PART VI: Discrete mathematics

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Chapter 35: Sets

35.1. (a) $\mathbb{N}^+$ (see (35.2)) is the set of all positive integers. Hence

$$S = \{x | x \in \mathbb{N}^+ \text{ and } 3 \leq x \leq 10\}$$

is the set

$$S = \{3, 4, 5, 6, 7, 8, 9, 10\}.$$

(b) The set

$$S = \{x | x \in \mathbb{N}^+ \text{ and } -2 \leq x \leq 4\}$$

is

$$S = \{1, 2, 3, 4\},$$

since -2, -1 and 0 do not belong to $\mathbb{N}^+$.
(c) $\mathbb{Z}$ is the set of all integers. Hence

$$s = \{x | x \in \mathbb{Z} \text{ and } -2 \leq x \leq 4\}$$

is the set

$$S = \{-2, -1, 0, 1, 2, 3, 4\}.$$

(d) $\mathbb{N}^-$ is the set of all negative integers. Hence

$$S = \{x | x \in \mathbb{N}^+ \text{ or } \mathbb{N}^{-1} \text{ and } -2 \leq x \leq 4\}$$

is the set

$$S = \{-2, -1, 1, 2, 3, 4\}.$$

(e) The set defined by

$$S = \{(1/x) | x \in \mathbb{N}^+ \text{ and } 3 \leq x \leq 8\}$$

is the set of rational numbers

$$S = \{\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \tfrac{1}{8}\}.$$

(f) The set defined by

$$S = \{x^2 | x \in \mathbb{N}^+ \text{ and } |x| \leq 3\}$$

is the set of integers

$$S = \{1, 4, 9\}.$$

(g) The set

$$S = \{(x + \mathrm{i}y) | x \in \mathbb{N}^+, y \in \mathbb{N}^+, 1 \leq 4, 2 \leq 5\}$$

is the set of complex numbers

$$\{1 + 2\mathrm{i}, 1 + 3\mathrm{i}, 1 + 4\mathrm{i}, 1 + 5\mathrm{i}, 2 + 2\mathrm{i}, 2 + 3\mathrm{i}, 2 + 4\mathrm{i}, 2 + 5\mathrm{i},$$

$$3 + 2\mathrm{i}, 3 + 3\mathrm{i}, 3 + 4\mathrm{i}, 3 + 5\mathrm{i}, 4 + 2\mathrm{i}, 4 + 3\mathrm{i}, 4 + 4\mathrm{i}, 4 + 5\mathrm{i}\}.$$

35.2. The sets specified are shown shaded in Venn diagrams :

35.3. We require the union of the two sets in each case.

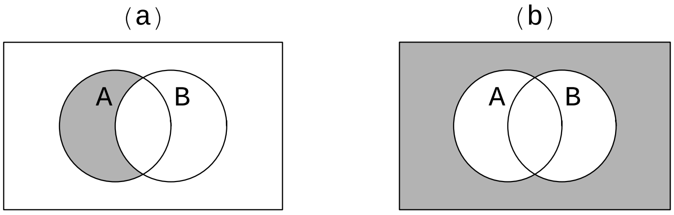

Figure 1: Problem 35.2(a): $A \cup \overline{B}$; (b) $\overline{A} \cap \overline{B}$.

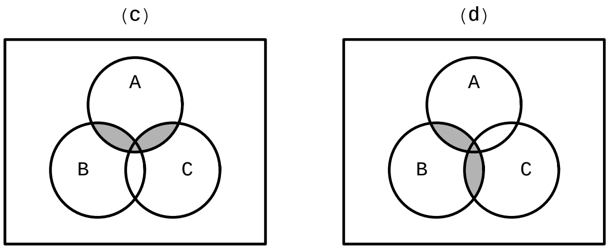

Figure 2: Problem 35.2(c): $A \cap (B \cup C)$; (d) $(A \cap B) \cup (B \cap C)$.

(a) $A = \{x | x \in \mathbb{R} \text{ and } -1 \leq x \leq 2\}$, $B = \{x | x \in \mathbb{R} \text{ and } -1 \leq x \leq 4\}$, where $\mathbb{R}$ is the set of all real numbers.

$$A \cup B = \{x | x \in \mathbb{R} \text{ and } -1 \leq x \leq 4\} = B.$$

(b) $A = \{x | x \in \mathbb{R} \text{ and } -1 \leq x < 0\}$, $B = \{x | x \in \mathbb{R} \text{ and } 0 < x < 1\}$.

$$A \cup B = \{x | x \in \mathbb{R} \text{ and } -1 < x < 1, x \neq 0\}.$$

(c) $A = \{1, 2, 3, 4\}$, $B = \{-4, -3, -2, -1\}$.

$$A \cup B = \{-4, -3, -2, -1, 1, 2, 3, 4\}.$$

(d) $A = \{y | y = \cos x, x \in \mathbb{R}, \text{ and } 0 \leq x \leq \frac{1}{2}\pi,$
$B = \{y | y = \sin x, x \in \mathbb{R}, \text{ and } -\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi.$

$$A \cup B = \{y | y \in \mathbb{R} \text{ and } -1 \leq y \leq 1\}.$$

35.4. We require the intersection of the two sets in each case.
(a) $A = \{x | x \in \mathbb{R}, \text{ and } -2 \leq x \leq 1\}$, $B = \{x | x \in \mathbb{R}, \text{ and } -1 \leq x \leq 2\}$. Then

$$A \cap B = \{x | x \in \mathbb{R} \text{ and } -1 \leq x \leq 1\}.$$

(b) $A = \{x | x \in \mathbb{N}^+ \text{ and } -5 \leq x \leq 2\}$, $B = \{x | x \in \mathbb{R}, \text{ and } -5 \leq x \leq 2\}$, where $\mathbb{N}^+$ is the set of all positive integers. Then

$$A \cap B = \{1, 2\}.$$

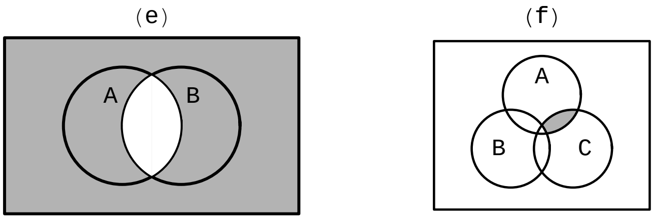

Figure 3: Problem 35.2(e): $\overline{A \cap B}$; (f) $(A \backslash B) \cap C$.

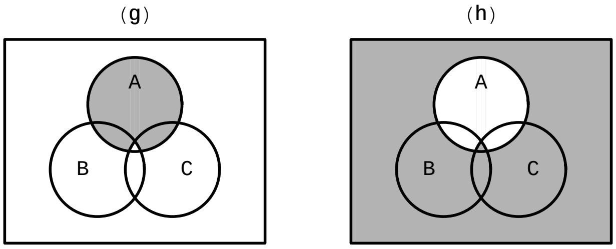

Figure 4: Problem 35.2(g): $A\backslash(B\cap C)$; (h) $\overline{(A\backslash B)\cup(B\backslash C)}$.

(c) $A=\{n|n=1/m \text{ and } m\in\mathbb{N}^+\}$, $B=\{n|n\in 1/m^2 \text{ and } m\in\mathbb{N}^{+1}\}$. The numbers $\{1/m\}$ or $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$ includes the numbers $\{1/m^2\}$ or $\{1,\frac{1}{4},\frac{1}{9},\ldots\}$. Hence

$$A\cap B=\{n|n=1/m^2 \text{ and } n\in\mathbb{N}^+\}.$$

(d) $A=\{x|x\in\mathbb{R} \text{ and } x^2-3x+2=0\}$, $B=\{x|x\in\mathbb{R} \text{ and } 2x^2+x-3=0\}$. The quadratic equations can be factorized as follows:

$$x^2-3x+2=(x-1)(x-2)=0,\quad 2x^2+x-3=(x-1)(2x+3)=0.$$

These quadratic equations have the common solution $x=1$. Hence

$$A\cap B=\{1\}.$$

(e) $A=\{x|x\in\mathbb{R} \text{ and } |x|\le 2\}$, $B=\{x|x\in\mathbb{R} \text{ and } |x-1|\le 1\}$. Since $|x|\le 2$ is equivalent to $-2\le x\le 2$, and $|x-1|\le 1$ is equivalent to $0\le x\le 2$, the intersection of the intervals is $0\le x\le 2$. Hence

$$A\cap B=\{x|x\in\mathbb{R} \text{ and } |x-1|\le 1\}=B.$$

35.5. Note that there can be many ways of representing the shaded set using standard operations.
(a) The shaded region lies in the intersection of sets B and C which does not lie in A. The intersection of B and C is $B\cap C$. The part of this set which does not lie in A is $(B\cap C)\backslash A$.

(b) $B\backslash(A\cup C)$; or $\overline{A\cup C}\cap B$.

(c) The smaller shaded region is the intersection of all the sets, namely $A\cap(B\cap C)$, whilst the larger region is given by (b), namely $B\backslash(A\cup C)$. Hence the shaded set is the union of these sets: $(A\cap(B\cap C))\cup(B\backslash(A\cup C))$.

(d) $B\cap(C\cap D)\backslash A$ or $(B\cap D)\backslash A$.

35.6. (a) The set S_1 consists of those products which pass test 1, S_2 those products which pass test 2. The union $S_1\cup S_2$ contains those products which pass at least one of the tests 1 and 2. Similarly, the union

$$S_1\cup S_2\cup S_3\cup\ldots\cup S_n$$

contains those products which pass at least one test. Hence the set of products which fail every test is the complement of this set. Hence

$$\text{(the set of products which fail all tests)}=S\backslash(S_1\cup S_2\cup S_3\cup\ldots\cup S_n).$$

(b) We require the set consisting of products which lie in S_1 but not in any of $S_2,S_3,\ldots,S_n$. Hence

$$\text{(the set of products which fail only test 1)}=S_1\backslash(S\cup S_2\cup S_3\cup\ldots\cup S_n).$$

(c) The products which pass all tests belong to the set $S_1\cup S_2\cup\ldots\cup S_n$. The set of products which fail some (or all) tests is

$$\overline{(S_1\cup S_2\cup S_3\cup\ldots\cup S_n)}.$$

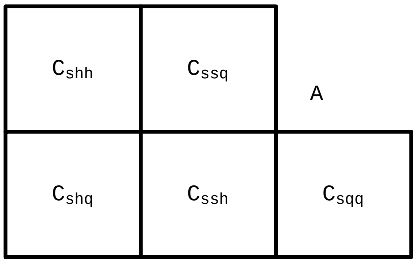

Figure 5: Problem 35.7

35.7. Let C_{shh} represent the set of students taking one science subject (s) and two humanities (h) subjects , C_{ssq} two science subjects and one social science (q) subject, and so on. A set view of the five possible sets of students is shown in the figure. The set A_1 is the union of C_{shh} and C_{shq}, the set B_1 is the union of C_{shq} and C_{ssh}, and B_2 is C_{sqq}. Hence

$$A_1 = C_{shh} \cup C_{shq} \cup C_{sqq}, \quad B_1 = C_{shq} \cup C_{ssh}, \quad B_2 = C_{sqq}.$$

(a) We require the union of C_{shq} and C_{ssq}. The set C_{shq} is the intersection of A_1 and B_1, thus

$$C_{shq} = A_1 \cap B_1.$$

The set C_{ssq} is in A but not in the union of A_1, B_1 and B_2. Hence

$$C_{ssq} = A \backslash (A_1 \cup B_1 \cup B_2).$$

Finally

$$C_{shq} \cup C_{ssq} = (A_1 \cap B_1) \cup (A \backslash (A_1 \cup B_1 \cup B_2)).$$

(b) The required set of students who take no humanities subjects is the difference between A and the set of students who take at least one humanities subject, namely $A \backslash (A_1 \cup B_1) \cup B_2$ or $[(A \backslash A_1) \backslash B_1] \cup B_2$.

(c) The set of students who take one subject from each faculty is C_{shq} which is the intersection of A_1 and B_1. Hence

$$C_{shq} = A_1 \cap B_1.$$

35.8. (a) The set $(A \backslash B) \cap C$ is obtained by excluding from A the elements which also belong to B, and then shading the common region of this set with C as shown in Figure 6(i). The same region is obtained for $(A \cap C) \backslash B$ by removing that part of B which is in the intersection of A and C.

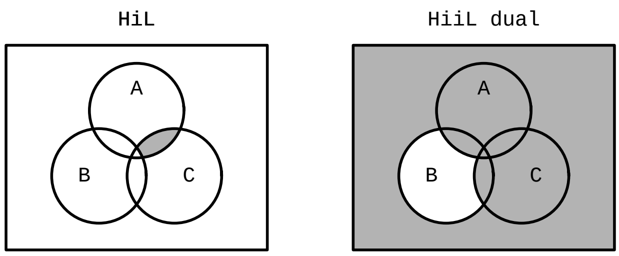

Figure 6: Problem 35.8(a): (i) $(A \backslash B) \cap C$; (ii) dual $(A \cup C) \backslash B$.

The identity includes the difference for which we have no rule given for duality. However we can express the difference as an intersection in the form $A \backslash B = A \cap \overline{B}$. The identity can then be expressed in terms of intersections as

$$(A \cap \overline{B}) \cap C = (A \cap C) \cap \overline{B}.$$

Hence the dual identity is

$$(A \cup \overline{B}) \cup C = (A \cup C) \cup \overline{B}.$$

Its Venn diagram is shown in Figure 6(ii).

(b) The set $A \cap (B \cup C)$ is the intersection of A with the union $B \cup C$, and is shown in Figure 7(i). The same set is given by the union of the intersections of A and B, and A and C, namely $(A \cap B) \cup (A \cap C)$. The dual identity is

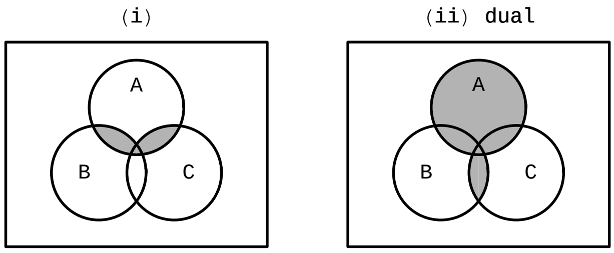

Figure 7: Problem 35.8(b): $A \cap (B \cup C)$; (ii) dual $(A \cup B) \cap (A \cup C)$.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

which is shown in Figure 7(ii).

35.9. Consider the Venn diagram shown in the figure. The identity

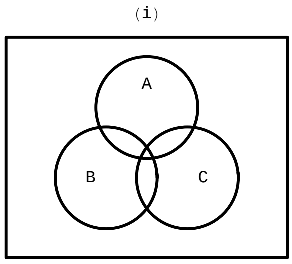

Figure 8: Problem 35.9

$$A \cap B \cap C = (A \cap B) \cup (B \cap C)$$

is not true in this Venn diagram, since $A \cap B \cap C$ is common to all the circles, that is it is represented by the central area as in Figure 35.4(b). On the other hand, $(A \cap B) \cup (B \cap C)$ includes regions common to both A and B and B and C. In other words the identity is only true for certain configurations in the Venn diagram. For this region the duality principle is not valid.

35.10. Given $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then the required cartesian products are

$$A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\},$$

$$B \times A = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\},$$

$$A^2 = A \times A = \{(1,1),(1,2),(2,1),(2,2)\},$$

$$B^2 = B \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}.$$

35.11. Given $A = \{1, 2, 3\}$, $B = \{0, 1\}$, $C = \{1, 2\}$, then the required cartesian products are

$$\begin{aligned} A \times B \times C &= \{1,2,3\} \times \{0,1\} \times \{1,2\} \\ &= \{(1,0,1),(1,0,2),(1,1,1),(1,1,2),(2,0,1),(2,0,2), \\ &\qquad (2,1,1),(2,1,2),(3,0,1),(3,0,2),(3,1,1),(3,1,2)\} \end{aligned}$$

$$\begin{aligned}
A^2 \times C &= \{1,2,3\} \times \{1,2,3\} \times \{1,2\} \\
&= \{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(1,3,1),(1,3,2), \\
&\quad (2,1,1),(2,1,2),(2,2,1),(2,2,2),(2,3,1),(2,3,2), \\
&\quad (3,1,1),(3,1,2),(3,2,1),(3,2,2),(3,3,1),(3,3,2)\}
\end{aligned}$$

The union $A \cup B = \{1,2,3\} \times \{0,1\} = \{0,1,2,3\}$. Hence

$$\begin{aligned}
(A \cup B) \times C &= \{0,1,2,3\} \times \{1,2\} \\
&= \{(0,1),(0,2),(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}
\end{aligned}$$

The intersection $A \cap B = \{1,2,3\} \cap \{0,1\} = \{1\}$. Hence

$$(A \cap B) \times C = \{1\} \times \{1,2\} = \{(1,1),(1,2)\}.$$

35.12. (See Example 35.7.) Figure 9 shows a Venn diagram of the process with sets P, Q and R representing the three quality checks. Let $\mathrm{n}(P), \mathrm{n}(Q), \ldots,$ be the numbers of components in sets $P, Q, \ldots$. We are given the following failure numbers:

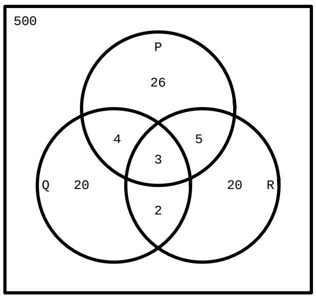

Figure 9: Problem 35.12

$$\mathrm{n}(P) = 38, \quad \mathrm{n}(Q) = 29, \quad \mathrm{n}(R) = 30,$$
$$\mathrm{n}(P \cap Q) = 7, \quad \mathrm{n}(Q \cap R) = 5, \quad \mathrm{n}(R \cap P) = 8,$$
$$\mathrm{n}(P \cap Q \cap R) = 3.$$

The numbers in the Venn diagram represent the number of failures in each undivided region. For example, 26 components fail check P only, since

$$\mathrm{n}(P) - \mathrm{n}(P \cap Q) - \mathrm{n}(P \cap R) + \mathrm{n}(P \cap Q \cap R) = 38 - 7 - 8 + 3 = 26 = N_P, \text{ (say)},$$

(the last term is added since otherwise $P \cap Q \cap R$ would be counted twice). Similarly, the numbers of components which just fail checks Q and R are

$$N_Q = 20, \quad N_R = 20.$$

(a) The number of components which pass all checks is

$$\begin{aligned}
N &= 500 - [\mathrm{n}(P) + \mathrm{n}(Q) + \mathrm{n}(R) - \mathrm{n}(P \cap Q) - \mathrm{n}(Q \cap R) \\
&\qquad - \mathrm{n}(R \cap P) + \mathrm{n}(P \cap Q \cap R)] \\
&= 500 - [38 + 29 + 30 - 7 - 5 - 8 + 3] \\
&= 500 - 80 = 420
\end{aligned}$$

(b) The number which fail just one check is

$$N_1 = N_P + N_Q + N_R = 26 + 20 + 20 = 66.$$

(c) From the Venn diagram, the number which fail just two checks is

$$N_2 = 4 + 5 + 2 = 11.$$

35.13. The method is a generalization of that for two sets given at the end of Section 35.3. In Figure 10, $U = A \cap B \cap C$ is the set of elements which are common to A, B and C, $X = A \backslash (B \cup C)$ are elements which are in A but not in B nor C, and so on. From the figure, for all the sets

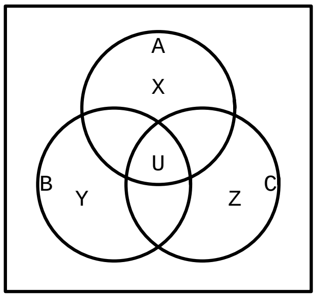

Figure 10: Problem 35.13; In the figure $X = A\backslash(B \cup C)$, $Y = B\backslash(C \cup A)$, $Z = C\backslash(A \cup B)$, $U = A \cap B \cap C$.

$$\begin{aligned} \mathrm{n}(A \cup B \cup C) &= \mathrm{n}(A\backslash B(\cup C)) + \mathrm{n}(B(\backslash(C \cup A)) + \mathrm{n}(C\backslash(A \cup B)) \\ &\quad \mathrm{n}(A \cap B) + \mathrm{n}(B \cap C) - 2\mathrm{n}(A \cap B \cap C). \qquad \text{(i)} \end{aligned}$$

For each set

$$\mathrm{n}(A) = \mathrm{n}(A\backslash(B \cup C)) + \mathrm{n}(A \cap B) + \mathrm{n}(A \cap C) - \mathrm{n}(A \cap B \cap C), \qquad \text{(ii)}$$

$$\mathrm{n}(B) = \mathrm{n}(B\backslash(A \cup C)) + \mathrm{n}(B \cap C) + \mathrm{n}(B \cap A) - \mathrm{n}(A \cap B \cap C), \qquad \text{(iii)}$$

$$\mathrm{n}(C) = \mathrm{n}(C\backslash(B \cup A)) + \mathrm{n}(C \cap A) + \mathrm{n}(C \cap A) - \mathrm{n}(A \cap B \cap C). \qquad \text{(iv)}$$

Eliminate $\mathrm{n}(A\backslash(B \cup C))$, $\mathrm{n}(B\backslash(A \cup C))$ and $\mathrm{n}(C\backslash(B \cup A))$ given by (ii), (iii) and (iv) in (i). The result is

$$\begin{aligned} \mathrm{n}(A \cup B \cup C) &= \mathrm{n}(A) + \mathrm{n}(B) + \mathrm{n}(C) + \mathrm{n}(A \cap B \cap C) \\ &\quad -\mathrm{n}(B \cap C) - \mathrm{n}(C \cap A) - \mathrm{n}(A \cap B) \end{aligned}$$

35.14. Let A contain p elements and B contain q elements so that $\mathrm{n}(A) = p$ and $\mathrm{n}(B) = q$. The set of ordered pairs in $A \times B$ consists of every combination of the elements in A with every element in B, which gives pq pairs. Hence

$$\mathrm{n}(A \times B) = \mathrm{n}(A)\mathrm{n}(B).$$

35.15. From the menu

$$\mathrm{n}(A) = 4, \quad \mathrm{n}(B), \quad \mathrm{n}(C) = 3.$$

For the full menu all possible choices are given by cartesian product $A \times B \times C$. The number of different full choices is

$$\mathrm{n}(A \times B \times C) = \mathrm{n}(A)\mathrm{n}(B)\mathrm{n}(C) = 4 \times 5 \times 3 = 60,$$

using a generalization of the previous problem.

The number of choices for the restricted menu is

$$\mathrm{n}(B \times C) = \mathrm{n}(B)\mathrm{n}(C) = 15.$$

The total number of different orders is therefore 75.

35.16. Given $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{2, 3, 4, 5\}$, then

$$B \cup C = \{3, 4\} \cup \{2, 3, 4, 5\} = \{2, 3, 4, 5\}, \quad B \cap C = \{3, 4\} \cap \{2, 3, 4, 5\} = \{3, 4\}.$$

Then

$$\begin{aligned} A \times (B \cup C) &= \{1, 2, 3\} \times \{2, 3, 4, 5\} \\ &= \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), \\ &\qquad (2, 4), (2, 5), (3, 2), (3, 3), (3, 4), (3, 5)\}. \end{aligned}$$

$$\begin{aligned} A \times (B \cap C) &= \{1, 2, 3)\} \times \{3, 4\} \\ &= \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), 3, 4)\}. \end{aligned}$$

Chapter 36: Boolean algebra: logic gates and switching functions

36.1. For all $a, b \in B$,

$$\begin{aligned} a * a \oplus b &= a * (a \oplus b) \\ &= (a \oplus 0) * (a \oplus b), \text{ (identity law (36.2))} \\ &= (a \oplus (0 * b), \text{ (distributive law (36.1))} \\ &= a \oplus 0, \text{ (identity law (36.2))} \\ &= a, \text{ (identity law again).} \end{aligned}$$

36.2. We have to show that

$$\overline{a \oplus b} = \overline{a} * \overline{b}. \tag{i}$$

Use results given in (36.1) and (36.2). Then

$$\begin{aligned} (a \oplus b) \oplus (\overline{a} * \overline{b}) &= (\overline{a} * \overline{b}) \oplus (a \oplus b), \text{ (commutative law)} \\ &= ((\overline{a} * \overline{b}) \oplus a) \oplus b \text{ (associative law)} \\ &= (a \oplus (\overline{a} * \overline{b})) \oplus b, \text{ (commutative law)} \\ &= ((a \oplus \overline{a}) * (a \oplus \overline{b}) \oplus b, \text{ (distributive law)} \\ &= (1 * (a \oplus \overline{b}) \oplus b, \text{ (identity law)} \\ &= (a \oplus \overline{b}) \oplus b, \text{ (identity law)} \\ &= a \oplus (\overline{b} \oplus b), \text{ (associative law)} \\ &= a \oplus 1 = 1. \text{ (identity law)} \end{aligned}$$

The duality principle referred to in Problem 36.1 states that there exists a dual theorem to (i) in which the operations $\oplus$ and $*$ are interchanged. Hence the dual result is

$$\overline{a * b} = \overline{a} \oplus \overline{b}.$$

36.3. (a) Using the distributive laws (36.1) and identity laws (36.2),

$$a * (\overline{a} \oplus b) = (a * \overline{a}) \oplus (a * b) = 0 \oplus (a * b) = a * b.$$

(b) By the distributive law

$$(a \oplus b) * (a \oplus \overline{b}) = a \oplus (b * \overline{b}) = a \oplus 0 = a.$$

(c) First, using the commutative law,

$$(a \oplus b) * (\overline{a} * \overline{b}) = (\overline{a} * \overline{b}) * (a \oplus b).$$

Now apply a distributive law from (36.1) so that

$$\begin{aligned}(\overline{a} * \overline{b}) * (a \oplus b) &= (\overline{a} * \overline{b} * a) \oplus (\overline{a} * \overline{b} * b) \\ &= (\overline{b} * \overline{a} * a) \oplus (\overline{a} * \overline{b} * b) \\ &= (\overline{b} * 0) \oplus (\overline{a} * 0) \\ &= 0 \oplus 0 = 0,\end{aligned}$$

using the identity law $c * 0 = 0$ for any c.

36.4. (a) Using the distributive law (36.1)

$$a * b \oplus a * \overline{b} = a * (b \oplus \overline{b}) = a * 1 = a, \text{ (identity law)}.$$

The truth table which confirms the result is shown in Table 1. The entries in the columns can be compiled using the binary operations in Table 36.1.

Table 1: Truth table for $a * b \oplus a * \overline{b}$

a	b	$\overline{b}$	$a * b$	$a * \overline{b}$	$a * b \oplus a * \overline{b} = a$
0	0	1	0	0	0
0	1	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1

(b) Noting the precedence rule for $\oplus$ and $*$ (see end of Section 36.1),

$$\begin{aligned}a \oplus \overline{a} * \overline{b} * c &= (a \oplus (\overline{a} * \overline{b})) * c), \\ &= ((a \oplus \overline{a}) * (a \oplus \overline{b})) * (a \oplus c), \text{ (distributive law)} \\ &= (1 * (a \oplus \overline{b})) * (a \oplus c), \text{ (complement law)} \\ &= (a \oplus \overline{b}) * (a \oplus c), \text{ (identity law)} \\ &= a \oplus (\overline{b} * c). \text{ (distributive law)}\end{aligned}$$

The truth table for $a \oplus \overline{a} * \overline{b} * c$ shown in Table 2 is calculated from the simpler equivalent form $a \oplus (\overline{b} * c)$.

Table 2: Truth table for $a \oplus (\overline{b} * c)$.

a	$\overline{b}$	c	$\overline{b} * c$	$a \oplus (\overline{b} * c)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
1	0	0	0	1
1	1	0	0	0
1	0	1	0	1
0	1	1	1	1
1	1	1	1	1

36.5. In Problem 36.4(b) it was shown that

$$a \oplus \overline{a} * \overline{b} * c = a \oplus (\overline{b} * c).$$

The figure shows logically equivalent sequences of gates for the Boolean forms.

36.6. A sequence of gates for $(a \oplus \overline{b}) * (a \oplus \overline{c})$ is shown in the figure

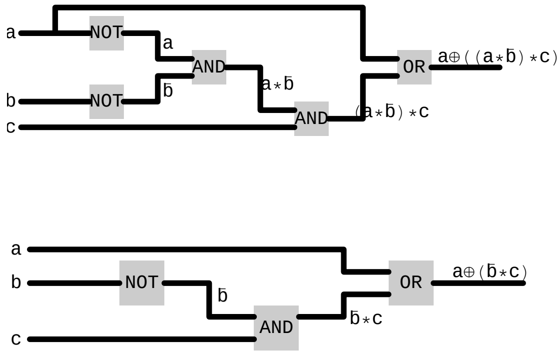

Figure 11: Problem 36.5: Sequences of gates for $a \oplus \overline{a} * \overline{b} * c$ and $a \oplus (\overline{b} * c)$.

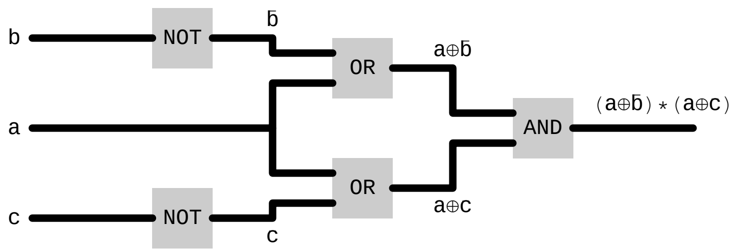

Figure 12: Problem 36.6: A sequence of gates for $(a \oplus \overline{b}) * (a \oplus \overline{c})$.

Table 3: Truth table for $(a \oplus \overline{b}) * (a \oplus \overline{c})$.

a	$\overline{b}$	$\overline{c}$	$a \oplus \overline{b}$	$a \oplus \overline{c}$	$(a \oplus \overline{b}) * (a \oplus \overline{c}))$
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	0	0
1	0	0	1	1	1
0	1	1	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

36.7. The product $*$ takes precedence over $\oplus$. Therefore we shall bracket the expression as follows:

$$\begin{aligned}
(a \oplus b) * (\overline{a} \oplus b) \oplus a &= ((b \oplus a) * (b \oplus \overline{a})) \oplus a, \text{ (commutative law)} \\
&= (b \oplus (a * \overline{a})) \oplus a, \text{ (distributive law)} \\
&= (b \oplus 0) \oplus a, \text{ (complement law)} \\
&= b \oplus a, \text{ (identity law)} \\
&= a \oplus b.
\end{aligned}$$

36.8. In (b), the product $*$ takes precedence over $\oplus$. In (b) the element b can take either the value 0 or 1. From the identity laws (36.2), $b * 1 = b$ and $b * 0 = 0$ so that $b * b = b$. Hence

$$a \oplus b * b = a \oplus (b * b) = a \oplus b.$$

36.9. The given truth table is

a	b	c	f
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Use the disjunctive normal form described in Section 16.4 for finding a Boolean expression for a given truth table. Identify the cases for which $f = 1$ in the final column: they occur in rows 1,2,5,6. In row 1, $a = 0$, $b = 0$, $c = 0$. Since these are all zero, form the Boolean expression $\overline{a}*\overline{b}*\overline{c}$. In row 2, $a = 0$, $b = 0$, $c = 1$. Replace a and b by their complements, and form the Boolean expression $\overline{a}*\overline{b}*c$. Similarly the Boolean expressions associated with rows 5 and 6 are $a*\overline{b}*\overline{c}$ and $a*\overline{b}*c$. A Boolean expression for the truth table is the sum of the Boolean expressions obtained for the rows 1,2,5,6, namely

$$f = \overline{a}*\overline{b}*\overline{c} \oplus \overline{a}*\overline{b}*c \oplus a*\overline{b}*\overline{c} \oplus a*\overline{b}*c.$$

36.10. (a) $f = (a*b)*(a\oplus b)$.

(b) $f = (a*b)\oplus(b*c)$.

(c) $f = (\overline{a*b})\oplus(c\oplus d)$.

(d) The exclusive-OR gate which a and b pass through is shown in Figure 36.10. Hence

$$f = \overline{(a*\overline{b})\oplus(\overline{a}*b)}\oplus(c*d).$$

36.11. $g = a*b$ and $f = (\overline{a*b})*(a\oplus b)$. If x (or a) is 1 and y (or b) is 1 then $f = 0$ and $g = 1*1 = 1$.

36.12. The various combinations of NOR gates which can replace the OR and AND gates are given Example 36.4 and Figure 36.8. They follow from the identities

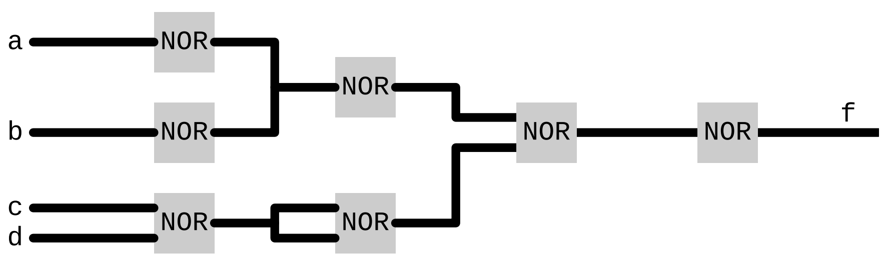

Figure 13: Problem 36.12:

$$a\oplus b = \overline{\overline{a\oplus b}\oplus\overline{a\oplus b}}, \quad a*b = \overline{\overline{a}\oplus\overline{b}}.$$

The simulations of the OR and AND gates in device in Figure 36.6 in the book are shown in Figure 13.

36.13. (See Section 36.4.) The first truth table is

a	b	f
0	0	0
0	1	1
1	0	1
1	1	1

In the column for f, the output has a '1' in rows 2,3, and 4. In row 2 we replace any zero element in a by its complement $\overline{a}$ and form its product with the other element b. Thus the Boolean expression for row 2 is $\overline{a} * b$. Similarly the Boolean expressions for rows 2 and 3 are $a * \overline{b}$ and $a * b$. Finally a Boolean expression for the truth table is the sum of these products, namely

$$f = \overline{a} * b \oplus a * \overline{b} \oplus a * b.$$

The second truth table is

a	b	c	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

In this truth table '1' appears in rows 1,4,5,7. Applying the method to row 1, the Boolean expression is $\overline{a} * \overline{b} * \overline{c}$. The Boolean expressions for rows 4,5, and 7 are $\overline{a} * b * c$, $a * \overline{b} * \overline{c}$ and $a * b * \overline{c}$. The disjunctive normal form for this truth table is the sum of these Boolean expressions, namely

$$f = \overline{a} * \overline{b} * \overline{c} \oplus \overline{a} * b * c \oplus a * \overline{b} * \overline{c} \oplus a * b * \overline{c}.$$

36.14. The NAND gate shown in Figure 36.4 provides the operation $f = \overline{a * b}$ on the elements a and b. The three figures show arrangements of NAND gates which can replace the AND, OR and

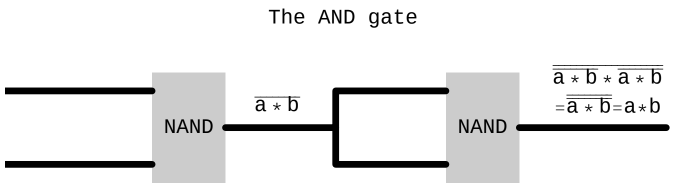

Figure 14: Problem 36.14:

NOT gates.

The AND gate is based on the identity

$$\overline{\overline{a * b} * \overline{a * b}} = \overline{\overline{a * b}} = a * b.$$

The OR gate uses the de Morgan law in the form

$$\overline{\overline{a} * \overline{b}} = a \oplus b.$$

Finally the NOT gate follows since $\overline{a * a} = \overline{a}$.

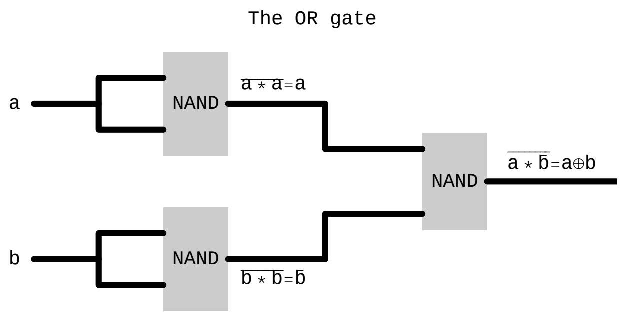

Figure 15: Problem 36.14:

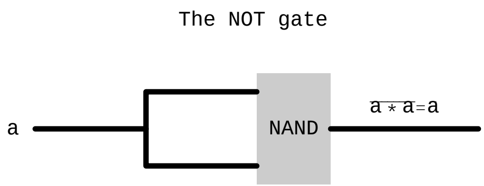

Figure 16: Problem 36.14:

36.15. Consult Section 36.5.
(a) Let a_1, a_2, a_3, a_4, a_5 be the states of the switches S_1, S_2, S_3, S_4, S_5, '1' for on and '0' for off. The switches S_1 and S_2 are in parallel: hence their state is represented by $a_3 \oplus a_2$. The switches S_3 and S_4 are in parallel but in series with S_5: therefore the state of this part of the circuit is given by $(a_3 * a_4) \oplus a_5$. this part of the whole switch is in parallel with S_1, S_2. Finally the state f of the switch is given by

$$f = (a_1 \oplus a_2) \oplus (a_3 \oplus a_4) * a_5.$$

(b) As in (a), let a_1, a_2, a_3, a_4, a_5 be the states of the switches S_1, S_2, S_3, S_4, S_5, '1' for on and '0' for off. Using the rules for series and parallel swithches, the final state f is given by

$$f = (((a_2 * a_3) \oplus a_4) * a_5) \oplus a_1.$$

36.16. For the sake of discussion suppose that all the switches are down, that is, $a_1 = a_2 = a_3 = 0$, and that the lighting is off, that is $f = 0$. These become the elements in the first row of a truth table for the state of the lighting. If a_1 is changed to up, the lights should go on making $f = 1$. We can now fill in the elements in row 2. If either a_2 or a_3 is changed to up then the lights should go out making $f = 0$. If all switches are up then the lights should go on giving row 5. Similar arguments if $a_2 = 1$ or $a_3 = 1$ initially, which specifies the remaining rows.

a_1	a_2	a_3	f
0	0	0	0
1	0	0	1
1	1	0	0
1	0	1	0
1	1	1	1
0	1	0	1
0	1	1	0
0	0	1	1

Applying the method of Section 36.4, the truth table can be represented by the Boolean expression

$$f = (\overline{a}_1 * \overline{a}_2 * a_3) \oplus (\overline{a}_1 * a_2 * \overline{a}_3) \oplus (a_1 * \overline{a}_2 * \overline{a}_3) \oplus (\overline{a}_1 * \overline{a}_2 * \overline{a}_3).$$

Chapter 37: Graph theory and its applications

37.1. (See Section 37.2.) The degree of a vertex is the number of edges which meet at the vertex. Therefore, from the figure in the book

$$\deg(a) = 4, \quad \deg(b) = 3, \quad \deg(c) = 3, \quad \deg(d) = 4, \quad \deg(e) = 4.$$

37.2. A *complete graph* is one in which every vertex is joined to every other vertex by just one edge. Five edges meet at each vertex and there are six vertices but each edge will be duplicated.

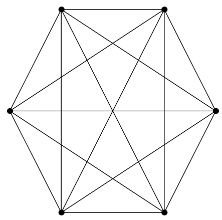

Figure 17: Problem 37.2: The complete graph with 6 vertices usually denoted by K_6.

Hence the complete graph, K_6, has $\frac{1}{2} \times 5 \times 6 = 15$ edges. In general, the complete graph K_n has $\frac{1}{2}n(n-1)$ edges.

37.3. The 21 connected unlabelled graphs with five vertices are shown in Figure 18. It can be

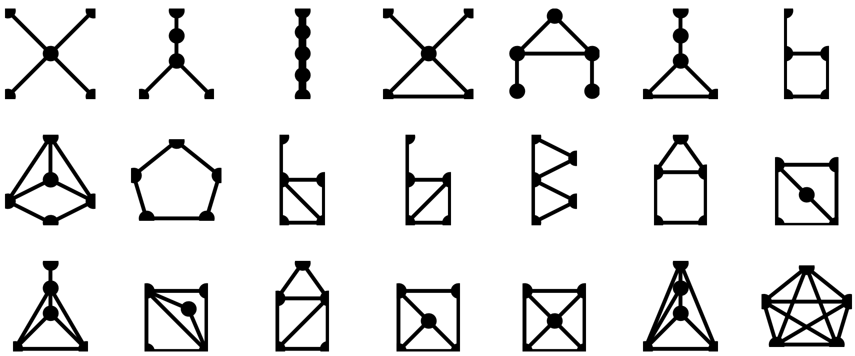

Figure 18: Problem 37.3: The complete set of connected graphs with five vertices.

seen from the figure that 20 of the graphs are planar: just the complete graph K_5 is non-planar.

37.4. The set of regular graphs is shown in the figure. The degrees 0,1,2,3,4,5 are possible with 2 and 3 repeated degrees. Five of the graphs are connected.

37.5. The graph shown in Figure 37.1 has the adjacency matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

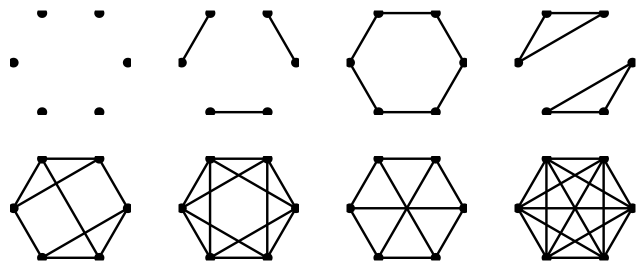

Figure 19: Problem 37.4: The complete set of regular graphs with six vertices.

The square of A is given by

$$A^2 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 5 & 1 & 3 & 2 \\ 1 & 6 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 3 \end{bmatrix}.$$

The relation between the edges and the vertices a, b, c, d, or, 1,2,3,4 is given by

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 5 & 1 & 3 & 2 \\ 1 & 6 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 3 \end{pmatrix} \end{array}.$$

Looking at the first row, the first element 5 gives the number of two edge paths which start and finish at vertex at a in Figure 37.1. One path goes from a to d and back to a, and 4 different paths go from a to b and back to a: these would appear as 5 loops on a graph with adjacency A^2. The second element 1 on the first row indicates that there is just one two edge path between a and b *via* d. The number 3 on the first row means that there are 3 different two-edge paths between a and c, and so on.

37.6. (a) The graph defined by the adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

is shown Figure 20 is the complete matrix with 5 vertices.

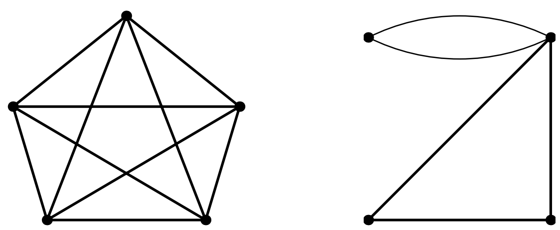

Figure 20: Problem 37.6(a),(b)

(b) The graph defined by

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

is also shown in Figure 20.

37.7. The adjacency matrices of the graphs shown in Figure 37.7 are as follows with the vertices listed alphabetically across and down from the top left-hand corner:
(a)

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

(b)

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix};$$

(c)

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

(d)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

(e)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

This is the adjacency matrix for the complete graph with 5 vertices.

The graph in (c) is not connected and is effectively two graphs with 3 and 2 vertices. The adjacency matrix has blocks of zeros as shown in the representation of A given below.

$$A = \begin{bmatrix} 0 & 2 & 0 & \vdots & 0 & 0 \\ 2 & 0 & 1 & \vdots & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix}$$

The blocks would not be quite so obvious if the vertices were taken in the order (say) a, d, b, c, e rather than a, b, c, d, e.

37.8. A cycle is a closed walk with no repeated edges or vertices. Label the graph, which is the complete graph with 4 vertices, as shown in Figure 21. Consider the vertex labelled a (we need only consider one vertex). There are six cycles through a given by

$$abda, \quad acda, \quad acba, \quad abcda, \quad abdca, \quad adbca$$

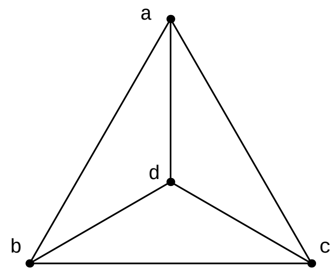

Figure 21: Problem 37.8

37.9. A trail between two vertices is a walk with no repeated edges: a path is a trail with additionally no repeated vertices. In Figure 37.57 (shown here as Figure 22) there are 13 walks which are trails of which the first 11 are paths. The full list is

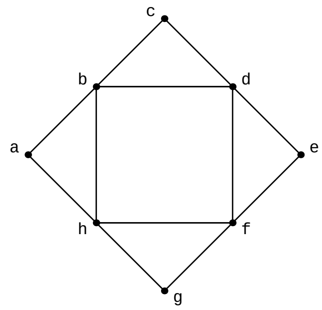

Figure 22: Problem 37.9

$$ahf, \quad ahgf, \quad abhf, \quad abdf, \quad abcdf, \quad abdef, \quad abhgf, \quad ahbdf,$$

$$abcdef, \quad ahbcdf, \quad ahbdef,$$

$$abcdbhf, \quad abcdbhgf.$$

37.10. The degree of every vertex in the graph given by Figure 37.57 is an even number, which means the the graph is eulerian. A possible eulerian trail is

$$abcdefghfdbha.$$

The graph is hamiltonian since we can find a cycle which passes through every vertex. A possible cycle is

$$abcdefgha.$$

37.11. A possible spanning tree (which is a connected graph, without cycles, which includes every vertex) for the graph is $abcdefgh$ as shown by the solid line in Figure 23. No vertex of the spanning tree has degree more than 2. The cotree of this spanning tree is set of edges in Figure 23 represented by dashed lines.

37.12. Figure 37.58 is reproduced in Figure 24.
(a) Every vertex has even degree which implies that the graph must be eulerian, that is, there exists a closed trail which includes every edge.
(b) A spanning tree with 6 branches is shown on the right in Figure 24.
(c) A cutset which disconnects a, b, g, f from c, d e is the dashed curve on the right in Figure 24.

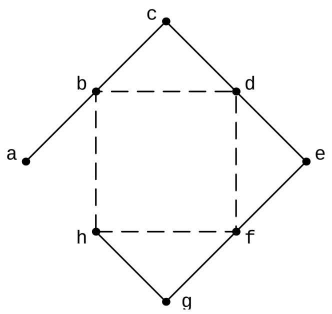

Figure 23: Problem 37.11

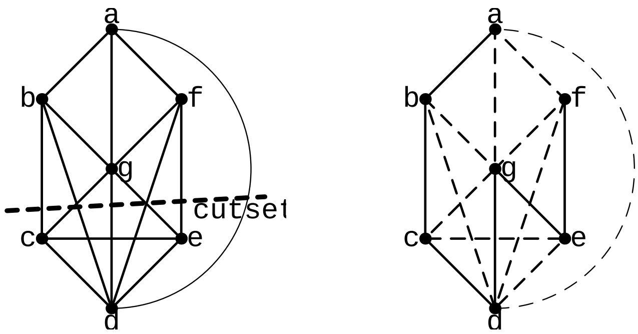

Figure 24: Problem 37.12: The figure on the left is Figure 37.58 from the book with a cutset (the dashed curve), and the figure on the right shows a spanning tree.

37.13. There are 3 trails between a and e, namely,

$$abfe, \quad abcde, \quad abfcde.$$

There are 3 more trails which visit e twice, namely

$$abfecde, \quad abcdebfe, \quad abfebcde.$$

The cycle $ebcde$ has 4 edges.

37.14. Figure 25 shows the corresponding graph for the circuit. It has 5 vertices which are in different positions to those of the ‘nodes’ in the circuit. For example, there is a vertex between the resistors R_2 and R_5 since the edges which represent them have different properties.

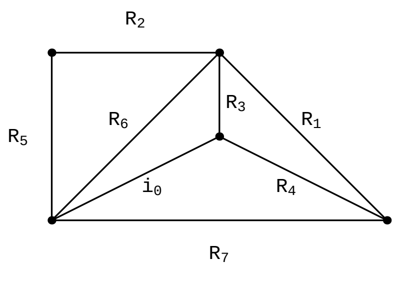

Figure 25: Problem 37.14

37.15. The circuit given by Figure 37.61 is reproduced in Figure 26. In the same figure, the diagram on the right shows a spanning tree with branches abc and ade. The curves C_1, C_2, C_3

and C_4 define fundamental cutsets for the circuit: each curve cuts just one edge of the spanning tree. The sum of the currents on *all* the edges crossing a cutset curve must balance. Hence, for

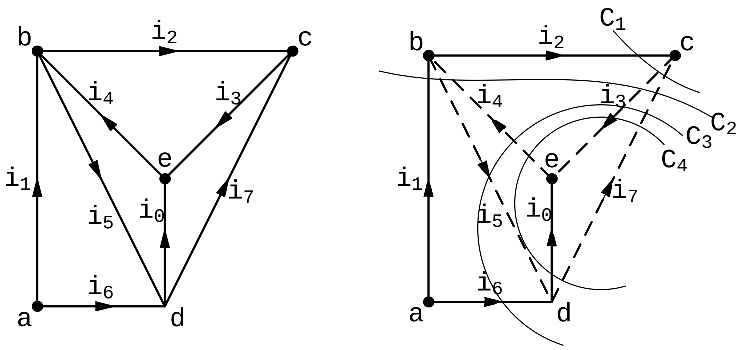

Figure 26: Problem 37.15

each of the curves:

$$C_1: \quad i_2 - i_3 + i_7 = 0, \tag{i}$$

$$C_2: \quad i_1 - i_5 + i_4 - i_3 + i_7 = 0, \tag{ii}$$

$$C_3: \quad -i_6 - i_5 + i_4 - i_3 + i_7 = 0, \tag{iii}$$

$$C_4: \quad -i_0 + i_4 - i_3 + i_7 = 0. \tag{iv}$$

Since all the resistors have resistance 1Ω, $i_1 = v_a - v_b$, $i_2 = v_b - v_c = v_b$ (since c is earthed) and so on. By Kirchhoff's law the voltage round each circuit is zero. Hence, in terms of voltages and currents these become for the following circuits:

$$abda: \quad i_1 + i_5 - i_6 = 0, \tag{v}$$

$$bceb: \quad i_2 + i_3 + i_4 = 0, \tag{vi}$$

$$bcdb: \quad i_2 - i_7 - i_5 = 0. \tag{vii}$$

Any further circuits will provide nothing new. We need to solve the set of linear equations (i)-(vii) for $i_1, \ldots, i_7$ in terms of the known current i_0. The full solution set is

$$i_1 = -\tfrac{1}{3}i_0, \quad i_2 = \tfrac{5}{6}i_0, \quad i_3 = \tfrac{5}{6}i_0, \quad i_4 = -\tfrac{11}{12}i_0, \quad i_5 = -\tfrac{2}{3}i_0,$$

$$i_6 = -\tfrac{10}{3}i_0, \quad i_7 = \tfrac{3}{4}i_0.$$

Looking at potential differences,

$$v_b - v_c = i_2 = \tfrac{5}{6}i_0, \quad v_b - v_d = i_5 = -\tfrac{2}{3}i_0,$$

$$v_a - v_b = i_1 = \tfrac{1}{3}i_0, \quad v_e - v_b = i_4 = -\tfrac{11}{12}i_0,$$

since the resistances are all 1Ω. Hence, since $v_c = 0$,

$$v_b = \tfrac{5}{6}i_0, \quad v_d = \tfrac{3}{2}i_0, \quad v_a = \tfrac{7}{6}i_0, \quad v_e = -\tfrac{1}{2}i_0.$$

(*Note*: the final list should be corrected to v_a, v_b, v_d, v_e in the question.)

37.16. (a) Figure 27 shows a graph representation of the circuit with nodes labelled a to f. The currents through the resistors R_1, R_2, R_3, R_4, R_5, R_6 are, respectively i_1, i_2, i_3, i_4, i_5, i_6. Figure 27 on the right shows a spanning tree and five fundamental cutsets C_1, C_2, C_3, C_4, C_5, C_6. The currents across these cutsets must have sum which is algebraically zero. Hence

$$C_1: \quad i_6 - i_Z - i_5 = i_6 - 2 - i_5 = 0, \tag{i}$$

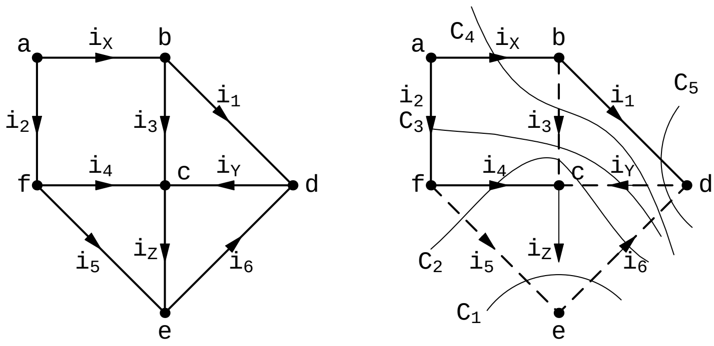

Figure 27: Problem 37.16(a)

$$C_2: \quad i_6 - i_Y - i_3 - i_4 - i_5 = i_6 - 2 - i_3 - i_4 - i_5 = 0, \tag{ii}$$

$$C_3: \quad i_6 - i_Y - i_3 - i_2 = i_6 - 2 - i_2 = 0, \tag{iii}$$

$$C_4: \quad i_6 - i_Y - i_3 + i_X = i_6 - 2 - i_3 + 1 = 0, \tag{iv}$$

$$C_5: \quad i_6 + i_1 = 0. \tag{v}$$

Apply Kirchhoff's law to the circuit $dbcfed$. The algebraic sum of the voltages around this circuit must be zero, so that, using Ohm's law,

$$-3i_1 + i_3 - i_4 + 2i_5 + i_6 = 0. \tag{vi}$$

Now solve eqns (i) to (vi) (this can be achieved quickly using *Mathematica*, but alternatively use elimination):

$$i_1 = 4, \quad i_2 = -1, \quad i_3 = 3, \quad i_4 = -1, \quad i_5 = 2, \quad i_6 = 4.$$

Let the voltages at the nodes be v_a to v_f. Then, for the resistors, the voltage changes across the resistors are

$$v_b - v_d = R_1 i_1 = 3i_1 = 12, \tag{vii}$$

$$v_a - v_f = R_2 i_2 = 2i_2 = -2, \tag{viii}$$

$$v_b - v_c = R_3 i_3 = i_3 = 3, \tag{ix}$$

$$v_f - v_c = R_4 i_4 = i_4 = -1, \tag{x}$$

$$v_f - v_e = R_5 i_5 = 2i_5 = 4, \tag{xi}$$

$$v_e - v_d = R_4 i_4 = i_6 = 4. \tag{xii}$$

(b) Figure 28 shows a graph of the circuit in Figure 37.62(b) with nodes labelled a to b. The currents through the resistors R_1 to R_8 are i_1 to i_8 respectively, in the directions shown. On the right in Figure 28 a spanning tree is shown together with the fundamental cutsets C_1, C_2, C_3, C_4, C_5, C_6. The sums of the currents across these curves are algebraically zero. Hence

$$C_1: \quad -i_7 - i_8 = 0, \tag{i}$$

$$C_2: \quad -i_7 - i_Y - i_6 = -i_7 - 2 - i_6 = 0, \tag{ii}$$

$$C_3: \quad -i_7 + i_X - i_3 - i_5 - i_6 = -i_7 + 1 - i_3 - i_5 - i_6 = 0, \tag{iii}$$

$$C_4: \quad -i_7 + i_X - i_3 + i_2 = -i_7 + 1 - i_3 + i_2 = 0, \tag{iv}$$

$$C_5: \quad -i_7 + i_X + i_1 = -i_7 + 1 + i_1 = 0, \tag{v}$$

$$C_6: \quad -i_7 + i_4 = 0. \tag{vi}$$

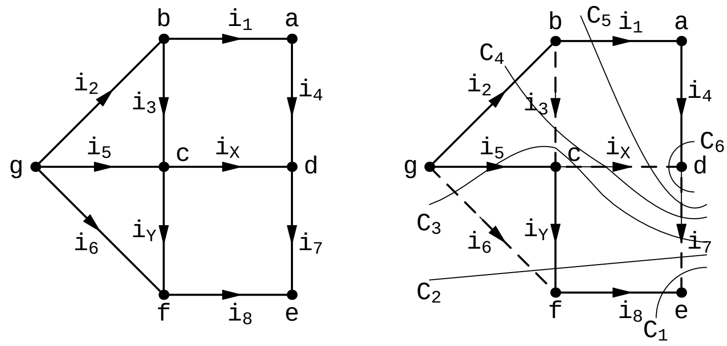

Figure 28: Problem 37.16(b)

We can also apply the Kirchhoff law to two loops, bcg and $abgfeda$, around which the sum of the voltages are algebraically zero. Using Ohm's law these are equivalent to

$$3i_3 - 2i_5 + 2i_2 = 0, \tag{vii}$$

$$-i_1 - 2i_2 + i_6 + i_8 - i_7 - i_4 = 0. \tag{viii}$$

Solve the eqns (i) to (viii) for the currents. As in (a) it easier using *Mathematica* to solve these linear equations. The answer is

$$i_1 = -\tfrac{6}{5},\ i_2 = 0,\ i_3 = \tfrac{6}{5},\ i_4 = -\tfrac{1}{5},\ i_5 = \tfrac{9}{5},\ i_6 = -\tfrac{9}{5},\ i_7 = -\tfrac{1}{5},\ i_8 = \tfrac{1}{5}.$$

The voltage differences can be found using Ohm's law.

(c) (*Note:* Unfortunately the resistance across the cell with voltage 2 has not been stated: it has been specified as R_6 in the answer below. The choice $R_6 = 1$ gives reasonable numbers for the currents.) Figure 29 shows a representation of the circuit with labelled nodes and currents as indicated, whilst the figure on the right shows a spanning tree for the graph together with 4 fundamental cutsets. The sums of the currents across these curves are algebraically zero. Hence

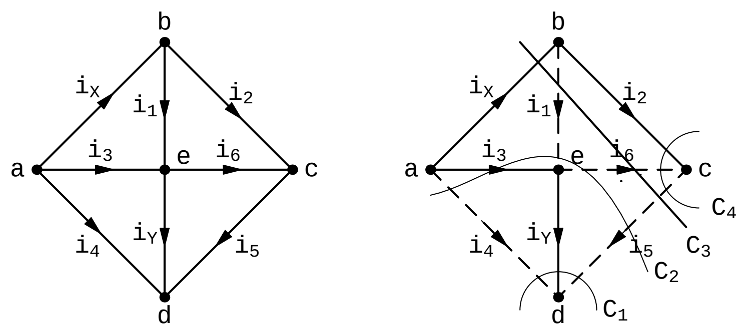

Figure 29: Problem 37.16(c)

$$C_1: \quad i_5 + i_Y + i_4 = i_5 + 2 + i_4 = 0, \tag{i}$$

$$C_2: \quad i_5 - i_6 + i_3 + i_4 = 0, \tag{ii}$$

$$C_3: \quad i_5 - i_6 + i_1 - i_X = i_5 - i_6 + i_1 - 2 = 0, \tag{iii}$$

$$C_4: \quad i_5 - i_6 - i_2 = 0. \tag{iv}$$

There are 6 unknown currents so that we require 2 further equations relating currents. Apply Kirchhoff's law to the loops *ecbe* and *ecdae* around which the sums of the voltages must be algebraically zero. Hence

$$R_6 i_6 - 2i_2 + i_1 = -2, \tag{v}$$

$$R_6 i_6 + 2i_5 - i_4 + 3i_3 = -2. \tag{vi}$$

Using *Mathematica* or similar software, the solutions of equations (i) to (vi) are

$$i_1 = \frac{2 + R_6}{3 + R_6}, \quad i_2 = \frac{4 - R_6}{3 + R_6}, \quad i_3 = \frac{2R_6 - 1}{3 + R_6},$$

$$i_4 = -1, \quad i_5 = -1, \quad i_6 = -\frac{7}{3 + R_6}.$$

37.17. Since the feedback loop through $H_1(s)$ is positive, this feedback loop can be replaced by a single device which produces an output

$$\frac{G_2(s)}{1 - G_2(s)H_1(s)},$$

as shown in Figure 37.29. Let $A(s)$ be the output from the first junction and let $F(s)$ be the feedback signal which has passed through the device $H_2(s)$. Hence

$$A(s) = P(s) - F(s). \tag{i}$$

The final output $Q(s)$ is occurs after $A(s)$ has passed through three devices in series, so that

$$Q(s) = G_1(s)\frac{G_2(s)}{1 - G_2(s)H_1(s)}G_3(s) = \frac{G_1(s)G_2(s)G_3(s)}{1 - G_2(s)H_1(s)}. \tag{ii}$$

The feedback signal is

$$F(s) = H_2(s)Q(s). \tag{iii}$$

Eliminate $A(s)$ and $F(s)$ between (i), (ii) and (iii) giving

$$Q(s) = \frac{G_1(s)G_2(s)G_3(s)P(s)}{1 - G_2(s)H_1(s) + G_1(s)G_2(s)G_3(s)H_2(s))},$$

confirming (37.30).

37.18. The 'plus' sign indicates a positive feedback through $H(s)$ unlike a similar device shown in Figure 37.26 which has negative feedback . Let $A(s)$ be the output from the sum operator and $F(s)$ be the positive feedback into it. Then, as in (37.27), (37.28) and (37.29)

$$A(s) = P(s) + F(s), \quad Q(s) = G(s)A(s), \quad F(s) = H(s)Q(s).$$

Elimination of $A(s)$ and $F(s)$ gives

$$Q(s) = \frac{G(s)}{1 - G(s)H(s)}P(s).$$

37.19. (a) In Figure 37.64(a), the negative feedback $H_1(s)$ on $G_2(s)$ is replaced by the device with transfer function

$$R(s) = \frac{G_2(s)}{1 + G_2(s)}H_1(s),$$

(see the first paragraph in Section 37.7). The three devices $G_1(s)$, $R(s)$ and $G_3(s)$ are in series which is equivalent to a device with transfer function

$$U(s) = \frac{G_1(s)G_2(s)G_3(s)}{1 + G_2(s)H_1(s)}.$$

Finally $H_2(s)$ induces a positive feedback on $U(s)$ so that the complete system is equivalent to a device with transfer function

$$\frac{G_1(s)G_2(s)G_3(s)/[1+G_2(s)H_1(s)]}{1-G_1(s)G_2(s)G_3(s)H_2(s)/[1+G_2(s)H_1(s)]} = \frac{G_1(s)G_2G_3(s)}{1+G_2(s)H_1(s)-G_1(s)G_2(s)G_3(s)H_2(s)}.$$

Hence

$$Q(s) = \frac{G_1(s)G_2G_3(s)P(s)}{1+G_2(s)H_1(s)-G_1(s)G_2(s)G_3(s)H_2(s)}.$$

(b) Figure 30 shows the signal-flow graph with the outputs $A(s)$, $B(s)$, $C(s)$, $D(s)$, $F(s)$, $J(s)$, $K(s)$, and $L(s)$ as shown. The figure should be viewed in conjunction with Figure 37.64(b) in the book. It has been assumed that the return through $H_2(s)$ is a positive feedback. First consider the feedback through $H_1(s)$: then

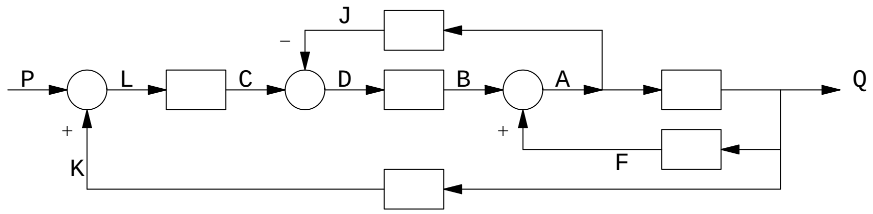

Figure 30: Problem 37.19(b)

$$J(s) = A(s)H_1(s), \tag{i}$$

$$D(s) = C(s) - J(s), \tag{ii}$$

$$B(s) = D(s)G_2(s). \tag{iii}$$

For the feedback through $H_2(s)$,

$$Q(s) = A(s)G_3(s), \tag{iv}$$

$$F(s) = Q(s)H_2(s), \tag{v}$$

$$A(s) = B(s) + F(s). \tag{vi}$$

Solve eqns (i) to (vi) to find $C(s)$ in terms of $Q(s)$. Between (i), (ii), (ii), (v), (vi), eliminate $B(s)$, $F(s)$ and $J(s)$ so that

$$C(s) = D(s) + A(s)H_1(s), \quad A(s) = D(s)G_2(s) + Q(s)H_2(s).$$

Now eliminate $A(s)$ and $D(s)$ between these equations and (iv): hence

$$C(s) = \frac{1+G_2(s)H_1(s)-G_3(s)H_2(s)}{G_2(s)G_3(s)}Q(s). \tag{vii}$$

For the positive feedback through $H_3(s)$

$$C(s) = L(s)G_1(s), \quad K(s) = Q(s)H_3(s), \quad L(s) = P(s) + K(s).$$

From these equations

$$C(s) = G_1(s)[P(s)+K(s)] = G_1(s)[P(s)+H_3(s)Q(s)]. \tag{viii}$$

Finally eliminate $C(s)$ between (vii) and (viii):

$$Q(s) = \frac{G_1(s)G_2(s)G_3(s)P(s)}{1+G_1(s)H_1(s)-G_3(s)H_2(s)-G_1(s)G_2(s)G_3(s)H_3(s)}.$$

37.20. It would be helpful to sketch the resulting signal-flow graph after each operation.
(a) Refer to Figure 37.64(a) and the rules in Section 37.7. x_3x_4 is a stem: hence

$$x_5 = G_2G_3x_3, \quad x_2 = G_2H_2x_3.$$

x_4 has now disappeared. x_2, x_3 is a cycle so that

$$x_3 = \frac{G_1x_1}{1 - G_1G_2H_2},$$

where x_2 has now disappeared. x_3, x_5 is another cycle, and

$$x_5 = \frac{G_1G_2G_3}{(1 - G_1G_2H_2)(1 + G_2G_3H_1)}x_1$$

on the edge x_1x_5.
(b) In Figure 37.64(b), x_4 is a stem so that

$$x_5 = G_1G_2x_2, \quad x_3 = G_1H_2,$$

x_2 and x_4 coalescing. x_5, x_3 and x_2 defines a cycle. Hence This is replaced by an edge with transfer

$$x_5 = -\frac{H_1H_3}{1 + G_1H_1H_2}x_2.$$

There remains one cycle between x_2 and x_5. Hence

$$x_6 = \frac{G_1G_2}{1 + \frac{G_1G_2H_1H_3}{1+G_1H_1H_2}}x_1 = \frac{G_1G_2(1 + G_1H_1H_2)}{1 + G_1H_1H_2(1 + G_2H_3)}x_1.$$

(c) x_3x_4 is a cycle so that

$$x_4 = \frac{G_5G_6}{1 - G_5H_1}x_2.$$

The edges x_5x_6 and x_6x_7 are in series, so that

$$x_7 = G_2G_3x_5.$$

The resulting edge x_5x_7 is part of a cycle: hence

$$x_7 = \frac{G_1G_2G_3}{1 + G_2G_3H_2}x_2.$$

Finally $x_2x_4x_8$ and $x_2x_7x_8$ are in parallel. Therefore

$$x_9 = \left[\frac{G_5G_6G_7}{1 - G_5H_1} + \frac{G_1G_2G_3G_4}{1 + G_2G_3H_2}\right]x_1.$$

(d) The edges between x_2 and x_6 are in the same configuration as those in (c) aside from notational changes. Hence they can be replaced by an edge with transfer

$$x_6 = \frac{G_1G_2G_3G_4}{1 + G_2G_3H_2}x_2.$$

There is a loop at x_7 which can be replaced by

$$x_7 = \frac{G_5}{1 - H_1}x_2.$$

Finally $x_2x_7x_8x_9$ are in series but parallel to the edge x_2x_6 whose transfer function is given above. Therefore

$$x_9 = \left[\frac{G_5G_6G_7}{1 - H_1} + \frac{G_1G_2G_3G_4}{1 + G_2G_3H_2}\right]x_1.$$

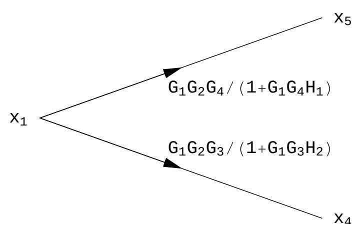

Figure 31: Problem 37.20(e)

(e) The stem $x_2x_5x_5x_4$ can be replaced by the two edges x_2x_5 with transfer function G_2G_4, and x_2x_4 with transfer function G_2G_3. Finally $x_1x_2x_5x_2$ and $x_1x_2x5x_2$ are cycles with transfer functions

$$x_5 = \frac{G_1G_2G_4}{1+G_2G_4H_1}x_1, \quad x_4 = \frac{G_1G_2G_3}{1+G_1G_3H_2},$$

respectively. The resulting stem is shown in Figure 31.

37.21. If v is the number of vertices, e the number of edges and f the number of faces of a planar graph, then Euler's theorem states that

$$v - e + f = 2.$$

(a) In the graph shown in Figure 37.66(a), $v = 12$, $e = 20$ and $f = 10$ (including the region exterior to the graph). Hence

$$v - e + f = 12 - 20 + 10 - 2$$

as required.

(b) For the graph in Figure 37.66(b), $v = 9$, $e = 16$ and $f = 9$. Hence

$$v - e + f = 9 - 16 + 9 = 2$$

as required.

37.22. Sufficient here to produce a planar representation of $K_{2,3}$ as shown in Figure 32.

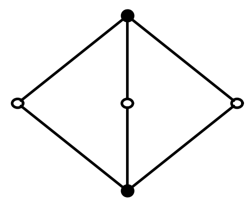

Figure 32: Problem 37.22: Planar drawing of $K_{2,3}$.

37.23. The complete graph K_4 is planar, and is represented by the plane drawing with vertices 1,2,3,4 and the solid edges in Figure 33. Add a fifth vertex labelled 5 either external to the graph, or within any face. Join this vertex to each of the vertices 1,2,3,4 avoiding edges if possible to produce K_5. A possible construction with dashed lines is shown in Figure 33. One edge has to cross an edge of K_4. Hence at least one edge crossing is required.

37.24. There are 24 paths between S and T listed in the table.

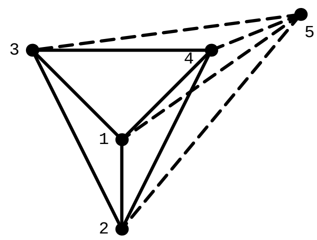

Figure 33: Problem 37.23

path	length	path	length	path	length
SAFT	12	$SAFET$	16	$SABFT$	14
$SABFET$	18	$SABET$	16	$SABDT$	13
$SABDET$	18	$SABCDT$	18	$SABCDET$	23
$SBFT$	15	$SBFET$	19	$SBET$	17
$SBDT$	14	$SBDET$	19	$SBCDT$	19
SBCDET	24	$SCDT$	16	$SCDET$	21

The shortest path is $SAFT$ with length 12, and the longest is $SBCDET$ with length 24.

37.25. (See Section 37.9.) Label the cell rows and columns as shown in Figure 34. Construct

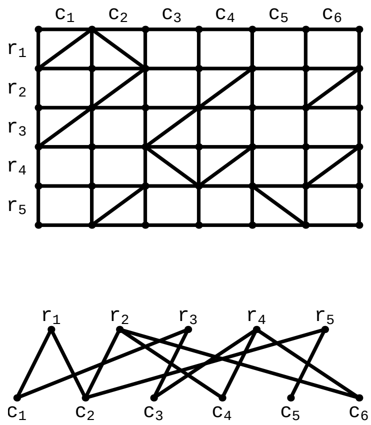

Figure 34: Problem 37.25

the bipartite graph $K_{5,6}$ with sets of vertices $\{r_1, r_2, r_3, r_4, r_5\}$ and $\{c_1, c_2, c_3, c_4, c_5, c_6\}$. If cell has a tie, then this is represented by an edge in the bipartite graph as shown in Figure 34. The graph is connected which means that the framework is braced. Any cycles in the graph indicate overbracing. From the graph $r_2c_4r_4c_6r_2$ is a cycle. We can remove any edge from this cycle without affecting the bracing. Remove edge r_4c_6. There is still a further cycle given by $c_1r_3c_3r_4c_4r_2c_2r_1c_1$, and we can remove any edge in this cycle. Remove r_1c_2. The graph has no further cycles, but is still connected.

37.26. The framework is shown on its side in Figure 35 to save space. The corresponding bipartite graph is also shown. Join r_1 to each of c_1 to c_6, which requires 6 edges. The vertex r_2 is left disconnected from the rest of the graph. Join it to any of the c's: the edge r_2c_6 is shown. The graph is connected and contains no cycles, and is a a minimum bracing. The number of edges is 7.

37.27. (a) The framework and its corresponding bipartite graph are shown in Figure 36. The bipartite graph is disconnected, which means the the framework is not braced. It can be braced by

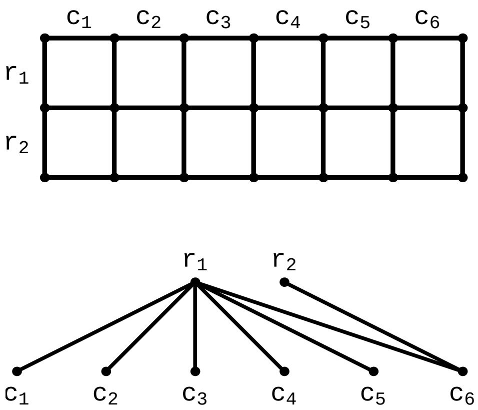

Figure 35: Problem 37.26

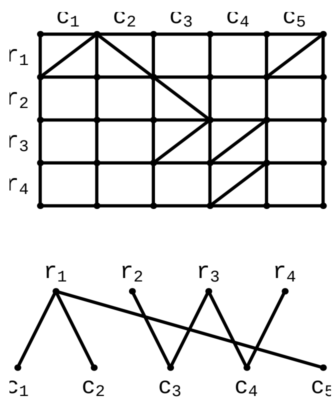

Figure 36: Problem 37.27(a)

the insertion of a single tie to make the graph connected: the tie r_4c_5 will connect the two separate subgraphs.

(b) The framework and its corresponding bipartite graph are shown in Figure 37. The bipartite graph is connected which means that the framework is braced. However, the graph does have a cycle $c_1r_1c_2r_3c_4r_2c_1$. Therefore the framework is overbraced and has at least one redundant tie. Any edge in the cycle could be removed.

(c) The bipartite graph derived from the framework is shown in Figure 38. The graph is connected which means that the framework is braced. Also the bracing is a minimum since the bipartite graph has no cycles.

37.28. Refer to Figure 39 in which the solid lines in the bipartite graph correspond to the framework in Figure 37.69(c). There are two vertices which have just one edge, namely c_1 and c_2. For every edge to be part of a cycle every vertex must be incident with at least two edges so that there is an incoming and outgoing edges. Insert the two edges c_1r_1 and c_5r_4 (the dashed lines in the figure). We can check that every edge is now an edge of at least one cycle.

37.29. Figure 37.70 in the book shows a junction with traffic signals controlling each lane of traffic. Represent the lanes of traffic by a vertices a to h as shown in Figure 40. If two lanes of traffic can move without impeding each other then join the two vertices by an edge: otherwise leave the vertices unconnected. For example, lanes a and f are can move simultaneously so a and f are

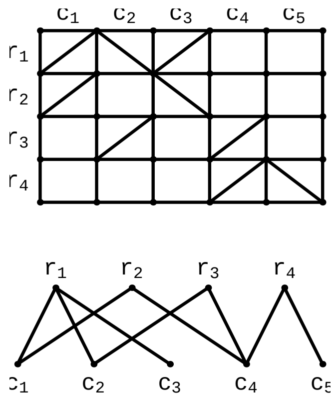

Figure 37: Problem 37.27(b)

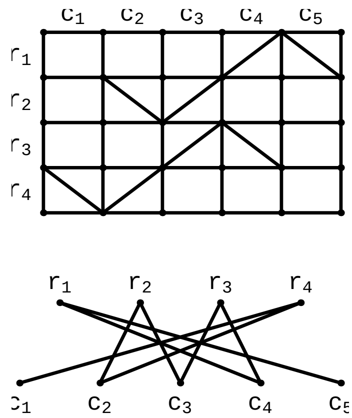

Figure 38: Problem 37.27(c)

joined, but f and g are should not move at the same time. Complete the compatibility graph C as shown in Figure 40. A complete subgraph in C is a subgraph in which each vertex in it is joined to each other vertex in the subgraph by just one edge. There is one subgraph with 4 vertices, and every triangle is a complete subgraph with 3 vertices. The full list of complete subgraphs with 3 or more vertices is

$$\{abef, abc, abe, abf, ace, acg, aef, aeh, bce, cde, cdg\}.$$

As given in the problem, the following complete subgraphs include all vertices: $\{abef, cdg, aeh\}$ (always include as many high order subgraphs as possible). With equal phasing of $\frac{1}{3}T$, the red/green sequence is as shown in the table.

time	green	red
$0 - \frac{1}{3}T$	a, b, c, d	e, f, g, h
$\frac{1}{3}T - \frac{2}{3}T$	c, d, g	a, b, e, f, h
$\frac{2}{3}T - T$	a, e, h	b, c, d, f, g

The lane-by-lane phases of the red/green sequence is listed in Figure 41. The solid lines represent the times when lights are at red and the dashed lines when they are at green. For example in cycle lane A is red between 0 and $\frac{1}{3}T$, and between $\frac{2}{3}T$ and T (which will be a continuous red

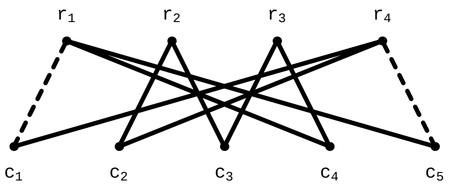

Figure 39: Problem 37.28

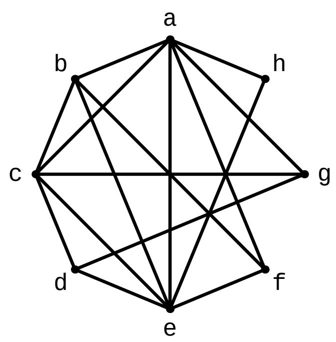

Figure 40: Problem 37.29: Compatibility graph C.

over each cycle). On the other hand lane g will be on red for just a third of the phase T between $\frac{1}{3}T$ and $\frac{2}{3}T$.

Let $t_a, t_b, \ldots$, etc be the waiting times respectively for lanes $a, b, \ldots$. Then, from Figure 41

$$t_a = \tfrac{1}{3}T, \quad t_b = \tfrac{2}{3}T, \quad t_c = t_d = \tfrac{1}{3}T, \quad t_e = t_f = t_g = t_h = \tfrac{1}{3}T.$$

Therefore this measure of the waiting time is

$$W = t_a + t_b + \cdots + t_h = \frac{1}{3}T + \frac{2}{3}T + \frac{2}{3}T + \frac{8}{3}T = \frac{13}{3}T.$$

Figure 42 shows the phases if the lights change at $0, \frac{1}{2}T, \frac{3}{4}T, T$ in each cycle. From this figure the waiting times for the lanes are

$$t_a = \frac{1}{4}T,\ t_b = \frac{1}{2}T,\ t_c = t_d = \frac{1}{4}T,\ t_e = \frac{3}{4}T,\ t_f = \frac{1}{2}T,\ t_g = t_h = \frac{3}{4}T.$$

Hence the total waiting time is

$$W = t_a + t_b + \cdots + t_h = 4T.$$

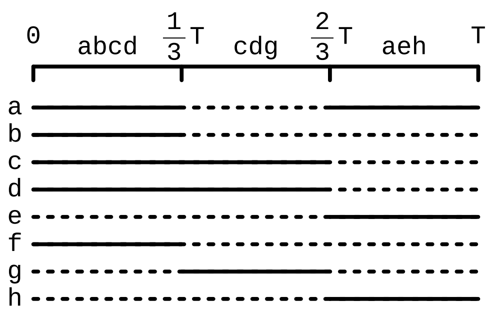

Figure 41: Problem 37.29: Traffic phasing with change times $0, \frac{1}{3}T, \frac{2}{3}T, T$.

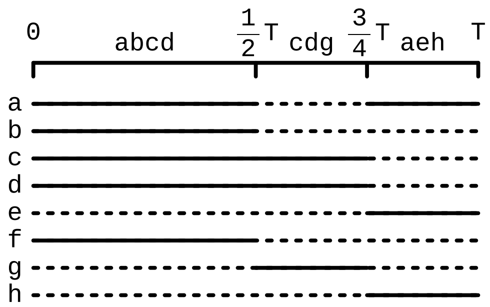

Figure 42: Problem 37.29: Traffic phasing with change times $0, \frac{1}{2}T, \frac{3}{4}T, T$.

Using this measure, the waiting time is improved by time $\frac{1}{3}T$ compared with that for time for the equal phasing.

Chapter 38: Difference equations

38.1. See Example 38.1. Let £P_0 be the initial sum, I the interest rate and £P_n be the investment after n time steps (in either years or months). Then

$$P_n = (1+I)^n P_0. \tag{i}$$

In this problem $P_0 = 1000$, $n = 10$ (in years) and $I = 0.06$. Hence, from (i)

$$P_{10} = (1+0.06)^{10} \times 1000 = £1790.85.$$

The interest rate is monthly (assume calendar). Hence $n = 120$ (in months), $P_{120} = 1790.85$ and $P_0 = 1000$. We now require I_m, the monthly interest rate. From (i),

$$P_{120} = (1+I_m)^{120} P_0, \text{ or } 1790.85 = (1+I_m)^{120} \times 1000.$$

Hence

$$I_m = \left(\frac{1790.85}{1000}\right)^{1/120} - 1 = 0.00487,$$

to 3 decimal places. As a percentage, this is 0.487%. This monthly rate is equivalent to 6% annually.

38.2. If £A is the annual repayment, £P is the amount borrowed, I is the (fixed) annual interest, and N is the period in years of the repayment period, then, following Example 38.2,

$$A = \frac{IP((1+I)^N}{(1+I)^N - 1}. \tag{i}$$

In this problem $P = 50000$, $I = 0.1$ and $N = 25$. Hence

$$A = \frac{0.1 \times 50000(1.1)^{25}}{1.1^{25} - 1} = £5508.4.$$

This means that over the period of the loan the total repayment is $25 \times 5508.4 = £137710$.
(a) Let Q_m be the amount of the loan outstanding after m years. Given $Q_0 = P$, then (see Example 38.2)

$$Q_m = \frac{A}{I} + (1+I)^m \left(Q_0 - \frac{A}{I}\right).$$

Given $A = 5508.4$, $I = 0.1$ and $m = 5$, then

$$Q_5 = £46896.20$$

Let A_1 be annual repayment for the remaining 20 years. Then applying (i) again with $P = 46896.20$, $I = 0.09$ and $N = 20$,

$$A_1 = \frac{0.1 \times 46896.20(1.09)^{20}}{1.09^{20} - 1} = £5137.31.$$

(b) In this case $A = 5508.40$, $N = 20$, $P = 46896.20$ and we must solve (i) for N. It follows from (i) that

$$(1 + I)^N = \frac{A}{A - IP}.$$

Hence, solving for N,

$$N = \frac{\ln[A/(A - IP)]}{\ln(1 + I)} \approx 16.9\text{years}.$$

Hence the mortgage term will be reduced by about 3.1 years.

38.3. As in (38.10), the fixed points of the difference equation $u_{n+1} = f(u_n)$ are any real solutions of $u = f(u)$.

(a) $u_{n+1} = u_n(2 - u_n)$. The fixed points satisfy

$$u = u(2 - u), \text{ or, } u(u - 1) = 0,$$

which has the solutions $u = 0$ and $u = 1$.

(b) $u_{n+1} = u_n(1 + u_n)(2 - 3u_n)$. The fixed points satisfy

$$u = u(1 + u)(2 - 3u), \text{ or } u(3u^2 + u - 1) = 0,$$

which has solutions $u = 0$ and $u = \frac{1}{6}(-1 \pm \sqrt{13})$.

(c) $u_{n+1} = \sin u_n$. Any fixed point satisfies $u = \sin u$. The straight line $z = u$ is a tangent to $z = \sin u$ at $(0, 0)$ in the (u, z) plane(sketch the tangent and the sine curve). There is just one intersection (at $u = 0$) between giving one solution $u = 0$.

(d) $u_{n+1} = \frac{1}{2}\sin u_n$. Any fixed point satisfies $2u = \sin u$. The straight line $z = 2u$ is steeper than the tangent to $z = \sin u$ at $(0, 0)$ (see (c) above). Hence there is just one solution $u = 0$.

(e) $u_{n+1} = \mathrm{e}^{u_n} - 1$. Any fixed point satisfies $u = \mathrm{e}^u - 1$. Sketch the line $z = u$ the curve $z = \mathrm{e}^u - 1$. The line is tangent to the exponential curve $z = \mathrm{e}^u - 1$ and the diagram shows that $\mathrm{e}^u - 1 \geq u$. Hence there can be only one solution $u = 0$.

38.4. Given the sequence $u_{n+1} = f(u_n)$ and u_0, we require $u_1, \ldots, u_5$. Computer programs to list these sequences are easy to construct. All these equations are forms of the linear difference equation $u_{n+1} = au_n + b$ discussed later in Section 38.5.

(a) $u_{n+1} = 2u_n(3 - u_n)$, $u_0 = 1$. Hence

$$u_1 = 2u_0(3 - u_0) = 4, \quad u_2 = 2u_1(3 - u_1) = -8,$$

and so on. The required sequence is

$$\{1, 4, -8, -176, -63008, -7940394176\}.$$

(b) $u_{n+1} = 2u_n(1 - u_n)$, $u_0 = 1$. It follows that $u_n = \frac{1}{2}$ for all n. In fact $u_n = \frac{1}{2}$ is a fixed point of the difference equation.

(c) $u_{n+1} = 3.2u_n(1 - u_n)$, $u_0 = \frac{1}{2}$. The required sequence is

$$\{0.5, 0.8, 0.0.5120\ldots, 0.7994\ldots, 0.5128\ldots, 0.7994\ldots\}.$$

It appears that the sequence is oscillating between two values 0.5128 and 0.7994.

(d) $u_{n+1} = 4u_n(1 - u_n)$, $u_0 = \frac{1}{2}$. The required sequence is

$$\{\frac{1}{2}, 1, 0, 0, 0, \ldots\}.$$

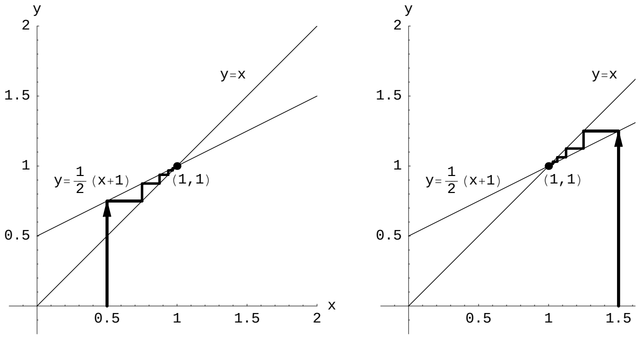

Figure 43: Problem 38.5(a)

By u_2, the sequence has reached the fixed point $u_n = 0$ of the difference equation.

38.5. (See Section 38.3, particularly (38.11).)
(a) The difference equation is $u_{n+1} = \frac{1}{2}u_n + \frac{1}{2}$ in the two cases $u_0 = \frac{1}{2}$ and $u_0 = \frac{3}{2}$. The equation has a fixed point at $u = 1$ corresponding to $(1, 1)$ in the (x, y) plane. The cobweb moves between the lines $y = x$ and $y = \frac{1}{2}x + \frac{1}{2}$ according to the rules explained in Section 3.8. For the two sample starting points less than $u = 1$ at $u_0 = \frac{1}{2}$, and greater than $u = 1$ at $u_0 = \frac{3}{2}$, the cobweb approaches the fixed point, indicating stability.

(b) $u_{n+1} = 2u_n - 2$ in the cases $u_0 = \frac{1}{2}$ and $u_0 = \frac{5}{2}$ (more interesting than the stated initial value $u_0 = \frac{3}{2}$ which is on the same side of the fixed point). The fixed point is at the solution of

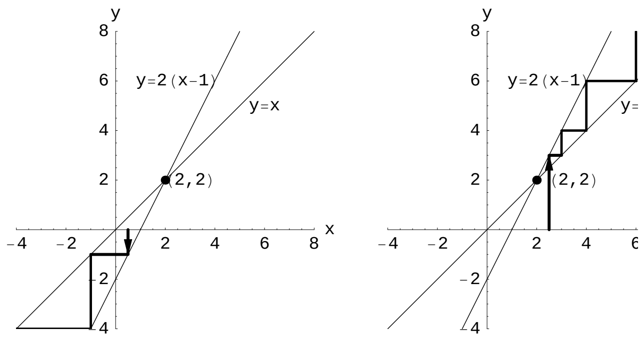

Figure 44: Problem 38.5(b)

$u = 2u - 2$, namely at $u = 2$. This corresponds to the point $(2, 2)$ on the cobweb diagram. The cobwebs for both $u_0 > 2$ and $u_0 < 2$ move away from the fixed point at $(2, 2)$. The fixed point is therefore unstable.

(c) $u_{n+1} = -u_n + 2$ in the cases $u_0 = \frac{1}{2}$ and $u_0 = \frac{3}{4}$. The fixed point occurs at the solution of $u = -u + 2$, namely at $u = 1$. This corresponds to the point $(1, 1)$ on the cobweb diagram. The lines $y = x$ and $y = -x + 2$ have slopes 1 and -1 respectively which means that a cobweb path starting from any initial point will generate a square cobweb path as shown in Figure 45. The fixed point $(1, 1)$ is stable in the sense that paths do not diverge from the fixed point but neither is the fixed point 'asymptotically' stable; that is, paths do not approach the fixed point as in (a).

(d) $u_{n+1} = -2\frac{1}{2}u_n + \frac{3}{2}$ in the cases $u_0 = \frac{1}{2}$ and $u_0 = \frac{3}{2}$. The fixed point occurs at the solution of $u = -\frac{1}{2}u + \frac{3}{2}$, namely where $u = 1$. This corresponds to the point $(1, 1)$ on the cobweb diagram. The two cobweb paths are shown in Figure 46. The fixed point is stable.

(e) $u_{n+1} = -2u_n + 3$ with initial values $u_0 = \frac{1}{2}$ and $u_0 = \frac{3}{2}$. The fixed point occurs at the solution of $u = -2u + 3$, namely at $u = 1$. The corresponds to the point $(1, 1)$ on the cobweb diagram. The

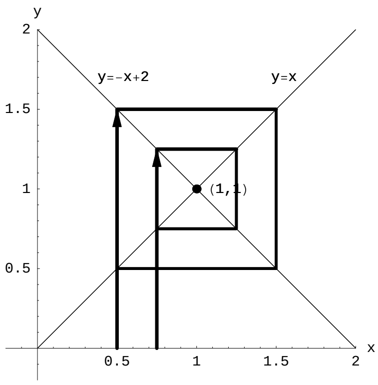

Figure 45: Problem 38.5(c)

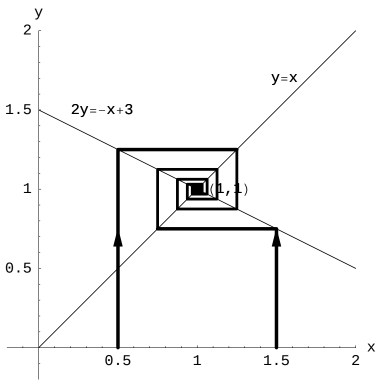

Figure 46: Problem 38.5(d)

cobweb paths diverge from the fixed point which is therefore unstable. Notice that the path with initial value $u_0 = \frac{3}{2}$ starts at the intersection of the line $y = -2x + 3$ and the x axis.

38.6. Let $n = 2^m$. Then the difference equation

$$f(n) = f(\tfrac{1}{2}n) + 1$$

becomes

$$f(2^m) = f(2^{m-1}) + 1.$$

Let $g(m) = f(2^m)$. Then $g(m)$ satisfies

$$g(m) = g(m-1) + 1.$$

The general solution of this equation is $g(m) = A + m$, where A is any constant. Since $m = \ln n/\ln 2$,

$$f(n) = g(\ln n/\ln 2) = A + \frac{\ln n}{\ln 2}.$$

The condition $f(1) = 0$ implies $A = 0$. The required solution is

$$f(n) = \frac{\ln n}{\ln 2}.$$

38.7. Let $n = 3^m$. Then the difference equation

$$f(n) = f(\tfrac{1}{3}n) + \tfrac{5}{8}$$

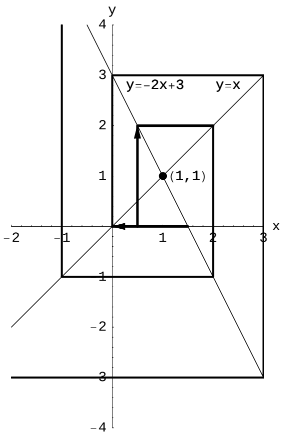

Figure 47: Problem 38.5(e)

becomes

$$f(3^m) = f(3^{m-1}) + \tfrac{5}{8}.$$

Let $g(m) = f(3^m) + \frac{5}{3}$. Then $g(m)$ satisfies

$$g(m) = g(m-1) + \frac{5}{3}.$$

The general solution of this equation is

$$g(m) = A + \frac{5m}{3} = A + \frac{5\ln n}{3\ln 3} = \frac{5\ln n}{3\ln 3},$$

since $f(1) = 0$.

38.8. (See Section 38.4.)

(a) $u_{n+2} + 2u_{n-1} - 3u_n = 0$. The characteristic equation is

$$p^2 + 2p - 3 = 0, \text{ or } (p+3)(p-1) = 0$$

which has solutions $p = -3$ and $p = 1$. Hence the general solution is

$$u_n = A(-3)^n + B \cdot 1^n = A(-3)^n + B.$$

(b) $u_{n+2} - 9u_n = 0$. The characteristic equation is

$$p^2 - 9 = 0, \text{ or } (p+3)(p-3) = 0$$

which has solutions $p = -3$ and $p = 3$. Hence the general solution is

$$u_n = A(-3)^n + B \cdot 3^n.$$

(c) $u_{n+2} + 9u_n = 0$. The characteristic equation is

$$p^2 + 9 = 0, \text{ or } (p+3\mathrm{i})(p-3\mathrm{i}) = 0$$

which has the imaginary solutions $p = -3\mathrm{i}$ and $p = 3\mathrm{i}$. Hence the general solution is

$$u_n = A(-3\mathrm{i})^n + B(3\mathrm{i})^n, \text{ or, } u_n = 3^n\left[C\cos\left(\frac{\pi n}{2}\right) + D\sin\left(\frac{\pi n}{2}\right)\right].$$

(d) $u_n - 4u_{n-1} + 5u_{n-2} = 0$. The characteristic equation is

$$p^2 - 4p + 5 = 0.$$

The roots are $p = 2 \pm \mathrm{i}$. The general solution is

$$u_n = A(2 - \mathrm{i})^n + B(2 + \mathrm{i})^n.$$

(e) $u_{n+2} - 4u_{n+1} + 4u_n = 0$. The characteristic equation is

$$p^2 - 4p + 4 = 0, \text{ or, } (p - 2)^2 = 0.$$

This equation has a repeated root $p = 2$, which is a special case. The general solution is

$$u_n = (A + Bn)2^n.$$

(f) $u_{n+3} - u_{n+2} + u_{n+q1} - u_n = 0$. The characteristic equation is the cubic equation

$$p^3 - p^2 + p - 1 = 0, \text{ or } (p - 1)(p + \mathrm{i})(p - \mathrm{i}) = 0.$$

Hence the general solution is

$$u_n = A + B\mathrm{i}^n + C(-\mathrm{i})^n,$$

or, in real form,

$$u_n = A + D\cos(\tfrac{1}{2}\pi n) + E\sin(\tfrac{1}{2}\pi n).$$

(g) $u_{n+3} - u_n = 0$. The characteristic equation is

$$p^3 - 1 = 0, \text{ or, } (p - 1)(p^2 + p + 1) = 0.$$

The roots are $p = 1$ and $p = \frac{1}{2}(-1 \pm \mathrm{i}\sqrt{3})$. Hence the general solution is

$$u_n = A + B[\tfrac{1}{2}(-1 + \mathrm{i}\sqrt{3})]^n + C[\tfrac{1}{2}(-1 - \mathrm{i}\sqrt{3})]^n,$$

or, in real form, since $-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i} = \cos(\frac{2}{3}\pi) + \mathrm{i}\sin(\frac{2}{3}\pi)$,

$$u_n = A + D\cos(\tfrac{2}{3}n\pi) + E\sin(\tfrac{2}{3}n\pi).$$

(h) $u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = 0$. The characteristic equation is

$$p^3 - 3p_2 + 3p - 1 = 0, \text{ or, } (p - 1)^2 = 0.$$

The root is repeated. Hence

$$u_n = A + Bn + cn^2.$$

(i) $u_{n+2} - u_{n+1} - u_n + u_{n-1} = 0$. The characteristic equation is

$$p^3 - p^2 - p + 1 = 0, \text{ or } (p + 1)(p - 1)^2 = 0.$$

Hence the general solution is

$$u_n = A + Bn + C(-1)^n.$$

38.9. $u_{n+2} - 6u_{n+1} + 13u_n = 0$, $u_0 = 0$, $u_1 = 1$. The characteristic equation is

$$p^2 - 6p + 13 = 0.$$

The roots of this equation are $p = 3 \pm 2\mathrm{i}$. In exponential form

$$3 \pm 2\mathrm{i} = \sqrt{13}[\cos(\alpha n) \pm \mathrm{i}\sin(\alpha n)].$$

where $\tan\alpha = \frac{2}{3}$. Hence the general solution can be expressed in the form

$$u_n = 13^{\frac{1}{2}n}[A\cos(\alpha n) + B\sin(\alpha n)].$$

From the initial conditions,

$$A = 0, \quad 1 = \sqrt{13} B \sin\alpha = \sqrt{13} B \frac{2}{\sqrt{13}} = 2B.$$

Therefore $B = \frac{1}{2}$. Hence the required solution is

$$u_n = \frac{1}{2} 13^{\frac{1}{2}n} \sin(\alpha n).$$

38.10. $u_n = A \cdot 2^n + b \cdot (-5)^n$ is a solution of a second-order difference equation with characteristic roots $p = 2$ and $p = -5$. These are the roots of

$$(p-2)(p+5) = p^2 + 3p - 10 = 0.$$

This is the characteristic equation of the difference equation of

$$u_{n+2} + 3u_{n+1} - 10u_n = 0.$$

38.11. Table 38.1 in Section 38.4 is helpful. It is helpful also to check the characteristic roots first to identify any special cases

(a) $u_{n+2} + 2u_{n+1} - 3u_n = f(n)$. The characteristic equation is $p^2 + 2p - 3 = 0$ which has solutions $p = 1$ and $p = -3$.

(i) $f(n) = 2^n$. Try $u_n = A2^n$. Then

$$\begin{aligned} u_{n+2} + 2u_{n+1} - 3u_n - 2^n &= A2^{n+2} + 2A2^{n+1} - 3A2^n - 2^n \\ &= 2^n(4A + 4A - 3A - 1) = 2^n(5A - 1). \end{aligned}$$

This is zero for all n if $A = \frac{1}{5}$. Hence a particular solution is

$$u_n = \tfrac{1}{5} 2^n.$$

(ii) $f(n) = n$. Special case. Try $u_n = An + Bn^2$. Then

$$\begin{aligned} u_{n+2} + 2u_{n+1} - 3u_n - n &= A(n+2) + B(n+2)^2 + 2A(n+1) \\ &\quad + 2B(n+1)^2 - 3An - 3Bn^2 \\ &= (8B - 1)n + (4A + 6B). \end{aligned}$$

This is zero for all n if $B = \frac{1}{8}$ and $A = -\frac{3}{16}$. Hence a particular solution is

$$u_n = -\frac{3}{16} n + \frac{1}{8} n^2.$$

(iii) $f(n) = 2$. Special case. Try $u_n = An$. Then

$$\begin{aligned} u_{n+2} + 2u_n - 3u_n - 2 &= A(n+2) + 2A(n+1) - 3An - 2 \\ &= (4A - 2). \end{aligned}$$

This is zero for all n if $A = \frac{1}{2}$. Hence a particular solution is $u_n = \frac{1}{2} n$.

(iv) $f(n) = (-3)^n$. Special case. Try $u_n = An(-3)^n$ (not $(-3)^n$). Then

$$\begin{aligned} u_{n+2} + 2u_n - 3u_n &= A(-3)^{n+2}(n+2) + 2(-3)^{n+1}(n+1) - 3(-3)^n n - (-3)^n \\ &= (-3)^n(12A - 1). \end{aligned}$$

This is zero for all n if $A = \frac{1}{12}$. Hence a particular solution is

$$u_n = \frac{1}{12} n(-3)^n.$$

(b) $u_{n+2}+2u_{n+1}+2u_n = f(n)$. The characteristic equation is $p^2+2p+2=0$ which has the solutions $p=-1\pm\mathrm{i}$.

(i) $f(n)=1$. Try $u_n=A$. Then

$$u_{n+2}+2u_{n+1}+2u_n-1 = A+2A+2A-1 = 5A-1,$$

This is zero if $A=\frac{1}{5}$. Hence a particular solution is $u_n=\frac{1}{5}$.

(ii) $f(n)=n+3$. Try $u_n=A+Bn$. Then

$$\begin{aligned} & u_{n+2}+2u_{n+1}+2u_n-n-3 \\ = \;& A+B(n+2)+2A+2B(n+1)+2A+2Bn-n-3 \\ = \;& 5A+4B-3+(5B-1)n. \end{aligned}$$

This is zero for all n if

$$5A+4B-3=0, \text{ and } 5B-1=0.$$

The solutions are $A=\frac{11}{25}$ and $B=\frac{1}{5}$. Hence a particular solution is

$$u_n=\tfrac{11}{25}+\tfrac{1}{5}n.$$

(iii) $f(n)=\cos(\frac{3}{4}\pi n)$. Replace the right-hand side by $\mathrm{e}^{\frac{3}{4}\pi \mathrm{i}n}$ and take the real part of the answer. Try $u_n=A\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}n}$, where $A=C+\mathrm{i}D$ is a complex constant. Then

$$\begin{aligned} & u_{n+2}+2u_{n+1}+2u_n-\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}n} \\ = \;& A\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}(n+2)}+2A\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}(n+1)}+A\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}n}-\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}n} \\ = \;& \mathrm{e}^{\frac{3}{4}\pi\mathrm{i}n}(C+\mathrm{i}D)[-\mathrm{i}-\sqrt{2}+\sqrt{2}\mathrm{i}+2C+2\mathrm{i}D-1] \\ = \;& \mathrm{e}^{\frac{3}{4}\pi\mathrm{i}n}[D(1-\sqrt{2})+C(2\sqrt{2})-1]+[C(\sqrt{2}-1)+(2-\sqrt{2})\mathrm{i}. \end{aligned}$$

This is zero for all n if

$$C=-\frac{2-\sqrt{2}}{2(3\sqrt{2}-4)}, \quad D=\frac{\sqrt{2}-1}{2(3\sqrt{2}-4)}.$$

Hence a particular solution is

$$\begin{aligned} u_n &= C\cos(\tfrac{3}{4}\pi n)-D\sin(\tfrac{3}{4}\pi n) \\ &= \frac{-(2-\sqrt{2})\cos(\frac{3}{4}\pi n)+(\sqrt{2}-1)\sin(\frac{3}{4}\pi n)}{2(3\sqrt{2}-4)} \end{aligned}$$

(c) $u_{n+3}-3u_{n+2}+3u_{n+1}+u_n=f(n)$. The characteristic equation has no simple solutions.

(i) $f(n)=1$. Try $u_n=A$. Then

$$u_{n+3}-3u_{n+2}+3u_{n+1}+u_n-1 = A-3A+3A+A-1 = 2A-1.$$

This is zero for all n if $A=\frac{1}{2}$.Hence a particular solution is $u_n=\frac{1}{2}$.

(ii) $f(n)=n$. Try $u_n=A+Bn$. Then

$$\begin{aligned} & u_{n+3}-3u_{n+2}+3u_{n+1}+u_n-n \\ = \;& A+B(n+3)-3A-3B(n+2)+3A+3B(n+1)+A+Bn-n \\ = \;& 2A+(2B-1)n. \end{aligned}$$

This is zero for all n if $A=0$ and $B=\frac{1}{2}$. Hence a particular solution is $u_n=\frac{1}{2}n$.

(iii) $f(n)=n^2$. Try $u_n=A+Bn+Cn^2$. Hence

$$\begin{aligned} & u_{n+3}-3u_{n+2}+3u_{n+1}+u_n-n^2 \\ = \;& A+B(n+3)+C(n+3)^2-3A-3B(n+2)-3C(n+2)^2 \\ & \quad +3A+3B(n+1)+3C(n+1)^2+A+Bn+Cn^2 \\ = \;& 2a+2bn+(2c-1)n^2. \end{aligned}$$

This is zero for all n if $a = 0$, $b = 0$ and $c = \frac{1}{2}$. Hence a particular solution is $u_n = \frac{1}{2}n^2$.
(d) $u_{n+2} - 6u_{n+1} + 9u_n = f(n)$. The characteristic equation is $p^2 - 6p + 9 = (p-3)^2 = 0$, which has the repeated solution $p = 3$.

(i) $f(n) = 2^n$. Try $u_n = A2^n$. Then

$$u_{n+2} - 6u_{n+1} + 9u_n - 2^n \quad = \quad A2^{n+2} - 6A2^{n+1} + 9A2^n - 2^n = 2^n(A-1)$$

This is zero for all n if $A = 1$. Hence a particular solution is $u_n = 2^n$.

(ii) $f(n) = 3$. Try $u_n = A$. Then

$$u_{n+2} - 6u_{n+1} + 9u_n - 3 = A - 6A + 9A - 3 = 4A - 3.$$

This is zero for all n if $A = \frac{3}{4}$. Hence a particular solution is $u_n = \frac{3}{4}$.

(iii) $f(n) = 3^n$. Special case. Therefore try $u_n = An^2 3^n$. Then

$$\begin{aligned} & u_{n+2} - 6u_{n+1} + 9u_n - 3^n \\ = \quad & A(n+2)^2 3^{n+2} - 6A(n+1)^2 3^{n+1} + 9An^2 3^n - 3^n \\ = \quad & 3^n[9A(n+2)^2 - 18A(n+1)^2 + 9An^2 - 1] \\ = \quad & 3^n(18A - 1). \end{aligned}$$

This is zero for all n if $A = \frac{1}{18}$. Hence a particular solution is $u_n = \frac{1}{18}n^2 3^n$.

(iv) $f(n) = n3^n$. Special case. Therefore try $u_n = (An^2 + Bn^3)3^n$. Hence

$$\begin{aligned} & u_{n+2} - 6u_{n+1} + 9u_n - 3^n n \\ = \quad & A[(n+2)^2 + B(n+2)^3]3^{n+2} - 6[A(n+1)^2 + B(n+1)^3]3^{n+1} \\ & \qquad +9[An^2 + Bn^3]3^n - 3^n n \\ = \quad & 3^n[(18A + 54B) + (54B - 1)n]. \end{aligned}$$

This is zero for all n if $A = -\frac{1}{18}$ and $B = \frac{1}{54}$. Hence a particular solution is

$$u_n = -\tfrac{1}{18}n^2 + \tfrac{1}{54}n^3.$$

38.12. The ball is dropped from height h_0. It hits the plate with speed $\sqrt{2gh_0}$ and rebound with speed $\varepsilon\sqrt{2gh_0}$ where ε is the the coefficient of restitution. The ball rebounds to a height $h_1 = \varepsilon^2 h_0$. In general after the nth impact the height reached is $h_n = \varepsilon^2 h_{n-1}$ in terms of the height h_{n-1} after the $(n-1)$th impact. This is a simple difference equation with initial condition h_0. To solve it:

$$h_n = \varepsilon^2 h_{n-1} = \varepsilon^4 h_{n-2} = \cdots = \varepsilon^{2n} h_0.$$

Assume that all impacts occurs at height zero. At the moment of impact, the plate is moving upwards with speed u, so the relative speed of the first impact is

$$v_0 + u = \sqrt{2gh_0} + u.$$

The ball rebounds with speed $\varepsilon(\sqrt{2gh_0} + u)$. It reaches a height h_1 where

$$\sqrt{2gh_1} = \varepsilon(\sqrt{2gh_0} + u).$$

Hence

$$h_1 = \frac{\varepsilon^2}{2g}(\sqrt{2gh_0} + u)^2.$$

In general, after n impacts the height h_n reached is given by

$$h_n = \frac{\varepsilon^2}{2g}(\sqrt{2gh_{n-1}} + u)^2.$$

This a nonlinear difference equation for h_n. Any fixed points of the equation satisfy

$$\sqrt{2gh} = \varepsilon(\sqrt{2gh} + u).$$

Hence

$$h = \frac{\varepsilon^2 u^2}{2g(1-\varepsilon)^2)}.$$

If h, u and ε satisfy this relation then the ball will rebound to the same height after each impact.

38.13. $D_n(x)$ is the $n \times n$ determinant defined by

$$D_n(x) = \begin{vmatrix} 2x & 1 & 1 & \dots & 0 \\ 1 & 2x & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2x \end{vmatrix}$$

Expand $D_n(x)$ by its first row:

$$D_n(x) = 2xD_{n-1}(x) - D_{n-2}(x).$$

This is a second-order difference equation. Its characteristic equation is

$$p^2 - 2xp + 1 = 0.$$

Assuming that $x \neq 1$, its solutions are

$$p_1, p_2 = x \pm \sqrt{(x^2 - 1)}.$$

Hence

$$D_n(x) = Ap_1^n + Bp_2^n.$$

The initial conditions are $D_1(x) = 2x$ and $D_2(x) = 4x^2 - 1$. Hence

$$D_1(x) = 2x = Ap_1 + Bp_2. \tag{i}$$

and

$$D_2(x) = 4x^2 - 1 = Ap_1^2 + Bp_2^2. \tag{ii}$$

Solve (i) and (ii) for A and B so that

$$A = \frac{2xp_2 - 4x^2 + 1}{p_1(p_2 - p_1)} = \frac{p_1^2}{2p_1\sqrt{(x^2-1)}} = \frac{p_1}{2\sqrt{(x^2-1)}},$$

$$B = \frac{2p_1 - 4x^2 + 1}{p_2(p_1 - p_2)} = -\frac{p_2^2}{2p_2\sqrt{(x^2-1)}} = -\frac{p_2}{2\sqrt{(x^2-1)}},$$

Hence, for $n \geq 1$,

$$\begin{aligned} D_n(x) &= Ap_1^n + Bp_2^n = \frac{p_1^n - p_2^n}{2\sqrt{(x^2-1)}} \\ &= \frac{(x + \sqrt{(x^2-1)})^n - (x - \sqrt{(x^2-1)})^n}{2\sqrt{(x^2-1)}} \end{aligned}$$

If $x = 1$, the characteristic equation is

$$p^2 - 2p + 1 = (p-1)^2 = 0.$$

which has the repeated solution $p = 1$. Hence

$$D_n = A + Bn.$$

Since $D_1 = 4$ and $D_2 = 3$, then $A = B = 1$ and

$$D_n = n + 1.$$

38.14. Consider the difference equation

$$u_{n+2} + u_{n+1} - 2u_n = 0,$$

and let $f(x) = \sum_{n=0}^{\infty} u_n x^n$. Multiply the equation by x^n and sum for all $n \geq 0$. Hence

$$\sum_{n=0}^{\infty} u_{n+2}x^n + \sum_{n=0}^{\infty} u_{n+1}x^n - 2\sum_{n=0}^{\infty} u_n x^n = 0,$$

or

$$\frac{1}{x^2}[f(x) - u_0 - u_1 x] + \frac{1}{x}[f(x) - u_0] - 2f(x) = 0,$$

or

$$\frac{1}{x^2}[f(x) - 1 + 2x] + \frac{1}{x}[f(x) - 1] - 2f(x) = 0.$$

Hence

$$f(x)(1 + x - 2x^2) = 1 - x, \text{ so that } f(x) = \frac{1}{1 + 2x}.$$

By the binomial theorem (which is only valid for $|x| < \frac{1}{2}$)

$$\frac{1}{1 + 2x} = 1 - 2x + (2x)^2 - (2x)^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n (2x)^n.$$

Therefore $u_n = (-2)^n$.

38.15. The Fibonacci sequence is a sequence in which any term is the sum of the two preceding terms, that is,

$$u_{n+2} = u_{n+1} + u_n, \quad (n = 1, 2, \ldots).$$

Assume that $u_1 = 1$, $u_2 = 2$. The characteristic equation is

$$p^2 - p - 1 = 0.$$

Its solutions are $p_1, p_2 = \frac{1}{2}[1 \pm \sqrt{5}]$. Therefore the general solution of the difference equation is given by

$$u_n = \frac{1}{2^n}[A(1 + \sqrt{5}) + B(1 - \sqrt{5})].$$

From the initial conditions

$$1 = \frac{1}{2}A(1 + \sqrt{5}) + \frac{1}{2}B(1 - \sqrt{5}),$$

$$2 = \frac{1}{4}A(1 + \sqrt{5})^2 + \frac{1}{4}B(1 - \sqrt{5})^2.$$

Solving these linear equations $A = -B = 1/(2\sqrt{5})$. Therefore the required solution is

$$u_n = \frac{1}{2^{n+1}\sqrt{5}}[(1 + \sqrt{5})^n - (1 - \sqrt{5})^n].$$

38.16. The difference equation

$$3u_{n+2} - 2u_{n+1} - u_n = 0$$

has the characteristic equation

$$3p^2 - 2p - 1 = 0, \text{ or } (3p + 1)(p - 1) = 0.$$

Its solutions are $p = -\frac{1}{3}$ and $p = 1$. Therefore

$$u_n = A + B(-\tfrac{1}{3})^n.$$

The intial conditions $u_1 = 2$ and $u_2 = 1$ imply

$$A - \tfrac{1}{3}B = 2, \quad A + \tfrac{1}{9}B = 1.$$

Hence $A = \frac{5}{4}$ and $B = -\frac{9}{4}$. The required solution is

$$u_n = \tfrac{5}{4} - \tfrac{9}{4}(-\tfrac{1}{3})^n.$$

As $n \to \infty$, then $(-\frac{1}{3})^n \to 0$ so that

$$u_n \to \frac{5}{4}.$$

38.17. In the symmetric random walk on $(0, N)$, the probability u_k that the walker reaches $x = 0$ before $x = N$ given that the walk starts at $x = k$ satisfies

$$u_{k+1} - 2u_k + u_{k-1} = 0.$$

Its characteristic equation is

$$p^2 - 2p + 1 = 0, \text{ or } (p-1)^2 = 0.$$

The equation has the repeated solution $p = 1$: therefore $u_k = A + Bk$. The boundary conditions are $u_0 = 1$ and $u_N = 0$. Hence $A = 1$ and $B = -\frac{1}{N}$. The probability that the walker reaches $x = 0$ first is

$$u_k = 1 - \frac{k}{N}.$$

If v_k is the probability that the walker reaches $x = N$ first, then v_k satisfies the same equation as u_k, but with $v_0 = 0$ and $v_1 = 1$. The solution is

$$v_k = 1 - u_k = \frac{k}{N}.$$

The expected duration d_k of the walk satisfies

$$d_{k+1} - 2d_k + d_{k-1} = -2.$$

This has the same characteristic equation as u_k. Hence the complementary function is $C + Dk$. Since $C + Dk$ is a solution of the homogeneous equation, we must try $d_k = Ek^2$. Therefore

$$d_{k+1} - 2d_k + d_{k-1} + 2 = E(k+1)^2 - 2Ek^2 + E(k-1)^2 + 2 = 2E + 2 = 0$$

if $E = -1$. The general solution is $d_k = C + Dk - k^2$. The boundary conditions are $d_0 = d_N = 0$ imply $C = 0$ and $D = N$. Hence the expected duration is given by

$$d_k = k(N - k).$$

38.18. Consider the difference equation

$$u_{n+2} = (n+2)(n+1)u_n.$$

$u_n = n!$ is a solution since

$$u_{n+2} - (n+2)(n+1)u_n = (n+2)! - (n+2)(n+1)n! = (n+2)! - (n+2)! = 0.$$

Let $u_n = v_n n!$. Then the difference equation becomes

$$v_{n+2}(n+2)! = (n+2)(n+1)n!v_n, \text{ or } v_{n+2} = v_n.$$

The linear difference equation for v_n has constant coefficients. Its characteristic equation $p^2 - 1 = 0$ with solutions $p = \pm 1$. Hence $v_n = A + B(-1)^n$ so that

$$u_n = An! + B(-1)^n n!.$$

The second independent solution is $B(-1)^n n!$.

38.19. Let

$$s_n = \sum_{k=1}^{n} k^3.$$

Then

$$s_n = \sum_{k=1}^{n} k^3 = \sum_{k=1}^{n-1} k^3 + n^3 = s_{n-1} + n^3.$$

The characteristic equation is $p - 1 = 0$. Therefore the complementary function is A. For a particular solution let $s_n = Bn + Cn^2 + Dn^3 + En^4$. Then

$$Bn + Cn^2 + Dn^3 + En^4 - B(n-1) - C(n-1)^2 - D(n-1)^3 - E(n-1)^4 = n^3.$$

Equate powers of n to zero: the result is

$$B - C + D - E = 0,$$

$$2C - 3D + 4E = 0,$$

$$3D - 6E = 0,$$

$$4E - 1 = 0.$$

Therefore

$$E = \frac{1}{4}, \quad D = \frac{1}{2}. \quad C = \frac{1}{4}, \quad B = 0.$$

Finally, we obtain the well-known formula

$$s_n = \sum_{k=1}^{n} k^3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4 = \frac{1}{4}n^2(n+1)^2.$$

38.20. Consider the difference equation

$$u_{n+2} + 2au_{n+1} + bu_n = 0.$$

Let

$$v_{n+1} = -u_n, \quad z_n = \begin{bmatrix} u_n \\ v_n \end{bmatrix}.$$

Then

$$z_{n+1} = \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -2au_n + bv_n \\ -u_n \end{bmatrix} = \begin{bmatrix} -2a & b \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} = Az_n,$$

where

$$A = \begin{bmatrix} -2a & b \\ -1 & 0 \end{bmatrix}.$$

It follows that

$$z_{n+1} = Az_n = A^2 z_{n-1} = \cdots = A^n z_1.$$

If $a = 1$ and $b = -8$, then

$$A = \begin{bmatrix} -2 & -8 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are given by

$$\left| \begin{array}{cc} -2-\lambda & -8 \\ -1 & -\lambda \end{array} \right].$$

Therefore λ satisfies

$$\lambda^2 + 2\lambda - 8 = (\lambda + 4)(\lambda - 2) = 0.$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -4$ and the corresponding eigenvectors are

$$\mathbf{s}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}.$$

The inverse of C is given by

$$C^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}.$$

Define the diagonal matrix D of eigenvalues, namely

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}.$$

By (13.5)

$$\begin{aligned} A^n &= CD^nC^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-4)^n \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}, \\ &= \frac{1}{6} \begin{bmatrix} 2^{n+1} - (-4)^{n+1} & -2^{n+3} - 2(-4)^{n+1} \\ -2^n + (-4)^n & 2^{n+2} + 2(-4)^n \end{bmatrix}. \end{aligned}$$

Finally

$$z_n = \frac{1}{6} \begin{bmatrix} 2^{n+1} - (-4)^{n+1} & -2^{n+3} - 2(-4)^{n+1} \\ -2^n + (-4)^n & 2^{n+2} + 2(-4)^n \end{bmatrix} \begin{bmatrix} u_1 \\ -u_0 \end{bmatrix}.$$

38.21. The logistic equation is

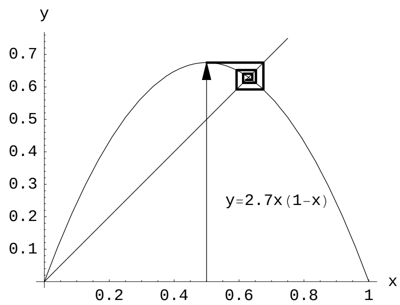

Figure 48: Problem 38.21(a)

$$u_{n+1} = \alpha u_n(1 - u_n).$$

The fixed points of the logistic equation occur where $u = \alpha u - \alpha u^2$, which has the solutions $u = 0$ and $u = (\alpha - 1)/\alpha$. It is the latter solution which is of interest.

(a) $\alpha = 2.7$ and $u_0 = 0.5$. The fixed point is at $(0.63, 0.63)$ in the cobweb diagram. Figure 48 shows the cobweb starting at $(0.5, 0)$ and approaching the fixed point which implies that it is stable.

(b) $\alpha = 2.9$ and $u_0 = 0.5$. The fixed point is at $(0.66, 0.66)$. Figure 49 shows the cobweb starting at $(0.5, 0)$. It approaches the fixed point which is therefore stable.

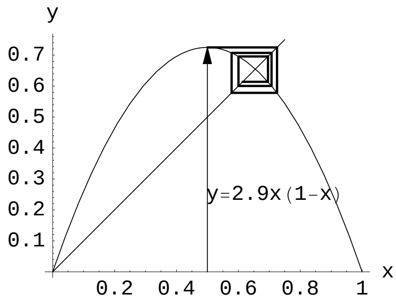

Figure 49: Problem 38.21(b)

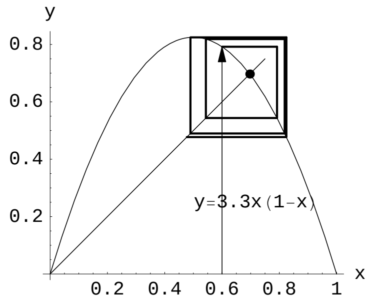

Figure 50: Problem 38.21(c)

(c) $\alpha = 3.3$ and $u_0 = 0.6$ (the initial value has been changed to improve the diagram). The fixed point is at $(0.70, 0.70)$. In this case the cobweb diverges from the fixed point indicating instability.

38.22. The logistic equation is

$$u_{n+1} = \alpha u_n(1 - u_n).$$

Let $f(x) = \alpha x(1 - x)$. Then $f'(x) = \alpha - 2\alpha x$ and at the origin $f'(0) = \alpha$. The behaviour of any cobweb starting at a positive small value of x will depend on the sign of the slope of $f(x)$ at the origin in relation to that of the straight line $y = x$. For stability we must have $0 < f'(0) < 1$, that is $0 < \alpha < 1$.

38.23. (See Section 38.5.) Consider the logistic equation

$$u_{n+1} = \alpha u_n(1 - u_n), \quad \alpha = 3.25.$$

Any period-2 solutions are given by solutions of

$$x = f(f(x)), \text{ where } f(x) = \alpha x(1 - x).$$

Hence

$$x = \alpha^2 x(1 - x) - \alpha^3 x^2(1 - x)^2.$$

This cubic equation factorizes into

$$(\alpha x + 1 - \alpha)(\alpha^2 x^2 - (\alpha + \alpha^2)x + 1 + \alpha) = 0.$$

The three solutions are

$$x_1 = \frac{\alpha - 1}{\alpha},$$

$$x_2 = \frac{1 + \alpha - \sqrt{(\alpha^2 - 2\alpha - 3)}}{2\alpha}, \quad x_3 = \frac{1 + \alpha + \sqrt{(\alpha^2 - 2\alpha - 3)}}{2\alpha}.$$

If $\alpha = 3.25$, then to 3 decimal places, $x_1 = 0.692$, $x_2 = 0.495$, $x_3 = 0.812$. u_n oscillates between x_2 and x_3.

38.24. Consider the difference equation $u_{n+1} = \alpha(\frac{1}{2} - |u_n - \frac{1}{2}|)$. The the graph of $y = f(x) = \alpha(\frac{1}{2} - |x - \frac{1}{2}|)$ for $\alpha = \frac{3}{2}$ (there is a misprint in early versions of the 3rd edition) is shown in

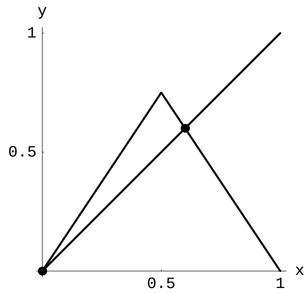

Figure 51: Problem 38.24: graph of $y = \alpha(\frac{1}{2} - |x - \frac{1}{2}|)$ with $y = x$.

Figure 51. The points where the line $y = x$ intersects the curve are the fixed points. One is obviously at the origin, and the other is where $x = \alpha(\frac{1}{2} - |x - \frac{1}{2}|)$. If $\alpha < 1$ there not a second fixed point, whilst if $\alpha > 1$ there is a second fixed point where (for $x > \frac{1}{2}$)

$$x = \alpha(\tfrac{1}{2} - (x - \tfrac{1}{2})) \text{ or, where } x = \tfrac{\alpha}{\alpha+1}.$$

For $\alpha = \frac{3}{2}$, the fixed point is at $(\frac{3}{5}, \frac{3}{5})$ in Figure 51.

The stability of the origin depends on the magnitude of $f'(0)$. For $x < \frac{1}{2}$, $f'(x) = \alpha$. Hence the origin is stable if $f'(0) = \alpha < 1$, and unstable if $f'(0) = \alpha > 1$. If $\alpha = 1$, then all values of x such that $0 \le x \le \frac{1}{2}$ are fixed points.

The graph of $y = f(f(x))$ where

$$f(f(x)) = \alpha[\tfrac{1}{2} - |\alpha(\tfrac{1}{2} - |x - \tfrac{1}{2}|) - \tfrac{1}{2}|]$$

and $\alpha = 2$, is shown in Figure 52. Over the interval $0 \le x \le 1$,

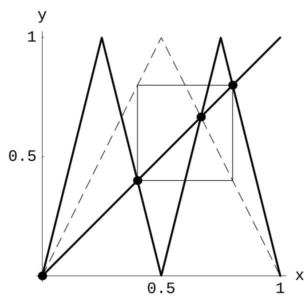

Figure 52: Problem 38.24

$$f(f(x)) = \begin{cases} 4x & 0 \le x \le \frac{1}{4} \\ -4x + 2 & \frac{1}{4} \le x \le \frac{1}{2} \\ 4x - 2 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4x + 4 & \frac{3}{4} \le x \le 1 \end{cases}.$$

The fixed points occur where $x = f(f(x))$ The solutions of $x = f(x)$ are still fixed points of $x = f(f(x))$. From Figure 52 two further fixed points occur where $x = -4x + 2$, namely at $x = \frac{2}{5}$, and where $x = -4x + 4$, namely at $x = \frac{4}{5}$. As shown in the figure it is possible to construct a periodic path between $(0.4, 0.4)$ and $(0.8, 0.8)$ with the other corners on $y = f(x)$. Hence, for $\alpha = 2$ a solution of

$$u_{n+1} = 2(\tfrac{1}{2} - |u_n - \tfrac{1}{2}|)$$

exists in which u_n alternates between the values 0.4 and 0.8.

38.25. The fixed points of $u_{n+1} = \alpha u_n(1 - u_n^3)$ occur where $x = \alpha x(1 - x^3)$. One fixed point occurs at $x = 0$. If $\alpha < 1$, then $x = 0$ is the only fixed point. There is a second fixed point at $x = x_1 = [(\alpha - 1)/\alpha]^{\frac{1}{3}}$, $(\alpha \neq 0)$. Since $f'(x) = \alpha - 4\alpha x^3$, then

$$f'(x_1) = \alpha - 4\alpha\left(\frac{\alpha - 1}{\alpha}\right) = 4 - 3\alpha.$$

The fixed point x_1 is stable if $f'(x_1) > -1$, that is, if $4 - 3\alpha > -1$, or $\alpha < \frac{5}{3}$. It is unstable if $f'(x_1) < -1$, that is if $\alpha > \frac{5}{3}$. A cobweb path for $u_{n+1} = 1.2u_n(1-u_n)$ starting at $u_0 = 0.3$ is shown

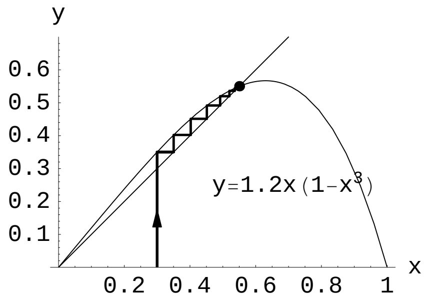

Figure 53: Problem 38.25: Cobweb for $u_{n+1} = 1.2u_n(1-u_n^3)$ with initial value $u_0 = 0.3$. The fixed point is at $(0.550, 0.550)$.

in Figure 54. The fixed point at $(0.550, 0.550)$ is stable. A cobweb path for $u_{n+1} = 1.4u_n(1 - u_n)$

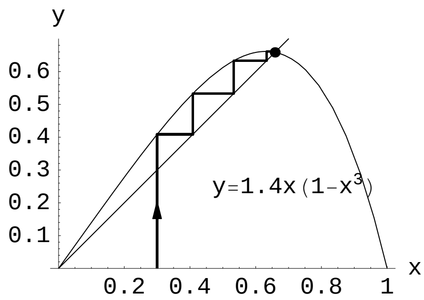

Figure 54: Problem 38.25: Cobweb for $u_{n+1} = 1.4u_n(1-u_n^3)$ with initial value $u_0 = 0.3$. The fixed point is at $(0.659, 0.659)$.

starting at $u_0 = 0.3$ is shown in Figure 54. The fixed point at $(0.523, 0.523)$ is stable. A cobweb path for $u_{n+1} = 1.8u_n(1 - u_n)$ starting at $u_0 = 0.3$ is shown in Figure 54. The fixed point at $(0.763, 0.763)$ is unstable since

$$f'(0.763) = -1.4 < -1.$$

It appears that there could exist a period-2 solution for this value of α but this would require further investigation.

38.26. Consider the logistic equation $u_{n+1} = \alpha u_n(1 - u_n)$ with $\alpha = 3.83$. Starting with $u_0 = 0.957417$, compute $u_1, u_2, \ldots$. The list up to u_{10} is shown in the table to 4 decimal places:

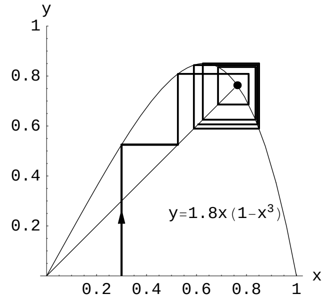

Figure 55: Problem 38.25: Cobweb for $u_{n+1} = 1.8u_n(1-u_n^3)$ with initial value $u_0 = 0.3$. The fixed point is at $(0.763, 0.763)$.

u_0	u_1	u_2	u_3	u_4	u_5
0.9574	0.1561	0.5047	0.9574	0.1561	0.5047
u_6	u_7	u_8	u_9	u_{10}	u_{11}
0.9574	0.1561	0.5047	0.9574	0.1561	0.5047

The 3-cycle can be seen (faintly) in Figure 38.11 in the book.

38.27. The difference equation is $u_{n+1} = \alpha u_n(1-u_n)^2$. Assume $\alpha > 0$. Fixed points are given by $u = \alpha u(1-u)^2$, which has the solutions $u = 0$ and $u = 1 \pm \alpha^{-\frac{1}{2}}$. Let $f(x) = \alpha x(1-x)^2$. Figure 56 shows the intersection of $y = f(x)$ and $y = x$ for the case $\alpha = 9$. Stability is determined by the

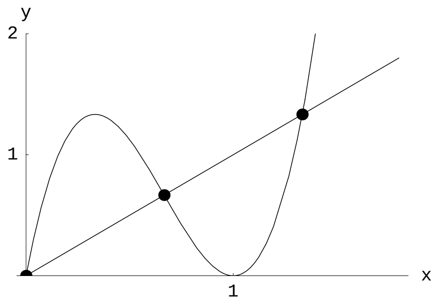

Figure 56: Problem 38.27

magnitude of $f'(x) = \alpha(1 - 4x + 3x^2)$.

(a) $\alpha = 9$. The fixed points are at $x = 0, \frac{2}{3}, \frac{4}{3}$. The derivatives are $f'(0) = 9 > 1$, $f'(\frac{2}{3}) = -3 < -1$, $f'(\frac{4}{3}) = 9 > 1$. Therefore $x = 0$ and $x = \frac{4}{3}$ are unstable and $x = \frac{2}{3}$ is stable.

(b) $\alpha = 4$. The fixed points are at $x = 0, \frac{1}{2}, \frac{3}{2}$. The derivatives are $f'(0) = 4 > 1$, $f'(\frac{1}{2}) = -1$, $f'(\frac{3}{2}) = \frac{7}{4} > 1$. Therefore $x = 0$ and $x = \frac{3}{2}$ are unstable. The fixed point at $x = \frac{1}{2}$ is a critical case at which period doubling should occur.

(c) $\alpha = \frac{9}{4}$. The fixed points are at $x = 0, \frac{1}{3}, \frac{5}{3}$. The derivatives are $f'(0) = \frac{9}{4} > 1$, $f'(\frac{1}{3}) = 0$, $f'(\frac{5}{3}) = 6 > 1$. Therefore $x = 0$ and $x = \frac{5}{3}$ are unstable, and $x = \frac{1}{3}$ is stable.

38.28. Let u_n satisfy $u_{n+1} = 4u_n(1-u_n)$. If $u_n = \sin^2(2^n C\pi)$, then

$$\begin{aligned}
u_{n+1} - 4u_n(1-u_n) &= \sin^2(2^{n+1}C\pi) - 4\sin^2(2^n C\pi) + 4\sin^4(2^n C\pi) \\
&= 4\sin^2(2^n C\pi)\cos^2(2^n C\pi) - 4\sin^2(2^n C\pi) + 4\sin^4(2^n C\pi) \\
&= 4\sin^2(2^n C\pi)[\cos^2(2^n C\pi) - 1] + 4\sin^4(2^n C\pi) \\
&= -4\sin^4(2^n C\pi) + 4\sin^4(2^n C\pi) \\
&= 0
\end{aligned}$$

verifying that it is a solution.