^{-5}\ \text{s}^{-1}} \approx 0.6931 \times 10^{5}\ \text{s}$

Section 3.8

43. As in Example 3.30, sketch graphs of **(i)** x^2, e^{-x}, x^2e^{-x}, **(ii)** x^{-2}, e^{x}, $x^{-2}e^{x}$

x	x^2	e^{-x}	x^2e^{-x}	x^{-2}	e^{x}	$x^{-2}e^{x}$
0	0	1.0000	0.0000	-	1.0000	-
1	1	0.3679	0.3679	1.0000	2.7183	2.7183
2	4	0.1354	0.5413	0.2500	7.3891	1.8473
3	9	0.0498	0.4481	0.1111	20.0855	2.2317
4	16	0.0183	0.2931	0.0625	54.5982	3.4124