The Chemistry Maths Book

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Solutions

Chapter 2. Algebraic functions

Section 2.1

1. Find the values of $y = 2 - 3x$ for **(i)** $x = 0$, **(ii)** $x = 2$, **(iii)** $x = -3$, **(iv)** $x = 2/3$

(i) $2 - 3\times 0 = 2 - 0 = 2$

(ii) $2 - 3\times 2 = 2 - 6 = -4$

(iii) $2 - 3\times(-3) = 2 + 9 = 11$

(iv) $2 - \cancel{3}\times\frac{2}{\cancel{3}} = 2 - 2 = 0$

2. Find the values of $y = 2x^2 + 3x - 1$ for **(i)** $x = 0$, **(ii)** $x = 1$, **(iii)** $x = -1$, **(iv)** $x = -2/3$

(i) $0 + 0 - 1 = -1$

(ii) $2 + 3 - 1 = 4$

(iii) $2\times(-1)^2 + 3\times(-1) - 1 = 2 - 3 - 1 = -2$

(iv) $2\times\left(-\frac{2}{3}\right)^2 + 3\times\left(-\frac{2}{3}\right) - 1 = \frac{8}{9} - 2 - 1 = \frac{8 - 18 - 9}{9} = -\frac{19}{9}$

3. Given $f(x) = x^3 - 3x^2 + 4x - 3$, find **(i)** $f(5)$, **(ii)** $f(0)$, **(iii)** $f(-2)$, **(iv)** $f(-2/3)$

(i) $f(5) = 5^3 - 3\times 5^2 + 4\times 5 - 3 = 125 - 75 + 20 - 3 = 67$

(ii) $f(0) = 0 - 0 - 0 - 3 = -3$

(iii) $f(-2) = (-2)^3 - 3\times(-2)^2 + 4\times(-2) - 3 = -8 - 12 - 8 - 3 = -31$

(iv) $f\left(-\frac{2}{3}\right) = -\frac{8}{27} - 3\times\frac{4}{9} - 4\times\frac{2}{3} - 3 = -\frac{8 + 36 + 72 + 81}{27} = -\frac{197}{27}$

4. If $f(x) = 2x^2 + 4x + 3$, what is **(i)** $f(a)$, **(ii)** $f(y^2)$?

(i) $f(a) = 2a^2 + 4a + 3$

(ii) $f(y^2) = 2(y^2)^2 + 4y^2 + 3 = 2y^4 + 4y^2 + 3$

5. If $f(x) = x^2 - 3x - 4$, what are **(i)** $f(a+3)$, **(ii)** $f(a^2+1)$, **(iii)** $f(x+1)$, **(iv)** $f(x^2-3x-4)$?

(i) $f(a+3) = (a+3)^2 - 3(a+3) - 4 = a^2 + 6a + 9 - 3a - 9 - 4$

$= a^2 + 3a - 4$

(ii) $f(a^2+1) = (a^2+1)^2 - 3(a^2+1) - 4 = a^4 + 2a^2 + 1 - 3a^2 - 3 - 4$

$= a^4 - a^2 - 6$

(iii) $f(x+1) = (x+1)^2 - 3(x+1) - 4 = x^2 + 2x + 1 - 3x - 3 - 4$

$= x^2 - x - 6$

(iv) $f(x^2-3x-4) = (x^2-3x-4)^2 - 3(x^2-3x-4) - 4$

$= x^4 - 6x^3 - 8x^2 + 9x^2 + 24x + 16 - 3x^2 + 9x + 12 - 4$

$= x^4 - 6x^3 - 2x^2 + 33x + 24$

6. If $f(x) = 2x - 1$ and $g(x) = 3x + 1$, express $f(g)$ as a function of x.

$f(g) = f(3x+1) = 2(3x+1) - 1$

$= 6x + 1$

Section 2.2

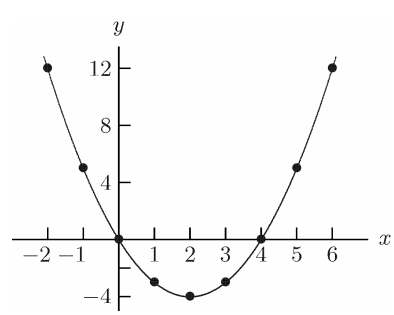
Make a table of (x, y) values and sketch a fully labelled graph of the quadratic:

7. $y = x^2 - 4x$

x	y
–2	12
–1	5
0	0
1	–3
2	–4
3	–3
4	0
5	5
6	12

We have $y = x(x-4)$ so that $y = 0$ when $x = 0$ and $x = 4$. Also, $y \to +\infty$ as $x \to \pm\infty$.

The sketch should look like

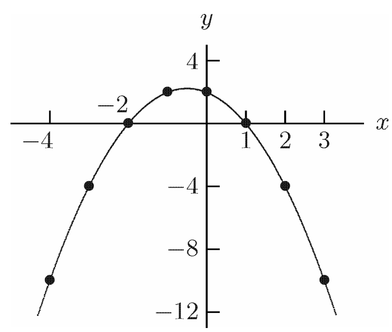


8 $y = -x^2 - x + 2$

x	y
–4	–10
–3	–4
–2	0
–1	2
0	2
1	0
2	–4
3	–10

We have $y = -(x^2 + x - 2) = -(x+2)(x-1)$ so that $y = 0$ when $x = -2$ and $x = 1$. Also, $y \to -\infty$ as $x \to \pm\infty$.

The sketch should look like



Section 2.3

Factorize:

9. $6x^2y^2 - 2xy^3 - 4y^2$

$2y^2$ is a common factor.

Therefore $6x^2y^2 - 2xy^3 - 4y^2 = 2y^2(3x^2 - xy - 2)$

10. $x^2 + 6x + 5 = (x+a)(x+b) = x^2 + (a+b)x + ab$ if $a + b = 6$ and $a \times b = 5$.

Therefore $a = 5$, $b = 1$

and $x^2 + 6x + 5 = (x+5)(x+1)$

11. $x^2 + x - 6 = x^2 + (a+b)x + ab$ if $a + b = 1$ and $a \times b = -6$.

Therefore $a = 3$, $b = -2$

and $x^2 + x - 6 = (x+3)(x-2)$

12. $x^2 - 8x + 15 = x^2 + (a+b)x + ab$ if $a + b = -8$ and $a \times b = 15$.

Therefore $a = -5$, $b = -3$

and $x^2 - 8x + 15 = (x-5)(x-3)$

13. $x^2 - 4 = x^2 - 2^2 = (x+2)(x-2)$

14. $4x^2 - 9 = (2x)^2 - 3^2 = (2x+3)(2x-3)$

15. $2x^2 + x - 6 = (2x+a)(x+b) = 2x^2 + (a+2b)x + ab$ if $a + 2b = 1$ and $a \times b = -6$.

Therefore $a = -3$ and $b = 2$

and $2x^2 + x - 6 = (2x-3)(x+2)$

16. $x^4 - 10x^2 + 9$

This a quadratic in x^2 . Thus $x^4 - 10x^2 + 9 = (x^2 + a)(x^2 + b)$ if $a = -9$ and $b = -1$.

Therefore $x^4 - 10x^2 + 9 = (x^2 - 9)(x^2 - 1)$.

Now $x^2 - 9 = (x+3)(x-3)$ and $x^2 - 1 = (x+1)(x-1)$.

Therefore $x^4 - 10x^2 + 9 = (x+3)(x-3)(x+1)(x-1)$

Simplify if possible:

17. $\dfrac{x}{3x^2 + 2x}$

x is a common factor.

Therefore $$\frac{x}{3x^2 + 2x} = \frac{\cancel{x}}{3x^{\cancel{2}} + 2\cancel{x}} = \frac{1}{3x+2}$$

18. $\dfrac{x+2}{x+4}$ No simplification is possible.

19. $\dfrac{x^2 - 4}{x - 2}$

The numerator is $x^2 - 4 = (x+2)(x-2)$.

Therefore $$\frac{x^2 - 4}{x - 2} = \frac{(x+2)\,\cancel{(x-2)}}{\cancel{x-2}} = x + 2$$

20. $\dfrac{x^2 + 3x + 2}{x + 2}$

The numerator is $x^2 + 3x + 2 = (x+2)(x+1)$ **.**

Therefore $$\frac{x^2 + 3x + 2}{x + 2} = \frac{\cancel{(x+2)}\,(x+1)}{\cancel{x+2}} = x + 1$$

21. $\dfrac{x^2-9}{x^2+5x+6}=\dfrac{(x+3)(x-3)}{(x+3)(x+2)}=\dfrac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x+2)}$

$=\dfrac{x-3}{x+2}$

22. $\dfrac{2x^2-3x+1}{x^2-3x+2}=\dfrac{(2x-1)\cancel{(x-1)}}{(x-2)\cancel{(x-1)}}$

$=\dfrac{2x-1}{x-2}$

Section 2.4

Find x as a function of y:

23. $y=x-2 \;\rightarrow x=y+2$

24. $y=\dfrac{1}{2}(3x+1) \;\rightarrow\; 2y=3x+1 \;\rightarrow\; 2y-1=3x$

Therefore $\quad x=\dfrac{1}{3}(2y-1)$

25. $y=\dfrac{1}{3}(2-x) \;\rightarrow\; 3y=2-x \;\rightarrow 3y-2=-x$

Therefore $\quad x=2-3y$

26. $y=\dfrac{x}{1-x} \;\rightarrow\; y(1-x)=x \;\rightarrow\; y-yx=x \;\rightarrow\; y=x+yx$

$\rightarrow\; y=x(1+y)$

Therefore $\quad x=\dfrac{y}{1+y}$

27. $y=\dfrac{2x+3}{3x-2} \;\rightarrow\; y(3x-2)=2x+3 \;\rightarrow\; 3xy-2y=2x+3$

$\rightarrow\; 3xy-2x=2y+3$

$\rightarrow\; x(3y-2)=2y+3$

Therefore $\quad x=\dfrac{2y+3}{3y-2}$

28. $y = \dfrac{x-1}{2x+1} \;\rightarrow\; y(2x+1) = x-1 \;\rightarrow\; 2xy + y = x - 1$

$$\rightarrow\; 2xy - x = -y - 1$$

$$\rightarrow\; x(2y-1) = -y-1$$

Therefore $\quad x = \dfrac{1+y}{1-2y}$

29. $y = \dfrac{x^2-1}{x^2+1} \;\rightarrow\; y(x^2+1) = x^2 - 1 \;\rightarrow\; yx^2 - x^2 = -1 - y$

$$\rightarrow\; x^2(y-1) = -(1+y)$$

Therefore $\quad x^2 = \dfrac{1+y}{1-y} \;\rightarrow\; x = \pm\sqrt{\dfrac{1+y}{1-y}}$

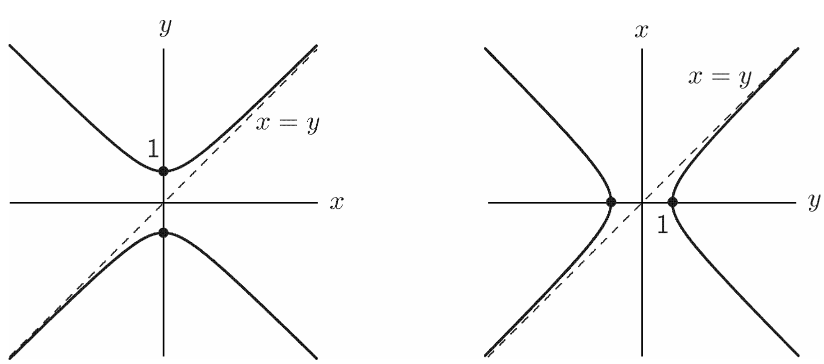
> For $y = f(x)$, **(i)** find *x* as a function of *y*, **(ii)** sketch graphs of $y = f(x)$ and $x = f^{-1}(y)$:

30. **(i)** $y^2 = x^2 + 1 \;\rightarrow\; x = f^{-1}(y) = \pm\sqrt{y^2-1}$

(ii) $y^2 = x^2 + 1 \;\rightarrow\; y = f(x) = \pm\sqrt{x^2+1}$

We have $y = \pm 1$ when $x = 0$, with no real value between $y = +1$ and $y = -1$. The graph of $y = f(x)$ has two branches, one for which $y \geq 1$ and $y \rightarrow +\infty$ as $x \rightarrow \pm\infty$, and one for which $y \leq 1$ and $y \rightarrow -\infty$ as $x \rightarrow \pm\infty$. The sketch of the $y = f(x)$ should look like graph (a) below.

The graph (b) of $x = f^{-1}(y)$ is identical to (a), but rotated around the line $x = y$



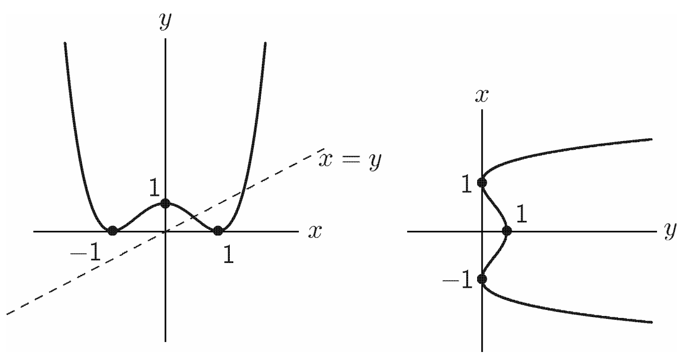
(a) $y = f(x)$ **(b)** $x = f^{-1}(y)$

31. **(i)** $y=(x^2-1)^2 \ \rightarrow\ \pm\sqrt{y}=x^2-1 \ \rightarrow\ x=f^{-1}(y)=\pm\sqrt{1\pm\sqrt{y}}$

(ii) $y=f(x)=(x^2-1)^2$

We have $y=1$ when $x=0$, and $y=0$ when $x=\pm 1$. The value of the function is positive for all values of x, and $y\rightarrow+\infty$ as $x\rightarrow\pm\infty$. The sketch of $y=f(x)$ should look like graph (a) below.

The graph (b) of $x=f^{-1}(y)$ is identical to (a), but rotated around the line $x=y$



(a) $y=f(x)$ **(b)** $x=f^{-1}(y)$

32. The virial equation of state of a gas can be approximated at low pressure as

$$pV_{\text{m}}=RT\left(1+\frac{B}{V_{\text{m}}}\right)$$

where *p* is the pressure, V_{m} is the molar volume, *T* is the temperature, *R* is the gas constant, and *B* is the second virial coefficient. Express *B* as an explicit function of the other variables.

$$pV_{\text{m}}=RT\left(1+\frac{B}{V_{\text{m}}}\right) \ \rightarrow\ \frac{pV_{\text{m}}}{RT}=1+\frac{B}{V_{\text{m}}}$$

$$\rightarrow\ \frac{B}{V_{\text{m}}}=\frac{pV_{\text{m}}}{RT}-1$$

Therefore $B=V_{\text{m}}\left[\frac{pV_{\text{m}}}{RT}-1\right]$

33. Kohlrausch's law for the molar conductivity Λ_{m} of a strong electrolyte at low concentration c is

$$\Lambda_{\mathrm{m}} = \Lambda_{\mathrm{m}}^{0} - \mathcal{K}\sqrt{c}$$

where Λ_{m}^{0} is the molar conductivity at infinite dilution and $\mathcal{K}$ is a constant. Express c as an explicit function of Λ_{m}.

$$\Lambda_{\mathrm{m}} = \Lambda_{\mathrm{m}}^{0} - \mathcal{K}\sqrt{c} \;\rightarrow\; \mathcal{K}\sqrt{c} = \Lambda_{\mathrm{m}}^{0} - \Lambda_{\mathrm{m}}$$

$$\rightarrow\; \sqrt{c} = \frac{\Lambda_{\mathrm{m}}^{0} - \Lambda_{\mathrm{m}}}{\mathcal{K}}$$

Therefore $$c = \left[\frac{\Lambda_{\mathrm{m}}^{0} - \Lambda_{\mathrm{m}}}{\mathcal{K}}\right]^2$$

34. The Langmuir adsorption isotherm

$$\theta = \frac{Kp}{1+Kp}$$

gives the fractional coverage θ of a surface by adsorbed gas at pressure p, where K is a constant. Express p in terms of θ.

$$\theta = \frac{Kp}{1+Kp} \;\rightarrow\; \theta(1+Kp) = Kp \;\rightarrow\; \theta + \theta Kp = Kp$$

$$\rightarrow\; Kp(1-\theta) = \theta$$

Therefore $$p = \frac{\theta}{K(1-\theta)}$$

35. In Example 2.12 on the van der Waals equation, verify the explicit expressions given for T and p, and the cubic equation in V.

For T: $$\left(p + \frac{n^2 a}{V^2}\right)(V-nb) - nRT = 0 \;\rightarrow\; \left(p + \frac{n^2 a}{V^2}\right)(V-nb) = nRT$$

Therefore $$\rightarrow\; T = \frac{1}{nR}\left(p + \frac{n^2 a}{V^2}\right)(V-nb)$$

For p: $$\left(p + \frac{n^2 a}{V^2}\right)(V-nb) - nRT = 0 \;\rightarrow\; p + \frac{n^2 a}{V^2} = \frac{nRT}{V-nb}$$

Therefore $$p = \frac{nRT}{V-nb} - \frac{n^2 a}{V^2}$$

Section 2.5

Expand (write out in full):

36. $\sum_{n=0}^{2}(n+1)x^n = (0+1)x^0 + (1+1)x^1 + (2+1)x^2$

$= 1 + 2x + 3x^2$

37. $\sum_{i=0}^{3} i x^{i-1} = 0x^{0-1} + 1x^{1-1} + 2x^{2-1} + 3x^{3-1}$

$= 1 + 2x + 3x^2$

38. $\sum_{k=1}^{3} k(k+1)x^{-k} = 1\times 2x^{-1} + 2\times 3x^{-2} + 3\times 4x^{-3}$

$= \frac{2}{x} + \frac{6}{x^2} + \frac{12}{x^3}$

39. $\sum_{n=0}^{3} n!\,x^{n^2} = 0!\,x^0 + 1!\,x^1 + 2!\,x^{2^2} + 3!\,x^{3^2}$

$= 1 + x + 2x^4 + 6x^9$

40. Find the equation and sketch the graph of the straight line that passes through the points: $(-2, -5)$ and $(1, 4)$

Let the line be $y = mx + c$. Then:

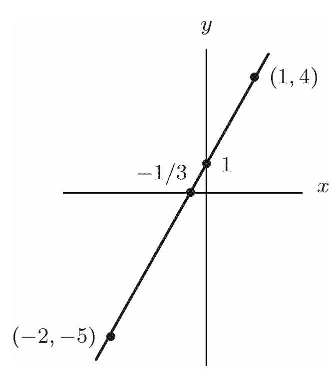
at point $(x_1, y_1) = (-2, -5)$, $\quad -5 = -2m + c$

at point $(x_2, y_2) = (1, 4)$, $\quad 4 = m + c$

Solution of the pair of simultaneous equations gives

$m = 3$ and $c = 1$.

Therefore $\quad y = 3x + 1$



41. Find the equation and sketch the graph of the straight line that passes through the points: $(-1, 6)$ and $(3, -2)$

Let the line be $y = mx + c$. Then:

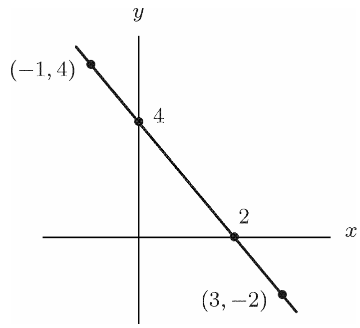
at point $(x_1, y_1) = (-1, 6)$, $6 = -m + c$

at point $(x_2, y_2) = (3, -2)$, $3 = \ \ m + c$

Solution of the pair of simultaneous equations gives

$m = -2$ and $c = 4$.

Therefore $y = -2x + 4$



42. Explain how $\mathcal{K}$ and Λ_m^0 in Kohlrausch's law (Exercise 33),

$$\Lambda_m = \Lambda_m^0 - \mathcal{K}\sqrt{c}$$

can be obtained graphically from the results of measurements of Λ_m over a range of concentration *c*.

Plot Λ_m against $\sqrt{c}$ for a straight line. The slope of the line is $-\mathcal{K}$ and the intercept with the Λ_m axis is Λ_m^0

43. The Debye equation

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{\rho}{M}\frac{N_A}{3\varepsilon_0}\left(\alpha + \frac{\mu^2}{3kT}\right)$$

relates the relative permittivity (dielectric constant) ε_r of a pure substance to the dipole moment μ and polarizability α of the constituent molecules, where ρ is the density at temperature *T*, and *M*, N_A, *k*, and ε_0 are constants. Explain how μ and α can be obtained graphically from the results of measurements of ε_r and ρ over a range of temperatures.

Plot $\frac{1}{\rho}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$ against $\frac{1}{T}$ for a straight line whose slope is $\frac{N_A \mu^2}{9M\varepsilon_0 k}$ and whose intercept is $\frac{N_A \alpha}{3M\varepsilon_0}$.

Hence μ and α.

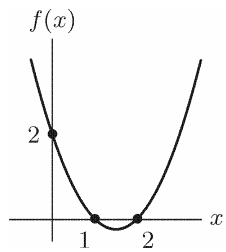
Find the roots and sketch the graphs of the quadratic functions:

44. $f(x) = x^2 - 3x + 2 = (x-1)(x-2)$

$= 0$ when $x = 1$ and $x = 2$

The roots of the quadratic are $x_1 = 1$ and $x_2 = 2$, for which $f(x) = 0$ and the graph of the function crosses the *x*-axis. Also, $f(0) = 2$ and $f(x) \to +\infty$ as $x \to \pm\infty$.

The sketch of the function should look like

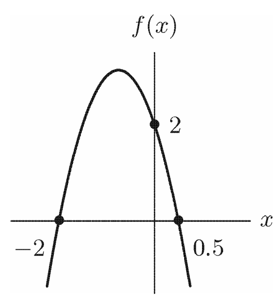


45. $f(x) = -2x^2 - 3x + 2 = -(2x-1)(x+2)$

$= 0$ when $x = 1/2$ and $x = -2$

The roots of the quadratic are $x_1 = 1/2$ and $x_2 = -2$. Also, $f(0) = 2$ and $f(x) \to -\infty$ as $x \to \pm\infty$.

The sketch of the function should look like

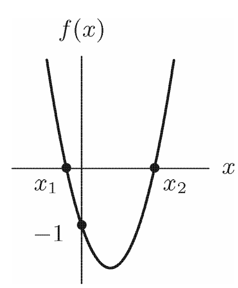


46. $f(x) = 3x^2 - 3x - 1$

$= 0$ when $x = \dfrac{3 \pm \sqrt{9+12}}{6} = \dfrac{1}{6}\left(3 \pm \sqrt{21}\right)$

The roots of the quadratic are $x_1 = (3 + \sqrt{21})/6$ and $x_2 = (3 - \sqrt{21})/6$. Also, $f(0) = -1$ and $f(x) \to +\infty$ as $x \to \pm\infty$.

The sketch of the function should look like

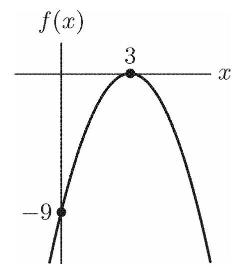


47. $f(x) = -x^2 + 6x - 9 = -(x-3)^2$

$= 0$ when $x = 3$

The quadratic has the double root $x_1 = x_2 = 3$. Also, $f(0) = -9$ and $f(x) \to -\infty$ as $x \to \pm\infty$.

The sketch of the function should look like

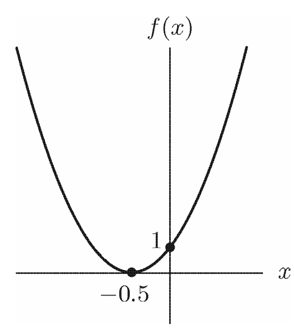


48. $f(x) = 4x^2 + 4x + 1 = (2x+1)^2$

$= 0$ when $x = -1/2$

The quadratic has the double root $x_1 = x_2 = -1/2$. Also, $f(0) = 1$ and $f(x) \to +\infty$ as $x \to \pm\infty$.

The sketch of the function should look like

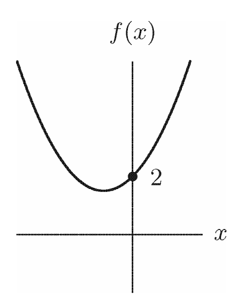


49. $f(x) = x^2 + x + 2$

$= 0$ when $x = \dfrac{-1 \pm \sqrt{1-8}}{2} = \dfrac{1}{2}\left(-1 \pm i\sqrt{7}\right)$

The roots of the quadratic are complex and the graph of the function does not cross the *x*-axis; $f(x) > 0$ for all real values of *x*. Also, $f(0) = 2$ and $f(x) \to +\infty$ as $x \to \pm\infty$.

The sketch of the function should look like

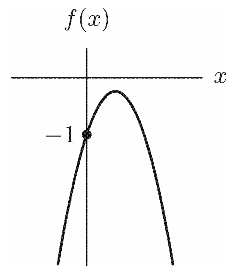


50. $f(x) = -3x^2 + 3x - 1$

$$= 0 \text{ when } x = \frac{3 \pm \sqrt{9-12}}{6} = \frac{1}{6}\left(3 \pm i\sqrt{3}\right)$$

The roots of the quadratic are complex and the graph of the function does not cross the *x*-axis; $f(x) < 0$ for all real values of *x*.. Also, $f(0) = -1$ and $f(x) \to -\infty$ as $x \to \pm\infty$.

The sketch of the function should look like



51. If $y = \dfrac{2x^2 + x + 1}{2x^2 + x - 1}$ find *x* as a function of *y*.

$$y = \frac{2x^2 + x + 1}{(2x^2 + x - 1)} \;\to\; y(2x^2 + x - 1) = 2x^2 + x + 1$$

$$\to y(2x^2 + x - 1) - (2x^2 + x + 1) = 0$$

$$\to 2(y-1)x^2 + (y-1)x - (y+1) = 0$$

$$\to 2x^2 + x - \frac{y+1}{y-1} = 0$$

This is a quadratic equation in *x*, with solutions

$$x = \frac{1}{4}\left[-1 \pm \sqrt{1 + 8\,\frac{y+1}{y-1}}\right]$$

52. The acidity constant K_a of a weak acid at concentration *c* is

$$K_a = \frac{\alpha^2 c}{1-\alpha}$$

where α is the degree of ionization. Express α in terms of K_a and *c* (remember that α, K_a, and *c* are positive quantities).

$$K_a = \frac{\alpha^2 c}{1-\alpha} \;\to\; K_a(1-\alpha) = \alpha^2 c$$

$$\to \alpha^2 c + K_a\alpha - K_a = 0$$

The quadratic in α has roots

$$\alpha = \frac{1}{2c}\left[-K_a \pm \sqrt{K_a{}^2 + 4K_a c}\right]$$

Now $K_a > 0$, and $c > 0$, so that $K_a{}^2 + 4K_a c > K_a$. Therefore, for positive degree of dissociation,

$$\alpha = \frac{1}{2c}\left[-K_a + \sqrt{K_a{}^2 + 4K_a c}\right]$$
$$= \frac{K_a}{2c}\left[\sqrt{1+\frac{4c}{K_a}} - 1\right]$$

53. Given that $x-1$ is a factor of the cubic x^3+4x^2+x-6, **(i)** find the roots, **(ii)** sketch the graph

(i) Because $x-1$ is a factor, the cubic can be factorized:

$$x^3+4x^2+x-6 = (x-1)(x^2+ax+b)$$
$$= x^3+(a-1)x^2+(b-a)x-b$$

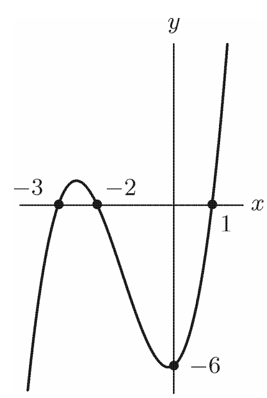
Then $a=5$ and $b=6$,

and $x^3+4x^2+x-6 = (x-1)(x^2+5x+6)$
$= (x-1)(x+2)(x+3)$

The roots of the cubic are $x = 1, -2, -3$.

(ii) The graph of the function $y = x^3+4x^2+x-6$, crosses the *x*-axis at $x = 1, -2,$ and -3 and the *y*-axis at $y = -6$. Also $y \to +\infty$ as $x \to +\infty$ and $y \to -\infty$ as $x \to -\infty$.

The sketch of the function should look like



54. Given that $x-1$ is a factor of the cubic x^3-6x^2+9x-4, **(i)** find the roots, **(ii)** sketch the graph.

(i) Because $x-1$ is a factor, the cubic can be factorized:

$$x^3-6x^2+9x-4=(x-1)(x^2+ax+b)=x^3+(a-1)x^2+(b-a)x-b$$

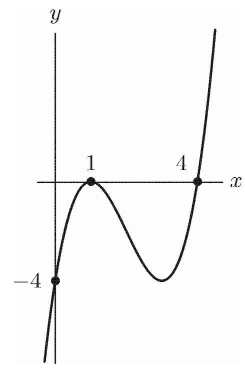
Then $a=-5$ and $b=4$,

and
$$\begin{aligned}x^3-6x^2+9x-4&=(x-1)(x^2-5x+4)\\&=(x-1)(x-1)(x-4)\end{aligned}$$

The roots of the cubic are $x=1$ (double), 4 .

(ii) The graph of the function $y=x^3-6x^2+9x-4$, touches the x-axis at $x=1$ and crosses it at $x=4$. It crosses the y-axis at $y=-4$. Also $y\to+\infty$ as $x\to+\infty$ and $y\to-\infty$ as $x\to-\infty$.

The sketch of the function should look like



55. Given that $x-1$ is a factor of the cubic x^3-3x^2+3x-1, **(i)** find the roots, **(ii)** sketch the graph.

(i) Because $x-1$ is a factor, the cubic can be factorized:

$$x^3-3x^2+3x-1=(x-1)(x^2+ax+b)=x^3+(a-1)x^2+(b-a)x-b$$

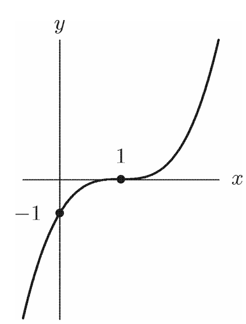
Then $a=-2$ and $b=1$,

and
$$\begin{aligned}x^3-3x^2+3x-1&=(x-1)(x^2-2x+1)\\&=(x-1)^3\end{aligned}$$

The cubic has the triple root $x=1$.

(ii) The graph of the function $y=x^3-3x^2+3x-1$, crosses the x-axis at $x=1$. It crosses the y-axis at $y=-1$. Also $y\to+\infty$ as $x\to+\infty$ and $y\to-\infty$ as $x\to-\infty$.

The sketch of the function should look like



56. Given that $x^2 - 1$ is a factor of the quartic $x^4 - 5x^3 + 5x^2 + 5x - 6$ **, (i)** find the roots, **(ii)** sketch the graph.

(i) Because $x^2 - 1$ is a factor, the quartic can be factorized:

$$\begin{aligned} x^4 - 5x^3 + 5x^2 + 5x - 6 &= (x^2 - 1)(x^2 + ax + b) \\ &= x^4 + ax^3 + (b-1)x^2 - ax - b \end{aligned}$$

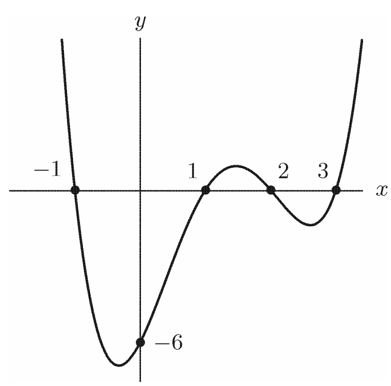
Then $a = -5$ and $b = 6$,

and
$$\begin{aligned} x^4 - 5x^3 + 5x^2 + 5x - 6 &= (x^2 - 1)(x^2 - 5x + 6) \\ &= (x^2 - 1)(x - 2)(x - 3) \\ &= (x + 1)(x - 1)(x - 2)(x - 3) \end{aligned}$$

The quartic has the roots $x = -1, 1, 2, 3$.

(ii) The graph of the function $y = x^4 - 5x^3 + 5x^2 + 5x - 6$, crosses the *x*-axis at $x = -1$, $x = 1$, $x = 2$, $x = 3$. It crosses the *y*-axis at $y = -6$. Also $y \to +\infty$ as $x \to \pm\infty$.

The sketch of the function should look like



Section 2.6

Use algebraic division to reduce the rational function to proper form:

57.	$\frac{2x-1}{x+3}$ → $= 2 - \frac{7}{x+3}$ ←		$x+3 \overline{)\,2x-1}$, quotient 2 $2x+6$ -7 remainder
58.	$\frac{3x^3-2x^2-x+4}{x+2}$ → $= 3x^2-8x+15-\frac{26}{x+2}$ ←		$x+2 \overline{)\,3x^3-2x^2-x+4}$, quotient $3x^2-8x+15$ $3x^3+6x^2$ $-8x^2-x+4$ $-8x^2-16x$ $15x+4$ $15x+30$ -26
59.	$\frac{x^3+2x^2-5x-6}{x+1}$ → $= x^2+x-6$ ←		$x+1 \overline{)\,x^3+2x^2-5x-6}$, quotient x^2+x-6 x^3+x^2 x^2-5x-6 x^2+x $-6x-6$ $-6x-6$ 0
60.	$\frac{2x^4-3x^3+4x^2-5x+6}{x^2-2x-2}$ → $= 2x^2+x+10+\frac{17x+26}{x^2-2x-2}$ ←		$x^2-2x-2 \overline{)\,2x^4-3x^3+4x^2-5x+6}$, quotient $2x^2+x+10$ $2x^4-4x^3-4x^2$ x^3+8x^2-5x+6 x^3-2x^2-2x $10x^2-3x+6$ $10x^2-20x-20$ $17x+26$

Section 2.7

61. Express $\dfrac{1}{(x-1)(x+2)}$ in terms of partial fractions.

Let $$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Therefore $$\frac{1}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

For this to be true for all values of x it is required that the numerators on the two sides of the equal sign be equal:

$$1 = A(x+2) + B(x-1)$$

The values of A and B can be obtained by making suitable choices of the variable x. Thus

when $x = 1$: $1 = 3A$ and $A = 1/3$

when $x = -2$: $1 = -3B$ and $B = -1/3$

Therefore $$\frac{1}{(x-1)(x+2)} = \frac{1}{3}\left[\frac{1}{x-1} - \frac{1}{x+2}\right]$$

62. Express $\dfrac{x+2}{x(x+3)}$ in terms of partial fractions.

Let $$\frac{x+2}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} = \frac{A(x+3)+Bx}{x(x+3)}$$

It is required that

$$x + 2 = A(x+3) + Bx$$

Then when $x = 0$: $2 = 3A$ and $A = 2/3$

when $x = -3$: $-1 = -3B$ and $B = 1/3$

Therefore $$\frac{x+2}{x(x+3)} = \frac{1}{3}\left[\frac{2}{x} + \frac{1}{x+3}\right]$$

63. Express $\dfrac{x-2}{x^2+3x+2}$ in terms of partial fractions.

Let $$\frac{x-2}{x^2+3x+2}=\frac{A}{x+2}+\frac{B}{x+1}=\frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$$

It is required that

$$x-2=A(x+1)+B(x+2)$$

Then when $x=-2$: $-4=-A$ and $A=4$

when $x=-1$: $-3=B$ and $B=-3$

Therefore $$\frac{x+2}{x(x+3)}=\frac{4}{x+2}-\frac{3}{x+1}$$

64. Express $\dfrac{2x^2-5x+7}{x(x-1)(x+2)}$ in terms of partial fractions.

Let $$\frac{2x^2-5x+7}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}$$

$$=\frac{A(x-1)(x+2)+Bx(x+2)+Cx(x-1)}{x(x-1)(x+2)}$$

It is required that

$$2x^2-5x+7=A(x-1)(x+2)+Bx(x+2)+Cx(x-1)$$

Then when $x=0$: $7=-2A$ and $A=-7/2$

when $x=1$: $4=3B$ and $B=4/3$

when $x=-2$: $25=6C$ and $B=25/6$

Therefore $$\frac{2x^2-5x+7}{x(x-1)(x+2)}=\frac{1}{6}\left[-\frac{21}{x}+\frac{8}{x-1}+\frac{25}{x+2}\right]$$

65. Express $\dfrac{x^2+2x-1}{(x+2)(x-1)^2}$ in terms of partial fractions.

Let $$\frac{x^2+2x-1}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$=\frac{A(x-1)^2+B(x-1)(x+2)+C(x+2)}{(x+2)(x-1)^2}$$

It is required that

$$x^2 + 2x - 1 = A(x-1)^2 + B(x-1)(x+2) + C(x+2)$$

Then when $x = 1$: $2 = 3C$ and $C = 2/3$

when $x = -2$: $-1 = 9A$ and $A = -1/9$

when $x = 0$: $-1 = A - 2B + 2C \rightarrow B = (1 + A + 2C)/2 = 10/9$

Therefore $$\frac{x^2 + 2x - 1}{(x+2)(x-1)^2} = \frac{1}{9}\left[-\frac{1}{x+2} + \frac{10}{x-1} + \frac{6}{(x-1)^2}\right]$$

Section 2.8

Solve the simultaneous equations:

66. (1) $x + y = 3$
(2) $x - y = 1$

To solve for x, add the equations:

$(1) + (2)$ $2x = 4 \rightarrow x = 2$

Substitution for x in (1) then gives $y = 1$.

Therefore $x = 2,\ y = 1$

and the lines cross at point $(x,\ y) = (2, 1)$.

67. (1) $3x - 2y = 1$
(2) $2x + 3y = 2$

To solve for y, multiply equation (1) by 2, and equation (2) by 3,

$(1')$ $6x - 4y = 2$
$(2')$ $6x + 9y = 6$

and subtract $(1')$ from $(2')$ to give $13y = 4 \rightarrow y = 4/13$. Substitution for y in (1) then gives $3x - 8/13 = 1 \rightarrow x = 7/13$.

Therefore $x = 7/13,\ y = 4/13$

and the lines therefore cross at point $(x,\ y) = (7/13,\ 4/13)$.

68. (1) $3x-2y=1$
(2) $6x-4y=6$

To solve, subtract twice (1) from (2):

(1) $3x-2y=1$
(2′) $0=4$

The second equation is not possible. The equations are said to be inconsistent and there is no solution. Graphically, the equations describe parallel lines.

69. (1) $3x-2y=1$
(2) $6x-4y=2$

In this case, doubling equation (1) gives equation (2) and there is effectively only one independent equation; both equations represent the same line. The equations are said to be linearly dependent and it is only possible to obtain a partial solution

$x=(1+2y)/3$ for all values of *y*.

70. (1) $x-2y+3z=3$
(2) $2x-y-2z=8$
(3) $3x+3y-z=1$

To solve, eliminate *x* from equations (2) and (3) by subtracting 2 × (1) from (2) and 3 × (1) from (3):

(1) $x-2y+3z=3$
(2′) $3y-8z=2$
(3′) $9y-10z=-8$

Now eliminate *y* from $(3')$ by subtracting $3\times(2')$ from $(3')$:

(1) $x-2y+3z=3$
(2′) $3y-8z=2$
(3″) $14z=-14$

The equations can now be solved in reverse order: $(3'')$ is $z=-1$, then $(2')$ is $3y+8=2$ so that $y=-2$, and (1) is $x+4-3=3$ so that $x=2$.

Therefore $x=2, y=-2, z=-1$ and the lines cross at $(x, y, z)=(2,-2,-1)$.

71. (1) $2x - y = 1$

(2) $x^2 - xy + y^2 = 1$

Equation (1) can be solved for *y* in terms of *x* and the result substituted in equation (2). Thus, from (1), $y = 2x - 1$, and (2) becomes

$$x^2 - (2x-1)x + (2x-1)^2 - 1 = 0$$
$$\rightarrow\ 3x^2 - 3x = 0$$
$$\rightarrow\ x(x-1) = 0$$

This has roots $x_1 = 0$ and $x_2 = 1$, with corresponding value of *y*, $y_1 = -1$ and $y_2 = 1$. In this case the two solutions are the points at which the straight line (1) crosses the ellipse (2); $(x_1, y_1) = (0, -1)$ and $(x_2, y_2) = (1, 1)$, as demonstrated in the figure below.

72. (1) $2x - y = 2$

(2) $x^2 - xy + y^2 = 1$

As in exercise 71, equation (1) can be solved for *y* in terms of *x* and the result substituted in equation (2). Thus, from (1), $y = 2x - 2$, and (2) becomes

$$x^2 - (2x-2)x + (2x-2)^2 - 1 = 0$$
$$\rightarrow\ 3x^2 - 6x + 3 = 0$$
$$\rightarrow\ (x-1)^2 = 0$$

This has the double root $x = 1$, with corresponding value $y = 0$. In this case the line (1) is tangent to (touches) the ellipse (2) at point $(x, y) = (1, 0)$, as demonstrated in the figure below.

73. (1) $2x - y = 3$

(2) $x^2 - xy + y^2 = 1$

As in Exercises 71 and 72, equation (1) can be solved for *y* in terms of *x* and the result substituted in equation (2). Thus, from (1), $y = 2x - 3$, and (2) becomes

$$x^2 - (2x-3)x + (2x-3)^2 - 1 = 0$$
$$\rightarrow\ 3x^2 - 9x + 8 = 0$$

The roots of the quadratic are complex,

$$x = \frac{9 \pm \sqrt{81-96}}{6} = \frac{9 \pm i\sqrt{15}}{6}$$

and the line dose not touch the ellipse, as demonstrated in the figure below.

Note

Exercises 71 to 73 are special cases of the line $y = 2x - a$ and the ellipse $x^2 - xy + y^2 = 1$. The general solution is $x = \frac{3a \pm \sqrt{12 - 3a^2}}{6}$. The line crosses the ellipse when $|a| < 2$, touches the ellipse when $a = \pm 2$, and misses the ellipse when $|a| > 2$, as demonstrated in the following figure.