PART V: Multivariable Calculus

Chapter 28: Differentiation of functions of two variables

28.1 If $z = f(x, y)$, then contours of the function are curves given by $f(x, y) = c$, a constant, for selected values of c.
(a) The contours are given by the straight lines $2x - 3y + 4 = c$.

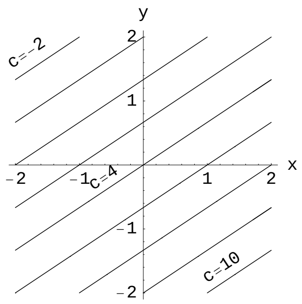

Figure 1: Problem 28.1(a)

(b) The contours are given by the straight lines $-x + 2y - 1 = c$.

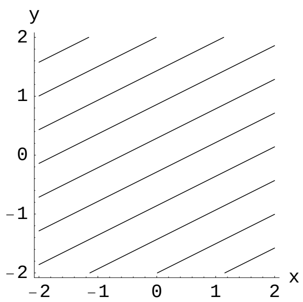

Figure 2: Problem 28.1(b)

(c) The contours are given by $(x - 1)(y - 1) = c$.
(d) The contours are given by $x^2 + \frac{1}{4}y^2 - 1 = c$, for $c > -1$. They form a family of ellipses (see Fig. 4).
(e) The contours are given by $x^2 + 2x + y^2 = c$ or $(x + 1)^2 + y^2 = c + 1$, which is a family of concentric circles.
(f) The contours are given by $(y/x) = c$ or $y = cx$, which is a family of straight lines through the origin.
(g) The contours are given by $y^2 - x^2 = c$ which is a family of hyperbolas.
(h) The contours are given by $y/x^3 = c$.
(i) The contours are given by $x^3 + 4y^2 = c$.
(j) The contours are given by $y/(x + y) = c$, or $y = cx/(1 - c)$, or $y = mx$, for any constant m. These are straight lines through the origin (not shown).

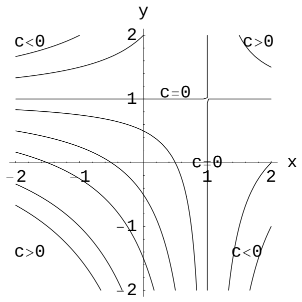

Figure 3: Problem 28.1(c)

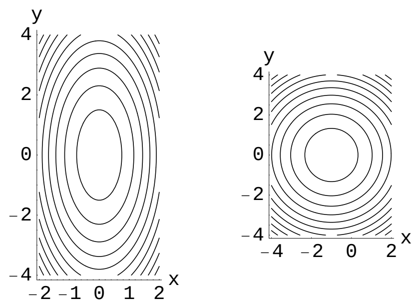

Figure 4: Problem 28.1(d); Problem 28.1(e): the contours are concentric circles centred at $(-1, 0)$.

28.2. In Figure 7, $P_0 : (x, y)$ is a point on a contour $\mathcal{C}_0$, and $\mathcal{C}_1$ is any closely adjacent contour associated with a slightly higher level than $\mathcal{C}_0$. P_1 is the point on $\mathcal{C}_1$ that is at the minimum distance from P_0; therefore P_0P_1 is in the direction of steepest ascent from P_0 on to the level on $\mathcal{C}_1$. The angle between P_0P_1 and $\mathcal{C}_0$ clearly approaches a right angle as the interval between the contours approaches zero. Therefore the path of steepest ascent crosses every intersecting contour at a right angle.

(a) $z = 2x - 3y + 4$. The contours are given by the parallel straight lines $2x - 3y + 4 = c$. The slopes of these lines are all $\frac{2}{3}$. The slope of the steepest ascent through $(1, 1)$ is therefore equal to $-\frac{3}{2}$, since $-\frac{3}{2} \times \frac{2}{3} = -1$ (see Figure 8).

(b) $z = x - y$. The contours are given by the parallel straight lines $x - y = c$, whose slope is equal to 1. The slope of the path of steepest ascent through $(1, 1)$ is therefore -1 (see Figure 8).

(c) $z = x^2y^2$. The contours are given by $x^2y^2 = c$ shown, for the first quadrant, in Figure 9. These are rectangular hyperbolas $y = \sqrt{c}/x$, $x > 0$, with an axis of symmetry $y = x$. This passes through

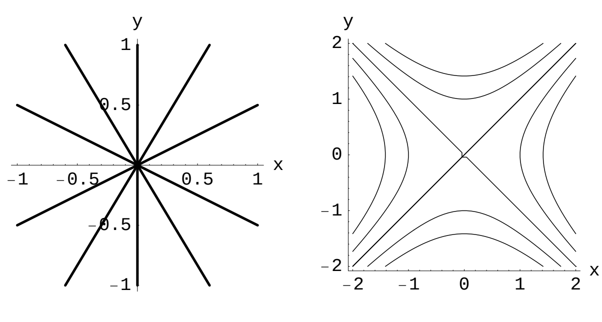

Figure 5: Problem 28.1(f): Problem 28.1(g)

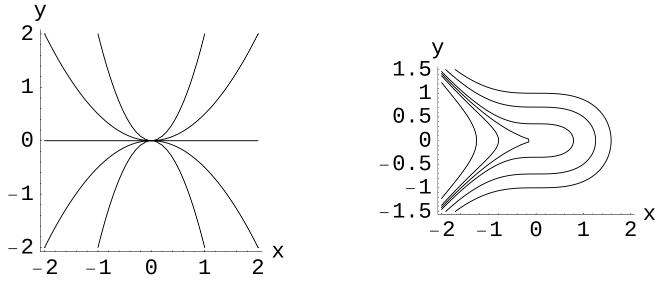

Figure 6: Problem 28.1(h); Problem 28.1(i).

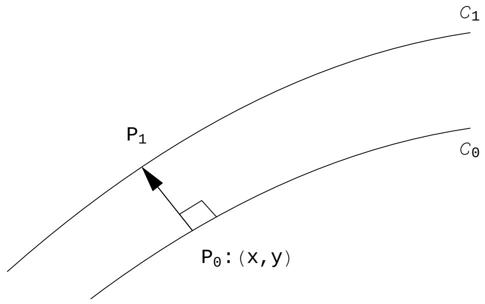

Figure 7: Problem 28.2

(1, 1) and is perpendicular to all the contours
(d) $z = (x-1)^2 + \frac{1}{4}(y-1)^2$. The contours are given by $(x-1)^2 + \frac{1}{4}(y-1)^2 = c > 0$, which is a family of ellipses centred on $(1, 1)$ and axes parallel to the x and y axes. As we move away from $(1, 1)$, z increases in all directions, but its steepest ascent directions are clearly in the x directions where the ellipses are closest (see Figure 9).
28.3 (a) $f(x, y) = 3x + 7y - 2$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x + 7y - 2) = 3$$

everywhere including the point $(2, 1)$. Similarly

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x + 7y - 2) = 7$$

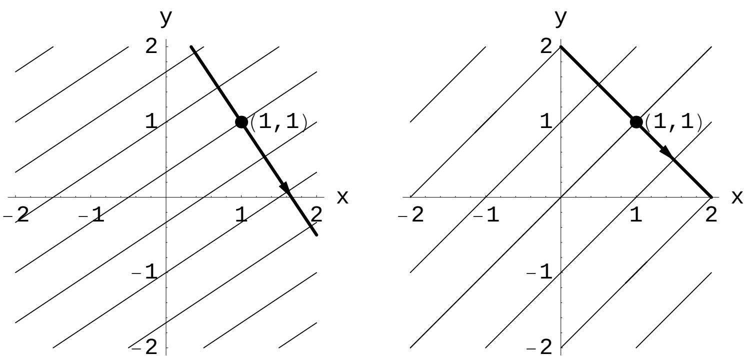

Figure 8: Problem 28.2(a); Problem 28.2(b)

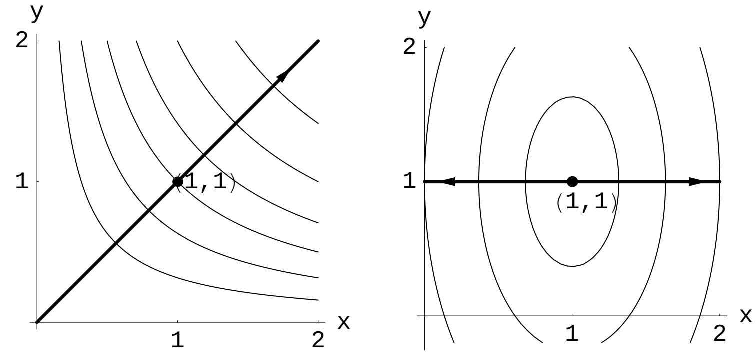

Figure 9: Problem 28.2(c); Problem 28.2(d).

everywhere including the point $(2,1)$.
(b) $f(x,y) = -2x + 3y + 4$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(-2x + 3y + 4) = -2$$

everywhere including the point $(2,1)$. Also

$$\frac{\partial}{\partial y}(-2x + 3y + 4) = 3$$

everywhere including the point $(2,1)$.
(c) $f(x,y) = 2x^2 - 3y^2 - 2xy - x - y + 1$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 4x - 2y - 1, \quad \frac{\partial f}{\partial y} = -6y - 2x - 1.$$

At the point $(2,1)$, $\partial f/\partial x = 5$ and $\partial f/\partial y = -11$.
(d) $f(x,y) = \frac{1}{8}x^3 + y^3 - 2y - 1$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{3}{8}x^2, \quad \frac{\partial f}{\partial y} = 3y^2 - 2.$$

At the point $(2,1)$, $\partial f/\partial x = \frac{3}{2}$ and $\partial f/\partial y = 1$.
(e) $f(x,y) = x^4y^2 - 1$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 4x^3y^2, \quad \frac{\partial f}{\partial y} = 2x^4y.$$

At $(2,1)$, $\partial f/\partial x = 32$ and $\partial f/\partial y = 48$.
(f) $f(x,y) = (x-1)(y-2)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = y - 2, \quad \frac{\partial f}{\partial y} = x - 1.$$

At $(2,1)$, $\partial f/\partial x = -1$ and $\partial f/\partial y = 1$.
(g) $f(x,y) = 1/(xy)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{1}{x^2y}, \quad \frac{\partial f}{\partial y} = -\frac{1}{xy^2}.$$

At $(2,1)$, $\partial f/\partial x = -\frac{1}{4}$ and $\partial f/\partial y = -\frac{1}{2}$.

(h) $f(x,y) = x/y$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{y}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}.$$

At $(2,1)$, $\partial f/\partial x = 1$ and $\partial f/\partial y = -2$.

(i) $f(x,y) = (x-y)/(x+y)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{2y}{(x+y)^2}, \quad \frac{\partial f}{\partial y} = -\frac{2x}{(x+y)^2}.$$

At $(2,1)$, $\partial f/\partial x = 2/9$ and $\partial f/\partial y = -4/9$.

(j) $f(x,y) = 3/(x^2+y^2)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{2x}{(x^2+y^2)^2}, \quad \frac{\partial f}{\partial y} = -\frac{2y}{(x^2+y^2)^2}.$$

At $(2,1)$, $\partial f/\partial x = -4/25$ and $\partial f/\partial y = -2/25$.

(k) $f(x,y) = (x^2+y^2)^{\frac{1}{2}}$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}, \quad \frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}.$$

At $(2,1)$, $\partial f/\partial x = 2/\sqrt{5}$ and $\partial f/\partial y = 1/\sqrt{5}$.

(l) $f(x,y) = (2x-3y+2)^3$. The chain rule (Section 3.3) is required. The partial derivatives are

$$\frac{\partial f}{\partial x} = 6(2x-3y+2)^2, \quad \frac{\partial f}{\partial y} = -9(2x-3y+2)^2.$$

At $(2,1)$, $\partial f/\partial x = 54$ and $\partial f/\partial y = -81$.

(m) $f(x,y) = \mathrm{e}^{x^2+y^2}$. Let $u = x^2+y^2$ so that $f = \mathrm{e}^u$. The chain rule then implies

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x} = \mathrm{e}^u.2x = 2x\mathrm{e}^{x^2+y^2}, \quad \frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial y} = \mathrm{e}^u.2y = 2y\mathrm{e}^{x^2+y^2}.$$

At $(2,1)$, $\partial f/\partial x = 4\mathrm{e}^5$ and $\partial f/\partial y = 2\mathrm{e}^5$.

(n) $f(x,y) = \cos(x^2-y^2)$. Use the chain rule. Let $u = x^2-y^2$ so that $f = \cos u$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x} = -\sin u.2x = -2x\sin(x^2-y^2),$$
$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial y} = -\sin u.(-2y) = 2y\sin(x^2-y^2).$$

At $(2,1)$, $\partial f/\partial x = -4\sin 3$ and $\partial f/\partial y = 2\sin 3$.

(o) $f(x,y) = \sin(x/y)$. Use the chain rule. Let $u = x/y$ so that $f = \sin u$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x} = \cos u.\frac{1}{y} = \frac{1}{y}\cos\left(\frac{x}{y}\right),$$
$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial y} = \cos u.-\frac{x}{y^2} = -\frac{x}{y^2}\cos\left(\frac{x}{y}\right).$$

At $(2,1)$, $\partial f/\partial x = \cos 2$ and $\partial f/\partial y = -2\cos 2$.

(p) $f(x,y) = \arctan(y/x)$. Use the chain rule. Let $u = y/x$ so that $f = \arctan u$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial x} = \frac{1}{1+u^2}.-\frac{y}{x^2} = -\frac{y}{x^2+y^2},$$

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\partial u}{\partial y} = \frac{1}{1+u^2}.\frac{1}{x} = \frac{x}{x^2+y^2}.$$

At $(2,1)$, $\partial f/\partial x = -1/5$ and $\partial f/\partial y = 2/5$.

28.4. (a) Given $z = g(ax+by)$ and $u = ax+b$. Then, by the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}g(u)}{\mathrm{d}u}\frac{\partial u}{\partial x} = g'(u)a = ag'(ax+by),$$

and

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}g(u)}{\mathrm{d}u}\frac{\partial u}{\partial y} = g'(u)b = bg'(ax+b).$$

If $z = g(u) = \cos u$ and $u = ax+by$, then $g'(u) = -\sin u$, and

$$\frac{\partial z}{\partial x} = g'(u)a = -a\sin u = -a\sin(ax+by),$$

$$\frac{\partial z}{\partial y} = g'(u)b = -b\sin u = -b\sin(ax+by).$$

To check these find $\partial/\partial x$ and $\partial/\partial y$ of e^{ax+by} and $\cos(ax+by)$ in the usual direct way.

If $z = g(u) = \mathrm{e}^u$ and $u = ax+by$, then $g'(u) = \mathrm{e}^u$, and

$$\frac{\partial z}{\partial x} = g'(u)a = a\mathrm{e}^u = a\mathrm{e}^{ax+by}, \quad \frac{\partial z}{\partial y} = g'(u)b = b\mathrm{e}^u = b\mathrm{e}^{ax+by}.$$

(b) In this case $z = g(u)$ where $u = \sin(xy)$, $g(u) = \mathrm{e}^u$ and $g'(u) = \mathrm{e}^u$. By the chain rule

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}g(u)}{\mathrm{d}u}\frac{\partial u}{\partial x} = g'(\sin xy)y\cos(xy) = \mathrm{e}^{\sin(xy)}y\cos(xy),$$

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}g(u)}{\mathrm{d}u}\frac{\partial u}{\partial y} = g'(\sin xy)x\cos(xy) = \mathrm{e}^{\sin(xy)}x\cos(xy).$$

Directly

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\mathrm{e}^{\sin(xy)}] = \mathrm{e}^{\sin(xy)}\frac{\partial}{\partial x}\sin(xy) = \mathrm{e}^{\sin(xy)}y\cos(xy).$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[\mathrm{e}^{\sin(xy)}] = \mathrm{e}^{\sin(xy)}\frac{\partial}{\partial y}\sin(xy) = \mathrm{e}^{\sin(xy)}x\cos(xy).$$

(c) $V = g(r)$ where $r = (x^2+y^2)^{\frac{1}{2}}$. In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. By the chain rule

$$\frac{\partial V}{\partial x} = g'(r)\frac{\partial r}{\partial x} = g'[(x^2+y^2)^{\frac{1}{2}}]\frac{x}{(x^2+y^2)^{\frac{1}{2}}},$$

$$\frac{\partial V}{\partial y} = g'(r)\frac{\partial r}{\partial y} = g'[(x^2+y^2)^{\frac{1}{2}}]\frac{y}{(x^2+y^2)^{\frac{1}{2}}}.$$

In terms of r and θ,

$$\frac{\partial V}{\partial x} = g'(r)\cos\theta, \quad \frac{\partial V}{\partial y} = g'(r)\sin\theta.$$

28.5. Given $r = (x^2+y^2)^{\frac{1}{2}}$ and $x = r\cos\theta$,

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}[(x^2+y^2)^{\frac{1}{2}}] = \frac{x}{(x^2+y^2)^{\frac{1}{2}}},$$

and

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta.$$

In terms of r and θ,

$$\frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta.$$

Hence, in terms of θ

$$\frac{\partial r}{\partial x}\frac{\partial x}{\partial r} = \cos\theta.\cos\theta = \cos^2\theta \neq 1,$$

in general.

Since $r = (x^2+y^2)^{\frac{1}{2}}$, the increment δr due to an increment δx when y is held constant is given by

$$\begin{aligned}\delta r &= ((x+\delta x)^2+y^2)^{\frac{1}{2}} - (x^2+y^2)^{\frac{1}{2}} \\ &= (x^2+y^2)^{\frac{1}{2}}\left(1+\frac{2x\delta x}{x^2+y^2}+\frac{(\delta x)^2}{x^2+y^2}\right)^{\frac{1}{2}} - (x^2+y^2)^{\frac{1}{2}} \approx \frac{x\delta x}{(x^2+y^2)^{\frac{1}{2}}} \\ &= \cos\theta\delta x\end{aligned}$$

in terms of r and θ.

For $x = r\cos\theta$, let δr be the incremental change in the direction r when θ is held constant. The corresponding change in x is

$$\delta x = (r+\delta r)\cos\theta - r\cos\theta = \delta r\cos\theta.$$

The incremental ratios $\delta r/\delta x$ differ since the directions $y =$ constant and $\theta =$ constant are different.

28.6. (a) For $z = \sin(x-y)$, let $u = x-y$. Then, by the chain rule, the partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}u}\sin u\frac{\partial}{\partial x}(x-y) = \cos u.1 = \cos(x-y),$$

and

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}u}\sin u\frac{\partial}{\partial y}(x-y) = \cos u.(-1) = -\cos(x-y),$$

Therefore

$$\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y} = \frac{\cos(x-y)}{-\cos(x-y)} = -1.$$

(b) This result shows that (a) is generally true for $z = g(x-y)$. As in (a), let $u = x-y$. Then, by the chain rule

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}u}g(u)\frac{\partial}{\partial x}(x-y) = g'(u).1 = g'(u),$$

and

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}u}g(u)\frac{\partial}{\partial y}(x-y) = g'(u)(-1) = -g'(u),$$

Therefore

$$\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y} = \frac{g'(u)}{-g'(u)} = -1.$$

28.7. Let $u = x/y$ in $z = g(x/y)$. Then, by the chain rule, the partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}u}g(u)\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = g'(u)\frac{1}{y},$$

and

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}u}g(u)\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = g'(u)\left(\frac{-x}{y^2}\right).$$

Therefore

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\frac{g'(u)}{y} - y\frac{xg'(u)}{y^2} = 0.$$

28.8. (a) $f(x,y) = ax + by + c$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = a, \quad \frac{\partial f}{\partial y} = b,$$

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0.$$

(b) $f(x,y) = x^2 + 2y^2 + 3xy - x + 1$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = 2x + 3y - 1, \quad \frac{\partial f}{\partial y} = 3x + 4y,$$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 3.$$

(c) $f(x,y) = \sin(x-y)$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = \cos(x-y), \quad \frac{\partial f}{\partial y} = -\cos(x-y),$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x-y), \quad \frac{\partial^2 f}{\partial y^2} = -\sin(x-y), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \sin(x-y).$$

(d) $f(x,y) = y/x$. The partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial f}{\partial y} = \frac{1}{x},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2y}{x^3}, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{1}{x^2}.$$

(e) $f(x,y) = \mathrm{e}^{2x+3y}$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 2\mathrm{e}^{2x+3y}, \quad \frac{\partial f}{\partial y} = 3\mathrm{e}^{2x+3y},$$

$$\frac{\partial^2 f}{\partial x^2} = 4\mathrm{e}^{2x+3y}, \quad \frac{\partial^2 f}{\partial y^2} = 9\mathrm{e}^{2x+3y}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6\mathrm{e}^{2x+3y}.$$

(f) $f(x,y) = (1/x) + (1/y)$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial f}{\partial y} = -\frac{1}{y^2},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0.$$

(g) $f(x,y) = \sin 3x + \cos 2y$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = 3\cos 3x, \quad \frac{\partial f}{\partial y} = -2\sin 2y,$$

$$\frac{\partial^2 f}{\partial x^2} = -9\sin 3x, \quad \frac{\partial^2 f}{\partial y^2} = -4\cos 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0.$$

(h) $f(x,y) = (3x-4y)^4$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 12(3x-4y)^3, \quad \frac{\partial f}{\partial y} = -16(3x-4y)^3,$$

$$\frac{\partial^2 f}{\partial x^2} = 108(3x-4y)^2, \quad \frac{\partial^2 f}{\partial y^2} = 192(3x-4y)^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -144(3x - 4y)^2.$$

(i) $f(x, y) = 1/(x + y)$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2}, \quad \frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{(x+y)^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{(x+y)^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{2}{(x+y)^2}.$$

(j) $f(x, y) = \ln(xy) = \ln x + \ln y$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{x}, \quad \frac{\partial f}{\partial y} = \frac{1}{y},$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{1}{y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0.$$

(k) $f(x, y) = 1/(x^2 + y^2)^{\frac{1}{2}}$. The first and second partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2)^{\frac{3}{2}}}, \quad \frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2x^2 - y^2}{(x^2+y^2)^{\frac{5}{2}}}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{-x^2 + 2y^2}{(x^2+y^2)^{\frac{5}{2}}}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{3xy}{(x^2+y^2)^{\frac{5}{2}}}.$$

28.9. If $r = (x^2 + y^2)^{\frac{1}{2}}$ and $z = \ln r$, then, using the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}r}\frac{\partial r}{\partial x} = \frac{1}{r}\frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{x}{r^2},$$

and

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\mathrm{d}z}{\mathrm{d}r}\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right) \\
&= \frac{1}{r}\left(-\frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}\right) \\
&= \frac{1}{r^2} - \frac{2x^2}{r^4}.
\end{aligned}$$

By symmetry in the variables x and y, we can write immediately

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{r^2} - \frac{2y^2}{r^4}.$$

Therefore, putting $x^2 + y^2 = r^2$,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r^2} - \frac{2x^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} = 0,$$

which is a **partial differential equation** called **Laplace's equation**. It has been verified that $z = \ln r = \frac{1}{2}\ln(x^2 + y^2)$ is a solution of Laplace's equation.

28.10. Given the surface $z = f(x, y)$ and a point $Q : (a, b, c = f(a, b))$ on the surface, then the tangent plane at Q is given by

$$z - c = \left(\frac{\partial f}{\partial x}\right)_{(a,b)}(x - a) + \left(\frac{\partial f}{\partial y}\right)_{(a,b)}(y - b).$$

The direction of a normal to the surface at Q is given by the vector

$$\left(\left(\frac{\partial f}{\partial x}\right)_{(a,b)}, \left(\frac{\partial f}{\partial y}\right)_{(a,b)}, -1\right).$$

(a) Surface $z = x^2 + y^2$ and point $Q : (1, 1, 2)$. The first partial derivatives are

$$\frac{\partial f}{\partial x} = 2x = 2 \text{ at } Q, \quad \frac{\partial f}{\partial y} = 2y = 2 \text{ at } Q.$$

Hence the tangent plane is

$$z - 2 = 2(x - 1) + 2(y - 1) \text{ or } 2x + 2y - z = 2.$$

A normal vector at Q is $(2, 2, -1)$.
(b) Surface $z = xy$ and point $Q : (2, 2, 4)$. The first partial derivatives are

$$\frac{\partial f}{\partial x} = y = 2 \text{ at } Q, \quad \frac{\partial f}{\partial y} = x = 2 \text{ at } Q.$$

Hence the tangent plane is

$$z - 4 = 2(x - 2) + 2(y - 2) \text{ or } 2x + 2y - z = 4.$$

A normal vector at Q is $(2, 2, -1)$.
(c) Surface $z = x/y$ and point $Q : (2, 1, 2)$. The first partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{y} = 1 \text{ at } Q, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2} = -2 \text{ at } Q.$$

Hence the tangent plane is

$$z - 2 = (x - 2) - 2(y - 1) \text{ or } x - 2y - z = -2.$$

A normal vector at Q is $(1, -2, -1)$.
(d) Surface $z = (29 - x^2 - y^2)^{\frac{1}{2}}$ and point $Q : (3, 4, 2)$. The first partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{x}{(29 - x^2 - y^2)^{\frac{1}{2}}} = -\frac{3}{2} \text{ at } Q, \quad \frac{\partial f}{\partial y} = -\frac{y}{(29 - x^2 - y^2)^{\frac{1}{2}}} = -2 \text{ at } Q.$$

Hence the tangent plane is

$$z - 2 = -\tfrac{3}{2}(x - 3) - 2(y - 4) \text{ or } 3x + 4y + 2z = 20.$$

A normal vector at Q is $(-\frac{3}{2}, -2, -1)$.
(e) Surface $z = x^2 + y^2 - 2x - 2y$ and the point $Q : (1, 2, -2)$. The first partial derivatives are

$$\frac{\partial f}{\partial x} = 2x - 2 = 0 \text{ at } Q, \quad \frac{\partial f}{\partial y} = 2y - 2 = 0 \text{ at } Q.$$

Hence the tangent plane is

$$z + 2 = 0, \text{ or } z = -2.$$

A normal vector is $(0, 0, -1)$.
(f) Surface $z = \mathrm{e}^{xy}$ and point $Q : (0, 0, 1)$. The first partial deritives are

$$\frac{\partial f}{\partial x} = y\mathrm{e}^{xy} = 0 \text{ at } Q, \quad \frac{\partial f}{\partial y} = x\mathrm{e}^{xy} = 0 \text{ at } Q.$$

Hence the tangent plane is

$$z - 1 = 0, \text{ or } z = 1.$$

A normal vector is $(0, 0, -1)$.

28.11. The surfaces are $z = x^2 + y^2$ and the plane $z = x - y + 2$. The point $Q : (1, 1, 2)$ lies on both surfaces. By (28.7), two normals are

$$\mathbf{n}_1 = \left[\left(\frac{\partial(x^2 + y^2)}{\partial x} \right)_Q, \left(\frac{\partial(x^2 + y^2)}{\partial y} \right)_Q, -1 \right] = [(2x)_Q, (2y)_Q, -1] = (2, 2, -1),$$

$$\mathbf{n}_2 = \left[\left(\frac{\partial(x - y + 2)}{\partial x} \right)_Q, \left(\frac{\partial(x - y + 2)}{\partial y} \right)_Q, -1 \right] = (1, -1, -1).$$

The scalar product

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (2, 2, -1) \cdot (1, -1, -1) = 2 - 2 + 1 = 1.$$

If θ is the angle between the normals, then, by (10.4),

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2| \cos\theta.$$

Therefore

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1}{\sqrt{(4 + 4 + 1)}\sqrt{(1 + 1 + 1)}} = \frac{1}{3\sqrt{3}}.$$

The angle between the normals is equal to the angle between the surfaces at Q, which is $1.37\ldots$ radians or $78.9\ldots^\circ$.

28.12. The stationary points of $f(x, y)$ occur at all simultaneous solutions of $\partial f/\partial x = 0$, $\partial f/\partial y = 0$. Let

$$\Delta(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2, \quad A(x, y) = \frac{\partial^2 f}{\partial x^2}.$$

Let $P : (a, b)$ is a stationary point. Then, from (28.9),

(i) P is a saddle if $\Delta(a, b) < 0$;
(ii) P is a maximum if $\Delta(a, b) > 0$ and $A(a, b) < 0$;
(iii) P is a minimum if $\Delta(a, b) > 0$ and $A(a, b) > 0$.

(a) $f(x, y) = (x - 1)(y + 2)$. The stationary point occurs where

$$\frac{\partial f}{\partial x} = y + 2 = 0, \quad \frac{\partial f}{\partial y} = x - 1 = 0,$$

that is, at $(1, -2)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = 1.$$

Hence $\Delta(1, -2) = -1 < 0$, so that $(1, -2)$ is a saddle.

(b) $f(x, y) = x^2 + y^2 - 2x + 2y$. The stationary point occurs where

$$\frac{\partial f}{\partial x} = 2x - 2 = 0, \quad \frac{\partial f}{\partial y} = 2y + 2 = 0,$$

that is, at $(1, -1)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Hence $\Delta(1, -1) = 4 > 0$ and $A(1, -1) = 2 > 0$, so that $(1, -2)$ is a minimum.

(c) $f(x,y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 - x + y + 3$. Stationary points occur where

$$\frac{\partial f}{\partial x} = x^2 - 1 = 0, \quad \frac{\partial f}{\partial y} = -y^2 + 1 = 0,$$

that is, where $(x-1)(x+1) = 0$ and $(y-1)(y+1) = 0$. There are four stationary points at $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = -2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

$(1,1)$. At this point $\Delta(1,1) = -4 < 0$; hence $(1,1)$ is a saddle.
$(1,-1)$. At this point $\Delta(1,-1) = 4 > 0$ and $A(1,-1) = 2 > 0$; hence $(1,-1)$ is a minimum.
$(-1,1)$. At this point $\Delta(-1,1) = 4 > 0$ and $A(-1,1) = -2 < 0$; hence $(-1,1)$ is a maximum.
$(-1,-1)$. At this point $\Delta(-1,-1) = -4 < 0$; hence $(-1,-1)$ is a saddle.

(d) $f(x,y) = \cos x + \cos y$. Stationary points occur where

$$\frac{\partial f}{\partial x} = -\sin x = 0, \quad \frac{\partial f}{\partial y} = -\sin y = 0,$$

The solutions of $\sin x = 0$ are $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, and the solutions of $\sin y = 0$ are $y = m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$. Therefore stationary points occur at $(n\pi, m\pi)$ for all the values of n and m given. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = -\cos x, \quad \frac{\partial^2 f}{\partial y^2} = -\cos y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Hence $\Delta(n\pi, m\pi) = \cos n\pi \cos m\pi = (-1)^n(-1)^m = (-1)^{n+m}$ and $A(n\pi, m\pi) = -\cos n\pi$. If $n+m$ is an odd integer, then the stationary point is a saddle. If $n+m$ is an even number then the stationary point is a maximum if, additionally, n is even (which is equivalent to m even), and a minimum if n is odd (which is equivalent to m odd).

(e) $f(x,y) = \ln(x^2 + x) + \ln(y^2 + y)$ (assume that $x > 0$ and $y > 0$, or that $x < -1$ and $y < -1$: otherwise the logarithms are not real). Stationary points occur where

$$\frac{\partial f}{\partial x} = \frac{2x+1}{x^2+x} = 0, \quad \frac{\partial f}{\partial y} = \frac{2y+1}{y^2+y} = 0.$$

The solution is $x = -\frac{1}{2}$, $y = -\frac{1}{2}$, but $f(x,y)$ is not real at this point. Hence $f(x,y)$ has no stationary points.

(f) $f(x,y) = e^{x^2+y^2-2x+2y}$. Stationary points occur where

$$\frac{\partial f}{\partial x} = e^{x^2+y^2-2x+2y}(2x-2) = 0, \quad \frac{\partial f}{\partial y} = e^{x^2+y^2-2x+2y}(2y+2) = 0,$$

that is, at $(1,-1)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = -2e^{x^2+y^2-2x+2y}(2x^2 - 4x + 3), \quad \frac{\partial^2 f}{\partial y^2} = 2e^{x^2+y^2-2x+2y}(2y^2 + 4y + 3),$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4e^{x^2+y^2-2x+2y}(x-1)(y+1).$$

Hence $\Delta(1,-1) = 4e^{-4} - 0 = 4e^{-4} > 0$ and $2e^{-2} > 0$. Therefore $(1,-1)$ is a minimum.

(g) $f(x,y) = xy + (1/x) + (1/y)$. Stationary points occur where

$$\frac{\partial f}{\partial x} = -\frac{1}{x^2} + y = 0, \quad \frac{\partial f}{\partial y} = x - \frac{1}{y^2} = 0,$$

Eliminate y between these equations giving $x = x^4$, which has the solutions $x = 0$ and $x = 1$. We discard $x = 0$ since $f(x, y)$ is infinite there. The only stationary point is at $(1, 1)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3}, \quad \frac{\partial^2 f}{\partial x \partial y} = 1.$$

Hence $\Delta(1, 1) = 4 - 1 = 3 > 0$ and $A(1, 1) = 2 > 0$. Therefore $(1, 1)$ is a minimum.

(h) $f(x, y) = x^3 + y^3 - 3xy = 1$. Stationary points occur where

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0, \quad \frac{\partial f}{\partial y} = 3y^2 - 3x = 0,$$

Eliminate y between these equations so that $x^4 = x$, which has the solutions $x = 0$ and $x = 1$. There are two stationary points at $(0, 0)$ and $(1, 1)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = -3.$$

$(0, 0)$. $\Delta(0, 0) = -(-3)^2 = -9 < 0$. Therefore $(0, 0)$ is a saddle.
$(1, 1)$. $\Delta(1, 1) = 6 \times 6 - (-3)^2 = 27 > 0$ and $A(1, 1) = 6 > 0$. Therefore $(1, 1)$ is a minimum.

(i) $f(x, y) = \sin x + \sin y$. Stationary points occur where

$$\frac{\partial f}{\partial x} = \cos x = 0, \quad \frac{\partial f}{\partial y} = \cos y = 0,$$

The solutions of $\cos x = 0$ are $x = \frac{1}{2}(2n + 1)\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$, and of $\cos y = 0$ are $y = \frac{1}{2}(2m + 1)\pi$, $(m = 0, \pm 1, \pm 2, \ldots)$. Hence $f(x, y)$ is stationary at $(\frac{1}{2}(2n + 1)\pi, \frac{1}{2}(2m + 1)\pi)$ for the values of n and m stated. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = -\sin x, \quad \frac{\partial^2 f}{\partial y^2} = -\sin y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Then

$$\begin{aligned} \Delta(\tfrac{1}{2}(2n + 1)\pi, \tfrac{1}{2}(2m + 1)\pi) &= \sin[\tfrac{1}{2}(2n + 1)\pi] \sin[\tfrac{1}{2}(2m + 1)\pi] \\ &= (-1)^n (-1)^m = (-1)^{n+m}, \end{aligned}$$

and

$$A(\tfrac{1}{2}(2n + 1)\pi, \tfrac{1}{2}(2m + 1)\pi) = (-1)^{n+1}$$

Therefore the stationary point is a saddle if $n + m$ is odd, a maximum if $n + m$ is even and n is even, and a minimum if $n + m$ is even and n is odd.

(j) $f(x, y) = xy^2 - x^2 y + x - y + 1$. Stationary points occur where

$$\frac{\partial f}{\partial x} = y^2 - 2xy + 1 = 0, \quad \frac{\partial f}{\partial y} = 2xy - x^2 - 1 = 0,$$

Adding the two equations we obtain $y^2 = x^2$, which implies $y = \pm x$. Substituting back into the first equation for $y = x$:

$$x^2 - 2x^2 + 1 = 0, \text{ or } x^2 = 1.$$

This leads to two stationary points $(1, 1)$ and $(-1, -1)$. Substituting back for $y = -x$ leads to

$$x^2 + 2x^2 + 1 = 0, \text{ or } 3x^2 + 1 = 0,$$

which has no real solutions.

The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = -2y, \quad \frac{\partial^2 f}{\partial y^2} = 2x, \quad \frac{\partial^2 f}{\partial x \partial y} = -2x + 2y.$$

$(1,1)$. $\Delta(1,1) = -4 - 0 = -4 < 0$. Hence $(1,1)$ is a saddle.
$(-1,-1)$. $\Delta(-1,-1) = -4 < 0$. Hence $(-1,-1)$ is a saddle also.
(k) $f(x,y) = x^2 - y^2 + 2xy$. The stationary points occur where

$$\frac{\partial f}{\partial x} = 2x + 2y = 0, \quad \frac{\partial f}{\partial y} = 2x - 2y = 0,$$

The only stationary point is at $(0,0)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 2.$$

Hence $\Delta(0,0) = -4 - 2^2 = -8 < 0$. Therefore $(0,0)$ is a saddle.
(l) $f(x,y) = (2 - x^2 - y^2)^2$. Stationary points occur where

$$\frac{\partial f}{\partial x} = -4x(2 - x^2 - y^2) = 0, \quad \frac{\partial f}{\partial y} = -4y(2 - x^2 - y^2) = 0,$$

This example is different in that *all* points on the circle $x^2+y^2 = 2$ have zero first partial derivatives, but they cannot be classified using the second derivatives test since $\Delta = 0$ on the circle. There is also an isolated stationary point at $(0,0)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 8x^2 - 4(2 - x^2 - y^2), \quad \frac{\partial^2 f}{\partial y^2} = 8y^2 - 4(2 - x^2 - y^2), \quad \frac{\partial^2 f}{\partial x \partial y} = 8xy.$$

Hence $\Delta(0,0) = 64 - 0 = 64 > 0$ and $A(x,y) = -8 < 0$ which means that $(0,0)$ is a maximum.
(m) $f(x,y) = x^4 + y^4 + y - x$. Stationary points occur where

$$\frac{\partial f}{\partial x} = 4x^3 - 1 = 0, \quad \frac{\partial f}{\partial y} = 4y^3 + 1 = 0,$$

at $(1/4^{\frac{1}{3}}, -1/4^{\frac{1}{3}})$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Hence $\Delta(1/4^{\frac{1}{3}}, -1/4^{\frac{1}{3}}) = 36 > 0$ and $A(1/4^{\frac{1}{3}}, -1/4^{\frac{1}{3}}) = 6 > 0$. Therefore the stationary point is a minimum.
(n) $f(x,y) = x^4 + y^4$. Stationary points occur where

$$\frac{\partial f}{\partial x} = 4x^3 = 0, \quad \frac{\partial f}{\partial y} = 4y^3 = 0,$$

Hence there is one stationary point at $(0,0)$. The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Obviously $\Delta(0,0) = 0$, which means that the second derivatives test fails. However, $f(0,0) = 0$ and $f(x,y) > 0$ for all $(x,y) \neq (0,0)$. Therefore the origin is a minimum.

28.13. Let $f(x,y) = ax^2 + 2hxy + by^2$. Stationary points occur where

$$\frac{\partial f}{\partial x} = 2ax + 2hy = 0, \quad \frac{\partial f}{\partial y} = 2hx + 2by = 0,$$

The linear equations equations $ax + hy = 0$, $hx + by = 0$ only have the solution $x = 0$, $y = 0$, unless $ab - h^2 = 0$, in which case $x = h\lambda$, $y = -a\lambda$ where λ is an arbitrary constant (we can assume that $a \neq 0$ and $b \neq 0$).

The second derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 2a, \quad \frac{\partial^2 f}{\partial y^2} = 2b, \quad \frac{\partial^2 f}{\partial x \partial y} = 2h.$$

• *Case* $ab - h^2 \neq 0$. $\Delta(0,0) = 4ab - 4h^2 = 4(ab - h^2) \neq 0$ and $A(0,0) = 2a$. If $ab - h^2 < 0$, then the origin is a saddle. If $ab - h^2 > 0$ and $a > 0$, then the origin is a minimum, whilst if $ab - h^2 > 0$ and $a < 0$, the origin is a maximum.
• *Case* $ab - h^2 = 0$. As we have seen, the stationary points occur at $x = h\lambda$, $y = -a\lambda$, which are parametric equations of a straight line through the origin. At all points along the line $f(x,y) = 0$.

28.14. Δ and A are defined in Problem 28.12.
(a) Put $c = 21 - a - b$ into abc, and let $f(a,b) = ab(21 - a - b)$. Then

$$\frac{\partial f}{\partial a} = -b(2a + b - 21) = 0, \quad \frac{\partial f}{\partial b} = -a(a + 2b - 21) = 0,$$

where $a = 0$, $b = 0$; $a = 0$, $b = 21$; $a = 21$, $b = 0$; and $a = 7$, $b = 7$. Therefore the stationary points occur at $(0,0)$, $(0,21)$, $(21,0)$ and $(7,7)$. The second derivatives are

$$\frac{\partial^2 f}{\partial a^2} = -2b, \quad \frac{\partial^2 f}{\partial b^2} = -2a, \quad \frac{\partial^2 f}{\partial a \partial b} = 21 - 2a - 2b.$$

•$(0,0)$. $\Delta(0,0) = -21^2 = -441 < 0$ which means that $(0,0)$ is a saddle.
•$(0,21)$. $\Delta(0,21) = -21^2 = -441 < 0$ which means that $(0,21)$ is a saddle.
•$(21,0)$. $\Delta(21,0) = -21^2 = -441 < 0$ which means that $(21,0)$ is also a saddle.
• $(7,7)$. $\Delta(7,7) = 14 \times 14 - 7^2 = 147 > 0$ and $A(7,7) = -14 < 0$. Hence $(7,7)$ is maximum.
Therefore abc is a maximum at $a = 7$, $b = 7$, $c = 7$.

(b) Put $c = 64/(ab)$ into $a + b + c$, and let $f(a,b) = a + b + 64/(ab)$. Stationary points occur where

$$\frac{\partial f}{\partial a} = 1 - \frac{64}{a^2 b} = 0, \quad \frac{\partial f}{\partial b} = 1 - \frac{64}{ab^2} = 0,$$

The difference of these equations leads to

$$64\left(\frac{1}{ab^2} - \frac{1}{a^2 b}\right) = \frac{64}{a^2 b^2}(a - b) = 0$$

Hence $a = b$ is the only solution. Substitute back to obtain $a = b = 4$. The second derivatives are

$$\frac{\partial^2 f}{\partial a^2} = \frac{128}{a^3 b}, \quad \frac{\partial^2 f}{\partial b^2} = \frac{128}{ab^3}, \quad \frac{\partial^2 f}{\partial a \partial b} = \frac{64}{a^2 b^2}.$$

Hence

$$\Delta(4,4) = \frac{128}{4^3 4}\frac{128}{4.4^3} - \frac{64^2}{4^4 4^4} = \frac{3 \times 64^2}{4^8} > 0,$$

and $A(4,4) = 128/4^4 = 1/512 > 0$ which means that $(4,4)$ is a minimum. Hence $a + b + c$ is a minimum where $a = 4$, $b = 4$, $c = 4$.

28.15. Let $f(x,y) = (2 - x^2 - y^2)^2$. The function is stationary where (see also Problem 28.12(l))

$$\frac{\partial f}{\partial x} = -4x(2 - x^2 - y^2), \quad \frac{\partial f}{\partial y} = -4y(2 - x^2 - y^2).$$

The function is stationary at all points on the circle $x^2 + y^2 = 2$, but these cannot be classified by using (28.9): on this circle $f(x,y) = 0$. The function is also stationary at $(0,0)$, where it is a (local) maximum as shown in Problem 28.12(l). Its value there is $f(0,0) = 4$. However, it is possible that the function has a greater value at some point on the boundary of the rectangle $-1 \le x \le 2$, $-1 \le y \le 1$ where $f(x,y)$ is not stationary. We can only check by calculating $f(x,y)$ on each edge of the rectangle. Thus

- $f(-1, y) = (1 - y^2)^2$. For $-1 \leq y \leq 1$, its maximum value is 1 at $y = 0$.
- $f(x, 1) = (1 - x^2)^2$. For $-1 \leq x \leq 2$, its maximum value is 9 at $x = 2$.
- $f(2, y) = (-2 - y^2)^2$. For $-1 \leq y \leq 1$, its maximum value is 9 at $y = \pm 1$.
- $f(x, -1) = (1 - x^2)^2$. For $-1 \leq x \leq 1$, its maximum value is 1 at $x = 0$.

Therefore the maximum value of $f(x, y) = (2 - x^2 - y^2)^2$ is 9, which occurs at the points $(2, 1)$ and $(2, -1)$.

28.16. The lines can be represented parametrically by

$$x = y = z = \lambda,$$

and

$$2x = y = z + 2 = \mu, \text{ or } x = \tfrac{1}{2}\mu, \quad y = \mu, \quad z = \mu + 2.$$

For any value of λ and μ these represent points on the lines. The distance $p(\lambda, \mu)$ between these general points is given by

$$p(\lambda, \mu) = [(\lambda - \tfrac{1}{2}\mu)^2 + (\lambda - \mu)^2 + (\lambda - \mu - 2)^2]^{\frac{1}{2}}.$$

At the stationary points of $p(\lambda, \mu)$

$$\frac{\partial p}{\partial \lambda} = \frac{1}{2p(\lambda, \mu)}[2(\lambda - \frac{1}{2}\mu) + 2(\lambda - \mu) + 2(\lambda - \mu - 2)] = 0,$$

or

$$6\lambda - 5\mu - 4 = 0 \qquad \text{(i)}$$

and

$$\frac{\partial p}{\partial \mu} = \frac{1}{2p(\lambda, \mu)}[-(\lambda - \frac{1}{2}\mu) - 2(\lambda - \mu) - 2(\lambda - \mu - 2)] = 0,$$

or

$$-10\lambda + 9\mu + 8 = 0. \qquad \text{(ii)}$$

The solution of the linear equations (i) and (ii) is $\lambda = -1$, and $\mu = -2$.

The second derivative test, which is lengthy in this case, is summarized. The second partial derivatives are

$$\frac{\partial^2 p}{\partial \lambda^2} = \frac{1}{2p(\lambda, \mu)^3}(\mu^2 + 4\mu + 16), \quad \frac{\partial^2 p}{\partial \mu^2} = \frac{1}{2p(\lambda, \mu)^3}(\lambda^2 + 2\lambda + 10),$$

$$\frac{\partial^2 p}{\partial \lambda, \mu} = \frac{1}{2p(\lambda, \mu)^3}(\lambda\mu + 2\lambda + \mu + 12).$$

Then, using the notation of Problem 28.12,

$$\Delta(-1, -2) = \frac{3}{\sqrt{2}} \cdot \frac{9}{4\sqrt{2}} - \left(-\frac{5}{2\sqrt{2}}\right)^2 = \frac{1}{4} > 0,$$

and $A(-1, -2) = 3/\sqrt{2} > 0$. Hence a minimum occurs at $\lambda = -1$, $\mu = -2$. The shortest distance joins the points $(-1, -1, -1)$ on the first line and $(-1, -2, 0)$ on the second line. This line is perpendicular to both lines.

28.17. Let $p(x, y)$ sum of the squares of the distances of $P : (x, y)$ from the N points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$. Then

$$p(x, y) = \sum_{r=1}^{N} [(x - x_r)^2 + (y - y_r)^2].$$

The function is stationary where

$$\frac{\partial p(x,y)}{\partial x} = 2\sum_{r=1}^{N}(x - x_r) = 2Nx - 2\sum_{r=1}^{N} x_r = 0,$$

and where

$$\frac{\partial p(x,y)}{\partial y} = 2\sum_{r=1}^{N}(y - y_r) = 2Ny - 2\sum_{r=1}^{N} y_r = 0.$$

Hence the stationary point is at

$$x = \frac{1}{N}\sum_{r=1}^{N} x_r, \quad y = \frac{1}{N}\sum_{r=1}^{N} y_r.$$

The second derivatives are

$$\frac{\partial^2 p(x,y)}{\partial x^2} = 2N, \quad \frac{\partial^2 p(x,y)}{\partial y^2} = 2N, \quad \frac{\partial^2 p(x,y)}{\partial x \partial y} = 0.$$

Hence, in the notation of Problem 28.12,

$$\Delta\left(\sum_{r=1}^{N} x_r/N, \sum_{r=1}^{N} y_r/N\right) = 4N^2 > 0 \text{ and } A\left(\sum_{r=1}^{N} x_r/N, \sum_{r=1}^{N} y_r/N\right) = 2N > 0.$$

Therefore the point $(\sum_{r=1}^{N} x_r/N, \sum_{r=1}^{N} y_r/N)$ minimizes the sum of the squares.

28.18. (a) Let the edges of the box be of lengths x, y, z. Then the surface area α is given by

$$\alpha = 2yz + 2zx + 2xy,$$

whilst the volume $V = xyz$ and V is fixed. Hence $z = V/(xy)$, and, eliminating z,

$$\alpha = 2(y + x)\frac{V}{xy} + 2xy.$$

This area is stationary where

$$\frac{\partial \alpha}{\partial x} = 2y - \frac{2V}{x^2} = 0, \quad \frac{\partial \alpha}{\partial y} = 2x - \frac{2V}{y^2} = 0.$$

From these equations, $V = x^2 y$ and $V = xy^2$. Hence $x = y = V^{\frac{1}{3}}$, and also $z = V/(xy) = V^{\frac{1}{3}}$: all the edges have the same lengths, which means that the box is a cube. The second derivatives are

$$\frac{\partial^2 \alpha}{\partial x^2} = \frac{4V}{x^3}, \quad \frac{\partial^2 \alpha}{\partial y^2} = \frac{4V}{y^3}, \quad \frac{\partial^2 \alpha}{\partial x \partial y} = 2.$$

Therefore

$$\Delta(V^{\frac{1}{3}}, V^{\frac{1}{3}}) = 4 \times 4 - 2^2 = 12 > 0, \text{ and } A(V^{\frac{1}{3}}, V^{\frac{1}{3}}) = 4 > 0,$$

which means that the area is a minimum.

(b) Let x and y be the lengths of the edges of the base of the box, and z its height. Then the surface area α is given by

$$\alpha = 2yz + 2zx + xy,$$

whilst the volume is $V = xyz$. Eliminating z:

$$\alpha = 2(y + x)\frac{V}{xy} + xy.$$

This area is stationary where

$$\frac{\partial \alpha}{\partial x} = y - \frac{2V}{x^2} = 0, \quad \frac{\partial \alpha}{\partial y} = x - \frac{2V}{y^2} = 0.$$

From these two equations it follows that $x = y = (2V)^{\frac{1}{3}}$, and that $z = (V/4)^{\frac{1}{3}}$. The base is square and the height of the box is half the edge-length of the base. The second derivatives are

$$\frac{\partial^2 \alpha}{\partial x^2} = \frac{4V}{x^3}, \quad \frac{\partial^2 \alpha}{\partial x^2} = \frac{4V}{y^3}, \quad \frac{\partial^2 \alpha}{\partial x \partial y} = 1.$$

Therefore

$$\Delta(V^{\frac{1}{3}}, V^{\frac{1}{3}}) = 2 \times 2 - 1 = 3 > 0, \text{ and } A(V^{\frac{1}{3}}, V^{\frac{1}{3}}) = 2 > 0,$$

which means that the area is a minimum.

(c) Let r be the radius of the cylinder and h its height. The volume V of the cylinder is given by $V = \pi r^2 h$.

(i) With a lid, its surface area is $\alpha = 2\pi r^2 + 2\pi r h$. Eliminating h:

$$\alpha = 2\pi r^2 + \frac{2V}{r}.$$

This is a function of a single variable r so that ordinary differentiation will do. Hence

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = 4\pi r - \frac{2V}{r^2} = 0$$

where $r = [V/(2\pi)]^{\frac{1}{3}}$. The corresponding height is $h = (4V/\pi)^{\frac{1}{3}}$. The second derivative is

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2} = 4\pi + \frac{4V}{r^3} = 12\pi > 0$$

for $r = [(V/(2\pi)]^{\frac{1}{3}}$. Hence the surface area is a minimum.

(ii) Without a lid, its surface area is $\alpha = \pi r^2 + 2\pi r h$. Eliminating h:

$$\alpha = \pi r^2 + \frac{V}{r}.$$

Its derivative is

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = 2\pi r - \frac{2V}{r^2} = 0$$

where $r = (V/\pi)^{\frac{1}{3}}$. The corresponding height is $h = (V/\pi)^{\frac{1}{3}}$ which equals the radius of the cylinder. The area is a minimum since

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2} = 2\pi + \frac{4V}{r^3} = 6\pi > 0$$

for $r = (V/\pi)^{\frac{1}{3}}$.

(d) Let x, y, z be the lengths of the edges. In this case α is fixed and $V = xyz$.

(i) As in (a), $\alpha = 2yz + 2zx + 2xy$. Therefore

$$z = \frac{\alpha - 2xy}{2(y + x)}. \qquad \text{(i)}$$

Eliminate z in the formula for V, so that

$$V = \frac{xy(\alpha - 2xy)}{2(y + x)}.$$

The volume V will be stationary where

$$\frac{\partial V}{\partial x} = \frac{y^2(\alpha - 2x^2 - 4xy)}{2(x + y)^2} = 0, \quad \frac{\partial V}{\partial y} = \frac{x^2(\alpha - 2y^2 - 4xy)}{2(x + y)^2} = 0.$$

Since neither x nor y will be zero for a maximum volume, we conclude that

$$\alpha - 2x^2 - 4xy = 0, \text{ and } \alpha - 2y^2 - 4xy = 0,$$

which means that $x = y$ restricted to positive solutions. Hence $x = y = \surd(\alpha/6)$, and from (i), $z = \surd(\alpha/6)$ also. The rectangular container of maximum volume is a cube.

The second derivatives are

$$\frac{\partial^2 V}{\partial x^2} = -\frac{y^2(\alpha + 2y^2)}{(x+y)^3}, \quad \frac{\partial^2 V}{\partial y^2} = -\frac{x^2(a + 2x^2)}{(x+y)^3},$$

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{xy(\alpha - 2x^2 - 6xy - 2y^2)}{(x+y)^3}.$$

Hence

$$\Delta[\surd(\alpha/6), \surd(\alpha/6)] = \frac{a}{8} > 0, \quad A[\surd(\alpha/6), \surd(\alpha/6)] = -\frac{\alpha}{6} < 0,$$

showing that the cube is a maximum.

(ii) As in (b), the area $\alpha = 2yz + 2zx + xy$ where the base edges are x and y. Then

$$z = \frac{\alpha - xy}{2(y+x)},$$

and

$$V = \frac{xy(\alpha - xy)}{2(y+x)}.$$

The volume will be stationary where

$$\frac{\partial V}{\partial x} = \frac{y^2(\alpha - x^2 - 2xy)}{2(x+y)^2} = 0, \quad \frac{\partial V}{\partial y} = \frac{x^2(\alpha - y^2 - 2xy)}{2(x+y)^2} = 0.$$

The required non-zero solutions are $x = y = \surd(\alpha/3)$. It follows that $z = \frac{1}{2}\surd(\alpha/3)$.

The second derivatives are

$$\frac{\partial^2 V}{\partial x^2} = -\frac{y^2(\alpha + y^2)}{(x+y)^3)}, \quad \frac{\partial^2 V}{\partial y^2} = -\frac{x^2(\alpha + x^2)}{(x+y)^3},$$

$$\frac{\partial^2 V}{\partial x \partial y} = -\frac{xy(-\alpha + x^2 + 3xy + y^2)}{(x+y)^3}.$$

Hence

$$\Delta(\surd(\alpha/3), \surd(\alpha/3)) = \frac{\alpha}{16}, \text{ and } A(\surd(\alpha/3), \surd(\alpha/3)) = -\frac{1}{2}\surd\left(\frac{\alpha}{3}\right).$$

Hence the box with dimensions

$$x = y = \surd(\alpha/3), \quad z = \frac{1}{2}\surd\left(\frac{\alpha}{3}\right)$$

has the maximum volume.

28.19. The table of data is

x	1	2	3	4	5
y	3.1	2.1	2.0	1.8	1.2

By (28.10), the least squares straight line fit to the points (x_n, y_n), $(n = 1, 2, \ldots, N)$ is $y = ax + b$, where a and b satisfy the linear equations

$$a\sum_{n=1}^{N} x_n^2 + b\sum_{n=1}^{N} x_n = \sum_{n=1}^{N} x_n y_n,$$

$$a\sum_{n=1}^{N} x_n + bN = \sum_{n=1}^{N} y_n.$$

In this problem $N = 5$, and

$$\sum_{n=1}^{5} x_n^2 = 55, \quad \sum_{n=1}^{5} x_n = 15, \quad \sum_{n=1}^{5} x_n y_n = 26.5, \quad \sum_{n=1}^{5} y_n = 10.2.$$

Therefore a and b satisfy

$$55a + 15b = 26.5, \quad 15a + 5b = 10.2,$$

which have the solution $a = -0.41$, $b = 3.27$. The required straight line is

$$y = -0.41x + 3.27.$$

28.20. The table of data is (to 2 decimal places in the third row)

t	0	2	3	5	8	10	12
P	12	23	26	60	170	300	690
$y = \ln P$	2.48	3.13	3.26	4.09	5.14	5.70	6.54

We assume that the growth takes the form $P = Ae^{bt}$. Take the logarithm of the relation, so that

$$\ln P = \ln A + bt = a + bt,$$

where $\ln A = a$. We estimate a least squares fit using data in the first and third rows in the table. Thus $N = 7$, and

$$\sum_{n=1}^{7} t_n^2 = 346, \quad \sum_{n=1}^{7} t_n = 40, \quad \sum_{n=1}^{7} t_n y_n = 216.35, \quad \sum_{n=1}^{7} y_n = 30.34.$$

Therefore a and b satisfy

$$346a + 40b = 216.35, \quad 40a + 7b = 30.34,$$

which have the solution $a = 0.37$, $b = 2.24$ to 2 decimal places. Since $A = e^a = 1.45$, the exponential fit to the data is $P = 1.45e^{2.24t}$.

28.21. By (28.10), the stationary values of the function $f(a, b)$, where

$$f(a, b) = \sum_{n=1}^{N} e_n^2 = \sum_{n=1}^{N} (y_n - ax_n - b)^2 \qquad \text{(i)}$$

(which is the sum of the squares of the errors in fitting the given points (x_n, y_n) by the straight line $y = ax + b$) are obtained by solving for a, b in the equations

$$a\sum_{n=1}^{N} x_n^2 + b\sum_{n=1}^{N} x_n = \sum_{n=1}^{N} x_n y_n, \qquad \text{(ii)}$$

,

$$a\sum_{n=1}^{N} x_n + bN = \sum_{n=1}^{N} y_n. \qquad \text{(iii)}$$

We shall show that $f(a, b)$ is a global minimum by showing that if we take any different pair of constants then the sum of the squares of errors is increased. Suppose that the new values are given by $a+\alpha$ and $b+\beta$, where α or β is different from zero. Then the new $\sum_{n=1}^{N} e_n^2$ is given by

$$\begin{aligned}
\sum_{n=1}^{N} e_n^2 &= \sum_{n=1}^{N}[y_n - (a+\alpha)x_n - (b+\beta)]^2 \\
&= \sum_{n=1}^{N}[(y_n - ax_n - b) - (\alpha x_n + \beta)]^2 \\
&= \sum_{n=1}^{N}(y_n - ax_n - b)^2 - 2\sum_{n=1}^{N}(\alpha x_n + \beta)(y_n - ax_n - b) \\
&\quad + \sum_{n=1}^{N}(\alpha x_n + \beta)^2 \\
&= f(a,b) - 2\alpha\sum_{n=1}^{N}(x_n y_n - ax_n^2 - bx_n) \\
&\quad + 2\beta\sum_{n=1}^{N}(y_n - ax_n - b) + \sum_{n=1}^{N}(\alpha x_n + \beta)^2.
\end{aligned}$$

The first term represents the value of f at the stationary point. The second is zero by eqn (ii). The third is zero from eqn (iii). The final term is the sum of squares, and is therefore always positive. Finally we have

$$\sum_{n=1}^{N} e_n^2 = f(a,b) + \sum_{n=1}^{N}(\alpha x_n + \beta_)^2 => f(a,b),$$

for all α and β, which proves that $f(a,b)$ is the smallest value attainable by f.

28.22. Let $\mathcal{L}_t\{z(x,t)\} = Z(x,s)$. The Laplace transform with respect to t of the partial differential equation

$$\frac{\partial z}{\partial t} + x\frac{\partial z}{\partial x} + z = 2x$$

is, using the derivative rule (24.12),

$$sZ(x,s) - sz(x,0) + x\frac{\partial Z(x,s)}{\partial x} + Z(x,s) = x\mathcal{L}\{1\} = \frac{2x}{s},$$

or

$$\frac{\partial Z(x,s)}{\partial x} + \frac{(1+s)}{x}Z(x,s) = \frac{1}{s}.$$

The Laplace transform of the boundary condition becomes $Z(0,s)=0$. The differential equation is a first-order equation with variable x of integrating-factor type (see Section 19.5). The integrating factor is

$$\exp\left[\int \frac{(1+s)\mathrm{d}x}{x}\right] = \exp[(1+s)\ln x] = x^{s+1}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{s+1}Z(x,s)) = \frac{2}{s}x^{s+1}.$$

Integrating

$$x^{s+1}Z(x,s) = \frac{2x^{s+2}}{s(s+2)} + C(s), \text{ or } Z(x,s) = \frac{2x}{s(s+2)} + \frac{C(s)}{x^{s+1}}.$$

Since $Z((0,s)=0$, we conclude that $C(s)=0$, so that

$$Z(x,s) = \frac{2x}{s(s+2)} = 2x\left(\frac{1}{2s} - \frac{1}{2(s+2)}\right).$$

Inverting the Laplace transform, using (24.6),

$$z(x,t) = x - x\mathrm{e}^{-2t},$$

which is the solution.

28.23. The height z of the grain in the silo is given by

$$z = f(x,y) = \{2a^2 - (x - \tfrac{1}{2}a)^2 - y^2\}/a = (\tfrac{7}{4}a^2 - x^2 + ax - y^2)/a$$

on the square region $OPQR$ defined by

$$0 \le x \le a, \quad 0 \le y \le a$$

(see Fig 10).

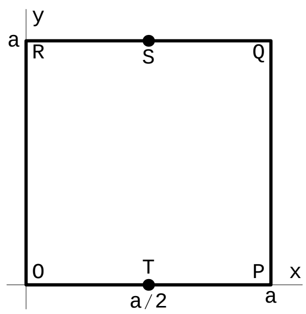

Figure 10: Problem 28.23

As in problem 23.15, the *overall* maximum or minimum might occur either:
(a) at points in the interior of the square where $\partial z/\partial x$ and $\partial z/\partial y$ are zero, and a local maximum or minimum occurs, or
(b) on certain points on the edges, excluding the corners, or
(c) at one or more corners.
In cases (b) and (c), $\partial z/\partial x$ and $\partial z/\partial y$ will not necessarily be zero, so we must examine the edges and corners separately from the interior o $OPQR$.

(a) *Points interior to OPQR*: $0 < x < a$, $0 < y < a$.
From (i),

$$\frac{\partial z}{\partial x} = \frac{-2x + a}{a}, \quad \frac{\partial z}{\partial y} = -\frac{2y}{a}.$$

These are both zero only at the point $T : (\frac{1}{2}a, 0)$. Since this point is on the edge OP we do not have to take note of it.

(b) *The edges excluding the corners.*
On OP,

$$z = f(x,0) = (\tfrac{7}{4}a^2 + ax - x^2)/a,$$

where $0 < x < a$. This is a function of the single variable x, and can be treated as in Section 15.4. There is a maximum, with value $z = 2a$, at $x = \frac{1}{2}a$: so $z = 2a$ at T in Fig 10.

On PQ, $z = f(a,y) = (\frac{7}{4}a^2 - y^2)/a$, $0 < y < a$.
There are no stationary points in this range.

On RQ, $z = f(a,y) = (\frac{3}{4}a^2 + ax - x^2)/a$, $0 < x < a$.
There is a maximum, with value $z = 2a$, at $x = \frac{1}{2}a$; so $z = 2a$ at S on Fig. 10.

On OR, $z = f(0,y) = (\frac{7}{4}a^2 - y^2)/a$, $0 < y < a$.
There are no stationary points in this range.

(c) *The corners* O, P, Q, R.
The values of z at these points are respectively $\frac{7}{4}a$, $\frac{7}{4}a$, $\frac{3}{4}a$, $\frac{3}{4}a$.

Finally, the overall maximum $z = 2a$ occurs on the edges at S and T, and the overall minimum value $z = \frac{3}{4}a$ at the corners Q and R.

28.24. By (28.3) the order in which the individual differentiations are carried out in evaluating an nth order partial derivative $\partial^n f/\partial x^r \partial y^{n-r}$ is immaterial. Therefore, If n is a fixed number, the number of *distinct* nth order derivatives which may be formed is equal to the order r of the x-derivative involved. We may have $r = 0, 1, 2, \ldots, n$. Therefore the number of possible distinct derivatives is $n + 1$.

Suppose that we had to write out all the possible nth order derivatives in all possible ways. Then the first differentiation could be either with respect to x or y, the second with respect to x or y and so on. Combining these possibilities, there would be $2 \times 2 \times \cdots \times 2 = 2^n$ possible forms for the derivatives.

28.25. Note that the correct form of the equation is $g(x, y) = H\{f(x, y)\}$, where H is a single-valued function of a single variable. By (28.4),

$$\frac{\partial g}{\partial x} = H'\{f(x,y)\}\frac{\partial f}{\partial x}, \quad \frac{\partial g}{\partial y} = H'\{f(x,y)\}\frac{\partial f}{\partial y}.$$

Therefore

$$\frac{\partial g}{\partial x}\Big/\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}\Big/\frac{\partial f}{\partial y} = H'\{f(x,y)\},$$

so that

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 0$$

for all x, y.

Chapter 29: Functions of two variables: geometry and formulae

29.1. Let $z = f(x, y)$, and δx, δy be given increments in x and y. The corresponding increment in z, δz about a 'representative point' (x, y), is defined (exactly) by

$$\delta z = f(x + \delta x, y + \delta y) - f(x, y). \tag{i}$$

By (29.2), the 'incremental approximation' to δz is

$$\delta z \approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \tag{ii}$$

for sufficiently small δx and δy, where the derivatives are evaluated at a representative point. The 'percentage error' E in the estimate (ii) to δz is given by

$$E = 100\left(\frac{\text{approximation (ii)} - \text{exact value (i)}}{\text{exact value (i)}}\right).$$

Note: In physical applications the numerical values of the increments that appear in problems may seem to be too large to qualify as 'sufficiently small' (for example, in Problem 29.5 an increment of 50 occurs). However, a mere change of units could reduce this number to a small value without changing the percentage error. A discussion of this point in one dimension follows Example 4.15.
(a) $z = x^2 + y^2$ at $(3, 1)$, and $\delta x = 0.1$, $\delta y = 0.3$. The partial derivatives are

$$\frac{\partial z}{\partial x} = 2x = 6, \quad \frac{\partial z}{\partial y} = 2y = 2$$

at $(3, 1)$. Hence, the incremental approximation is

$$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y = (6 \times 0.1) + (2 \times 0.3) = 1.2.$$

The exact value is

$$\delta z = [(3.1)^2 + (1.3)^2] - [3^2 + 1^2] = 1.3.$$

The percentage error E in δz is given by

$$E = 100\left(\frac{1.2 - 1.3}{1.3}\right) = -7.7\%.$$

(b) $z = \sin xy$ at $(0.5, 1.2)$ and $\delta x = 0.1$, $\delta y = -0.05$. The partial derivatives are

$$\frac{\partial z}{\partial x} = y\cos xy = 1.2\cos(0.6) = 0.990, \quad \frac{\partial z}{\partial y} = x\cos xy = 0.5\cos(0.6) = 0.413$$

at $(0.5, 1.2)$ to 3 significant figures. The incremental approximation is

$$\delta z \approx (0.990 \times 0.1) - (0.413 \times 0.05) = 0.0783,$$

and the exact value of δz is

$$\delta z = \sin(0.6 \times 1.15) - \sin(0.5 \times 1.2) = 0.0719.$$

The percentage error in δz is

$$E = 100\left(\frac{0.0783 - 0.0719}{0.0719}\right) = 8.9\%.$$

(c) $z = \mathrm{e}^{x^2+3y^2}$ at $(1, 1)$ and $\delta x = 0.1$, $\delta y = 0.2$. The partial derivatives are

$$\frac{\partial z}{\partial x} = 2x\mathrm{e}^{x^2+3y^2} = 109.2, \quad \frac{\partial z}{\partial y} = 6y\mathrm{e}^{x^2+3y^2} = 327.6$$

at $(1, 1)$. Hence the incremental approximation is

$$\delta z \approx 109.2 \times 0.1 + 327.6 \times 0.2 = 76.44,$$

and the exact value of δz is

$$\delta z = e^{1.1^2+(3\times 1.2^2)} - \mathrm{e}^4 = 197.54.$$

The percentage error in δz is

$$E = 100\left(\frac{76.44 - 197.54}{197.54}\right) = -61.3\%.$$

In this case, δy is too large for the incremental formula to give a useful result.
(d) $z = 1/(x^2 + y^2)^{\frac{1}{2}}$ at $(2, 1)$ where $\delta x = -0.2$, $\delta y = 0.1$. The partial derivatives are

$$\frac{\partial z}{\partial x} = -\frac{x}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{2}{5\sqrt{5}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{1}{5\sqrt{5}}$$

at $(2, 1)$. Hence the incremental approximation is

$$\delta z \approx -\frac{2}{5\sqrt{5}} \times (-0.2) - \frac{1}{5\sqrt{5}} \times 0.1 = 0.0268328.,$$

and the exact value of δz is

$$\delta z = \frac{1}{(1.8^2 + 1.1^2)^{\frac{1}{2}}} - \frac{1}{(2^2 + 1^2)^{\frac{1}{2}}} = 0.0268319.$$

The percentage error in δz is

$$E = 100\left(\frac{0.0268328 - 0.0268319}{0.0268319}\right) = 0.0034\%.$$

29.2. Given $z = x^2 - y^2$ and the points $P : (1.0, 2.1)$ and $Q : (1.2, 2.0)$. The partial derivatives of z are

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = -2y.$$

(a) At P, $\partial z/\partial x = 2.0$ and $\partial z/\partial y = -4.2$. Between P and Q, $\delta x = 1.2 - 1.0 = -0.2$ and $\delta y = 2.0 - 2.1 = -0.1$. Hence, by (29.2), the change in z from P to Q is given approximately by

$$\delta z \approx \left(\frac{\partial z}{\partial x}\right)_P \delta x + \left(\frac{\partial z}{\partial y}\right)_P \delta y = (2.0 \times 0.2) + (-4.2) \times (-0.1) = 0.82.$$

(b) At Q, $\partial z/\partial x = 2.4$ and $\partial z/\partial y = -4$. Between Q and P, $\delta x = 1.0 - 1.2 = -0.2$ and $\delta y = 2.1 - 2.0 = 0.1$. Hence, by (29.2), the change in z from Q to P is approximated by

$$\delta z \approx \left(\frac{\partial z}{\partial x}\right)_Q \delta x + \left(\frac{\partial z}{\partial y}\right)_Q \delta y = [2.4 \times (-0.2)] + [(-4) \times 0.1] = -0.88.$$

(c) Although the δx and δy increments from P to Q are minus the increments from Q to P, the partial derivatives at P and Q differ slightly, leading to small discrepancies in the estimates of $|\delta z|$.

29.3. The error $E(\delta x, \delta y)$ in the approximation to δf at the point (x_0, y_0) is given by

$$E(\delta x, \delta y) = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \delta y - f(x_0 + \delta x, y_0 + \delta y).$$

(a) $f(x, y) = xy$ near $(2, 1)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x.$$

Hence the error in δf is

$$E(\delta x, \delta y) = 2 + (1 \times \delta x) + (2 \times \delta y) - (2 + \delta x)(1 + \delta y) = -\delta x \delta y.$$

(b) $f(x, y) = x/y$ near $(2, 1)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{y}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}.$$

Hence

$$E(\delta x, \delta y) = 2 + (1 \times \delta x) + (2 \times \delta y) - \frac{2 + \delta x}{1 + \delta y} = \frac{\delta y(\delta x + 2\delta y)}{1 + \delta y}.$$

29.4. (See Example 29.5.) The focal length f is given by

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \text{ so that } f = \frac{uv}{u + v}.$$

The partial derivatives are

$$\frac{\partial f}{\partial u} = \frac{v^2}{(u + v)^2}, \quad \frac{\partial f}{\partial v} = \frac{u^2}{(u + v)^2}.$$

The measured values of u and v are $u = 0.31(\pm 0.01)$ and $v = 0.56(\pm 0.03)$. Choose as the reference values the central values $u = 0.31$ and $v = 0.56$, from which we obtain

$$\frac{\partial f}{\partial u} = 0.127 \quad \frac{\partial f}{\partial v} = 0.414.$$

Therefore

$$\Delta f \approx 0.127\Delta u + 0.414\Delta v,$$

which takes its greatest magnitude 0.013 when $\Delta u = 0.01$ and $\Delta v = 0.03$. At $u = 0.31$, $v = 0.56$, $f = 0.20$. Hence, the greatest possible error in estimating f is $\Delta f = 0.20(\pm 0.01)$ to two decimal places. The percentage error is 5%.

29.5. If the errors in d and l are Δd and Δl respectively, then by (29.3), the resultant error $\Delta\eta$ in η is estimated by

$$\Delta\eta \approx \frac{\partial\eta}{\partial d}\Delta d + \frac{\partial\eta}{\partial p}\Delta p. \tag{i}$$

Taking reference values for the derivatives at the central points $d = 0.002$, $l = 0.1$, $p = 5000$ and $q = 1.66$,

$$\frac{\partial\eta}{\partial d} = \frac{\pi p d^3}{32ql} = 23.7, \text{ and } \frac{\partial\eta}{\partial p} = \frac{\pi d^4}{128ql} = 23.7 \times 10^{-7}.$$

From (i), and the error ranges specified, the error $\Delta\eta$ of the greatest magnitude occurs when $\Delta d = \pm 0.0001$ and $\Delta p = \pm 50$. In that case

$$\Delta\eta \approx \pm 0.00249,$$

leadiong to an error of about 21%. (For a comment on the apparently large size of the increment Δp, see the note at the opening of Problem 29.1.)

29.6. (See also Example 29.4.) The solution $x(b, c)$ and its derivatives are given by

$$x = \frac{1}{2}[-b + (b^2 - 4c)^{\frac{1}{2}}]; \tag{i}$$

$$\frac{\partial x}{\partial b} = -\frac{1}{2} + \frac{b}{2(b^2 - 4c)^{\frac{1}{2}}}, \quad \frac{\partial x}{\partial c} = -\frac{1}{(b^2 - 4c)^{\frac{1}{2}}}, \tag{ii}$$

where the exact values of b and c are $b = 20.4$ and $c = 95.5$; and the rounded values are $b = 20$ and $c = 96$. Denote the rounding errors in b, c and x by Δb, Δc, Δx, where

$$\Delta b = -0.4, \quad \Delta c = 0.5.$$

The incremental approximation is

$$\delta x \approx \frac{\partial x}{\partial b}\delta b + \frac{\partial x}{\partial c}\delta x. \tag{iii}$$

We may choose the 'representative values' of b, c (at which the derivatives in (iii) are to be evaluated) to be the *rounded* values $b = 20$, $c = 96$. In this case the arithmetic is much simpler. Then the incremental formula requires δb, δc and δx to stand for '(exact value)-(rounded value)'. Therefore we should put

$$\delta b = 0.4 = -\Delta b, \quad \delta c = -0.5 = -\Delta c, \quad \delta x = \Delta x, \tag{iv}$$

along with

$$\frac{\partial x}{\partial b} = 2, \quad \frac{\partial x}{\partial c} = -0.25 \tag{v}$$

into the incremental approximation (iii). Then

$$\Delta x = -\delta x \approx -(2 \times 0.4) - [(-0.25) \times (-0.5)] = -0.925.$$

The rounding error Δx is about -0.9, and the percentage error (based on the rounded approximation to (i)) is about 11%.

29.7. The area of a triangle with base b and base angles A and C is given by

$$S = \frac{b^2 \tan A \tan C}{2(\tan A + \tan C)}.$$

The partial derivatives of S with respect to A and C are

$$\frac{\partial S}{\partial A} = \frac{b^2 \sec^2 A \tan^2 C}{2(\tan A + \tan C)^2}, \quad \frac{\partial S}{\partial C} = \frac{b^2 \sec^2 C \tan^2 A}{2(\tan A + \tan C)^2}.$$

The incremental approximation is

$$\delta S \approx \frac{\partial S}{\partial A}\delta A + \frac{\partial S}{\partial C}\delta C.$$

The nominal values are $b = 2$, $A = 30°$ and $C = 60°$, so that at these values $\partial S/\partial A = \frac{3}{2}$ and $\partial S/\partial C = \frac{1}{2}$. We now put $\delta C = 5 \times 60/100 = 3° = 0.0524$ radians. Thus

$$\delta S \approx \frac{3}{2}\delta A + \frac{1}{2}\delta C = \frac{3}{2}\delta A + \frac{1}{2}0.0524.$$

S will take the same value if $\delta S = 0$. Therefore

$$\delta A = -\frac{1}{3} \times 0.0524 = -0.0175 \text{ radians}.$$

Hence A must be reduced by 0.0175 radians or 1°.

29.8. We are given $S = ahr^3/p^2$, where a is fixed but h, r and p are measured and contain errors. Take the logarithms of S:

$$V = \ln S = \ln\left[\frac{ah^3}{p^2}\right] = \ln a + \ln h + 3\ln r - 2\ln p.$$

Using the incremental formula for a single variable (see 4.4) applied to the individual terms:

$$\delta V \approx \frac{\delta S}{S} \approx \frac{\delta h}{h} + 3\frac{\delta r}{r} - 2\frac{\delta p}{p}.$$

29.9. (a) The directional derivative at the point P in the direction θ on the surface $z = f(x, y)$ is given by

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \left(\frac{\partial f}{\partial x}\right)_P \cos\theta + \left(\frac{\partial f}{\partial y}\right)_P \sin\theta.$$

Along the contour through P, the derivative $\mathrm{d}z/\mathrm{d}s = 0$, so

$$\tan\theta = -\left(\frac{\partial f}{\partial x}\right)_P \Big/ \left(\frac{\partial f}{\partial y}\right)_P.$$

Following Example 29.9, the directions of steepest ascent and descent are perpendicular to the contour directions. These are the directions in which $\mathrm{d}(\mathrm{d}z/\mathrm{d}s)/\mathrm{d}\theta) = 0$. The steepest ascent occurs along the direction in which $\mathrm{d}z/\mathrm{d}x$ is a maximum, namely

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) < 0,$$

by eqn (4.2) (or it can be deduced from the contour heights).

(a) $z = f(x, y) = x^2 + y^2$ at $(1, 2)$, direction $\theta = 30°$. The partial derivatives of f are

$$\frac{\partial f}{\partial x} = 2x = 2, \quad \frac{\partial f}{\partial y} = 2y = 4, \text{ at } (1, 2).$$

Therefore the directional derivative is

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 2\cos 30^\circ + 4\sin 30^\circ = 2\frac{\sqrt{3}}{2} + 4\frac{1}{2} = \sqrt{3} + 2.$$

The directions of the contour at P are given by the solutions of

$$\tan\theta = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2}{4} = -\frac{1}{2},$$

which are the directions -26.6° and 153.4°.

The directions of steepest ascent and descent are perpendicular to these directions. They are respectively 63.4° and -116.6°.

(b) $f(x, y) = x^2y^2$ at $(2, 1)$, direction $\theta = -45^\circ$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 2xy^2 = 4, \quad \frac{\partial f}{\partial y} = 2x^2y = 8, \text{ at } (2, 1).$$

Therefore the directional derivative is

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 4\cos(-45^\circ) + 8\sin(-45^\circ) = 4\frac{1}{\sqrt{2}} - 8\frac{1}{\sqrt{2}} = -2\sqrt{2}.$$

The directions of the contours at P are given by the solutions of $\tan\theta = \frac{1}{2}$, which are the directions $\theta = 26.6^\circ$ and $\theta = -153.4^\circ$.

To find the direction θ of steepest ascents at $(2, 1)$,

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 4\cos\theta + 8\sin\theta,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) = -4\sin\theta + 8\cos\theta, \quad \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) = -4\cos\theta - 8\sin\theta.$$

The steepest paths occur where $\tan\theta = 2$, in directions $\theta = 63.43^\circ$ and $\theta = -116.57^\circ$. For $\theta = 63.43^\circ$,

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) = -4\frac{1}{\sqrt{5}} - 8\frac{2}{\sqrt{5}} = -4\sqrt{5} < 0.$$

Hence $\mathrm{d}z/\mathrm{d}s$ is a maximum which means that this direction is the steepest ascent.

(c) $f(x, y) = x^2y - xy^2 + 2$ at $(-1, 1)$, direction $\theta = 120^\circ$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 2xy - y^2 = -3, \quad \frac{\partial f}{\partial y} = x^2 - 2xy = 3, \text{ at } (-1, 1).$$

The directional derivative is

$$\frac{\mathrm{d}z}{\mathrm{d}s} = -3\cos 120^\circ + 3\sin 120^\circ = \frac{3}{2} + \frac{3\sqrt{3}}{2} = \frac{3}{2}(1 + \sqrt{3}).$$

The directions of the contours are $\theta = 45^\circ$ and $\theta = -135^\circ$. The steepest ascent is in the direction $\theta = 135^\circ$.

(d) $f(x, y) = \sin xy$ at $(\frac{1}{2}, \pi)$, direction $\theta = -90^\circ$. The partial derivatives are

$$\frac{\partial f}{\partial x} = y\cos xy = 0, \quad \frac{\partial f}{\partial y} = x\cos xy = 0, \text{ at } (\tfrac{1}{2}, \pi)$$

The directional derivative is

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 0,$$

independently of the direction θ, because $(\frac{1}{2}, \pi)$ is a stationary point of $f(x, y)$. The second derivative test (28.9) fails for this stationary value. Since $f(x, y)$ can never exceed 1, and $f(\frac{1}{2}, \pi) = 1$, the contour through $(\frac{1}{2}, \pi)$ is given by $xy = \frac{1}{2}\pi$, which is a rectangular hyperbola. There is no steepest ascent at $(\frac{1}{2}, \pi)$.

(e) $f(x, y) = \cos(x^2 - y)$ at $(0, -\pi)$, direction $\theta = 0$. The partial derivatives are

$$\frac{\partial f}{\partial x} = -2x\sin(x^2 - y) = 0, \quad \frac{\partial f}{\partial y} = \sin(x^2 - y) = 0, \text{ at } (0, -\pi)$$

The directional derivative is

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 0.$$

independently of the direction θ, because $(0, -\pi)$ is a stationary point of $f(x, y)$. As in (d) above, the second derivative test (28.9) fails. The contour through $(0, -\pi)$ is given by

$$\cos(x^2 - y) = \cos\pi = -1,$$

which is the parabola $y = x^2 - \pi$. Since

$$f(x, y) - f(0, -\pi) = \cos(x^2 - 1) + 1 \geq 0$$

there will be a steepest ascent in the directions $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$ perpendicular to the contour through $(0, -\pi)$: there are no steepest descents.

(f) $f(x, y) = \mathrm{e}^{x-y}$ at $(1, 1)$, direction $\theta = -45°$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \mathrm{e}^{x-y} = 1, \quad \frac{\partial f}{\partial y} = -\mathrm{e}^{x-y} = -1, \text{ at } (1, 1).$$

The directional derivative is

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 1 \times \frac{1}{\sqrt{2}} + 1 \times \frac{1}{\sqrt{2}} = \sqrt{2}.$$

The contour through $(1, 1)$ is in the directions given by $\tan\theta = 1$, which are $\theta = 45°$ and $\theta = -135°$. The steepest ascent is in the direction $\theta = -45°$.

29.10. The implicit-differentiation formula (29.6) for the slope of $f(x, y) = c$ at any point (x, y) is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y}.$$

(a) $f(x, y) \equiv xy = 1$ at $(2, \frac{1}{2})$. The partial derivatives are

$$\frac{\partial f}{\partial x} = y = \frac{1}{2}, \quad \frac{\partial f}{\partial y} = x = 2, \text{ at } \left(2, \frac{1}{2}\right).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{1}{2}}{2} = -\frac{1}{4}.$$

(b) $f(x, y) \equiv x^2 + y^2 = 25$ at $(3, 4)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x = 6, \quad \frac{\partial f}{\partial y} = 2y = 8, \text{ at } (3, 4).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{6}{8} = -\frac{3}{4}.$$

(c) $f(x, y) \equiv 1/x - 1/y$ at $(1, 2)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = -\frac{1}{x^2} = -1, \quad \frac{\partial f}{\partial y} = \frac{1}{y^2} = \frac{1}{4}, \text{ at } (1, 2).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(-1)}{\frac{1}{4}} = 4.$$

(d) $f(x, y) \equiv \frac{1}{10}x^2 + \frac{1}{15}y^2 = 1$ at $(2, 3)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{x}{5} = \frac{2}{5}, \quad \frac{\partial f}{\partial y} = \frac{2y}{15} = \frac{2}{5}, \text{ at } (2, 3).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{2}{5}}{\frac{2}{5}} = -1.$$

(e) $f(x, y) \equiv x^3 + 2y^3 = 3$ at $(1, 1)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 3x^2 = 3, \quad \frac{\partial f}{\partial y} = 6y^2 = 6, \text{ at } (1, 1).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{6} = -\frac{1}{2}.$$

(f) $f(x, y) \equiv x^3y + 3x^2 - y^2 - 19 = 0$ at $(2, 1)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 3x^2y + 6x = 24, \quad \frac{\partial f}{\partial y} = x^3 - 2y = 6, \text{ at } (2, 1).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{24}{6} = -4.$$

(g) $f(x, y) \equiv xy^2 - x^2y + 6 = 0$ at $(3, 2)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = y^2 - 2xy = -8, \quad \frac{\partial f}{\partial y} = 2xy - x^2 = 3, \text{ at } (3, 2).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{-8}{3} = \frac{8}{3}.$$

(h) $f(x, y) \equiv x^2 + y^2 = 4$ at $(2\cos\theta, 2\sin\theta)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x = 4\cos\theta, \quad \frac{\partial f}{\partial y} = 2y = 4\sin\theta, \text{ at } (\cos\theta, \sin\theta).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4\cos\theta}{4\sin\theta} = -\cot\theta.$$

(i) $f(x, y) \equiv x^2/a^2 + y^2/b^2 = 1$ at $(a\cos t, b\sin t)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{2x}{a^2} = \frac{2}{a}\cos t, \quad \frac{\partial f}{\partial y} = \frac{2y}{b^2} = \frac{2}{b}\sin t, \text{ at } (a\cos t, b\sin t).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{b}{a}\cot t.$$

(j) $f(x, y) \equiv x\cos y - y\sin x = 0$ at $(\frac{1}{2}\pi, 0)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = \cos y - y\cos x = 1, \quad \frac{\partial f}{\partial y} = -x\sin y - \sin x = -1, \text{ at } (\tfrac{1}{2}\pi, 0).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{1}{-1}\right) = 1.$$

(k) $f(x,y) \equiv y^2 - 4ax = 0$ at $(at^2, 2at)$. The partial derivatives are

$$\frac{\partial f}{\partial x} = -4a, \quad \frac{\partial f}{\partial y} = 2y = 4at, \text{ at } (at^2, 2at).$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{-4a}{4at}\right) = \frac{1}{t}.$$

29.11. The result can be verified by direct differentiation since $V = RT/P$ implies

$$\left(\frac{\partial V}{\partial P}\right)_T = -\frac{RT}{P^2},$$

and $T = PV/R$ implies

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{V}{R}, \quad \left(\frac{\partial T}{\partial V}\right)_P = \frac{P}{R}.$$

Hence

$$\begin{aligned}\left(\frac{\partial V}{\partial P}\right)_T + \left(\frac{\partial T}{\partial P}\right)_V \Big/ \left(\frac{\partial T}{\partial V}\right)_P &= -\frac{RT}{P^2} + \frac{V}{R}\Big/\frac{P}{V} \\ &= -\frac{RT}{P^2} + \frac{RT}{P^2} = 0,\end{aligned}$$

since $V = RT/P$.

The result to be proved is that

$$\left(\frac{\partial V}{\partial P}\right)_T = -\left(\frac{\partial T}{\partial P}\right)_V \Big/ \left(\frac{\partial T}{\partial V}\right)_P,$$

which bears a close formal resemblance to the implicit-differentiation formula (29.6). If we rewrite that formula in the more explicit form

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{f(x,y),\text{constant}} = -\left(\frac{\partial f}{\partial x}\right)_y \Big/ \left(\frac{\partial f}{\partial y}\right)_x,$$

an exact match is obtained by putting $x = P$, $y = V$ and $T = f(P, V)$ (ignoring R, which is constant throughout). The result therefore holds good for a general relationship between P, V and R; not simply to $PV = RT$. Several such thermodynamic formulae are proved in Section 31.2.

29.12. Require the tangent line to the given curve at the point (x_1, y_1) on the curve. Use (29.6) for the slope.

(a) The circle $f(x,y) \equiv x^2 + y^2 = a^2$. The slope at (x_1, y_1) is given by

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{2x}{2y} = -\frac{x_1}{y_1}.$$

Hence the tangent at (x_1, y_1) is (see (2.8))

$$y - y_1 = m(x - x_1) = -\frac{x_1}{y_1}(x - x_1), \text{ or } x_1x + y_1y = a^2.$$

(b) The ellipse $f(x,y) \equiv x^2/a^2 + y^2/b^2 = 1$. The slope at (x_1, y_1) is given by

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{a^2}\frac{b^2}{2y} = -\frac{x_1b^2}{y_1a^2}.$$

Hence the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{x_1b^2}{y_1a^2}(x - x_1), \text{ or } \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

(c) $f(x,y) \equiv a^2x^2 - b^2y^2 = c$. The slope at (x_1, y_1) is given by

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{2a^2x}{-2b^2y}\right) = \frac{a^2x_1}{b^2y_1}.$$

Hence the tangent at (x_1, y_1) is (since $x_1y_1 = 1$),

$$y - y_1 = \frac{a^2x_1}{b^2y_1}(x - x_1), \text{ or } a^2x_1x - b^2y_1y = c.$$

(d) The rectangular hyperbola $f(x,y) \equiv xy = 1$. The slope at (x_1, y_1) is given by

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} = -\frac{y_1}{x_1}.$$

Hence the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1), \text{ or } y_1x + x_1y = 2x_1y_1 = 2.$$

(e) $f(x,y) \equiv x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. The slope at (x_1, y_1) is given by

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}}.$$

Hence the tangent at (x_1, y_1) is

$$y - y_1 = -\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}}(x - x_1), \text{ or } y_1^{\frac{1}{3}}x + x_1^{\frac{1}{3}}y = x_1^{\frac{1}{3}}y_1^{\frac{1}{3}}.$$

(f) $F(x,y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. The slope at (x_1, y_1) is given by

$$m = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial y} = -\frac{2ax + 2hy + 2g}{2hx + 2by + 2f} = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}.$$

Hence the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}(x - x_1),$$

or

$$ax_1x + h(y_1x + x_1y) + by_1y + fy_1 + gy_1 + c = 0.$$

29.13. The curves $f(x,y) = \alpha$ and $g(x,y) = \beta$ intersect at (a, b). By (29.6) the slopes of the two curves are respectively

$$m_1 = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y}, \quad m_2 = -\frac{\partial g}{\partial x}\bigg/\frac{\partial g}{\partial y}$$

at (a, b). The curves intersect at right angles if $m_1m_2 = -1$, that is, if

$$\left(-\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y}\right)\left(-\frac{\partial g}{\partial x}\bigg/\frac{\partial g}{\partial y}\right) = -1, \text{ or, } \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} = 0.$$

(a) $f(x,y) \equiv x^2 + y^2 = \alpha$, $g(x,y) \equiv y/x = \beta$. Then

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} = (2x)\left(-\frac{y}{x^2}\right) + 2y\left(\frac{1}{x}\right) = 0$$

for all x and y. Hence the curves always intersect at right angles: the two families of curves are said to be *orthogonal*.

(b) $f(x,y) \equiv x^2 - y^2 = \alpha$, $g(x,y) \equiv xy = \beta$. Then

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} = (2x)y + (-2y)x = 0$$

for all x and y. The two systems of curves are orthogonal.
(c) $f(x,y) \equiv y^3 - x^3 = \alpha$, $g(x,y) \equiv 1/y + 1/x = \beta$. Then

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} = (-3x^2)\left(-\frac{1}{x^2}\right) + (3y^2)\left(-\frac{1}{y^2}\right) = 0$$

for all x and y. The two systems of curves are orthogonal.
(d) $f(x,y) \equiv (x^2 + y^2)/x = \alpha$, $g(x,y) \equiv (x^2 + y^2)/y = \beta$. Then

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} = \left(1 - \frac{y^2}{x^2}\right)\left(\frac{2x}{y}\right) + \left(\frac{2y}{x}\right)\left(-\frac{x^2}{y^2} + 1\right) = 0$$

for all x and y. Hence the two system of curves are orthogonal.

29.14. At any point on the curve $f(x,y) \equiv y^3 - x^3 = 1$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y} = \frac{3x^2}{3y^2} = \frac{x^2}{y^2},$$

using (29.6).

The differential equation

$$y^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^2$$

is a separable first-order equation (see Section 22.3) with solution

$$\int y^2 \mathrm{d}y = \int x^2 \mathrm{d}x + C, \text{ or } \frac{1}{3}y^3 = \frac{1}{3}x^3 + C,$$

where C is a constant. For the curve given in the problem, $C = \frac{1}{3}$. All the solutions are given by $y^3 - x^3 = A$, where A is any constant.

29.15. (a) Let $f(x,y) = x^2 + 2y^2$. Then the slope at any point on the curve $x^2 + 2y^2 = c$ is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y} = -\frac{2x}{4y} = -\frac{x}{2y}$$

using (29.6). The family of curves is generated by the differential equation

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = -x.$$

The computed contours (they are ellipses) are shown in Figure 11.
(b) Let $f(x,y) = x^2 + xy - y^3$. Then the slope at any point on the curve $x^2 + xy - y^3 = c$ is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y} = -\frac{2x + y}{x - 3y^2}$$

using (29.6). The family of curves is generated by the differential equation

$$(x - 3y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = -(2x + y).$$

The computed contours are shown in Figure 12.

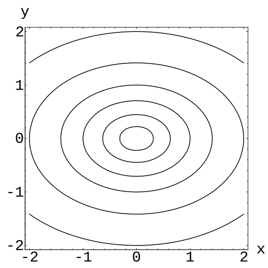

Figure 11: Problem 29.15a

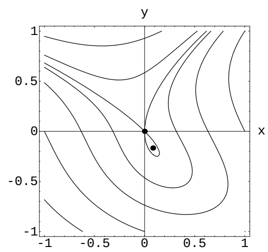

Figure 12: Problem 29.15b

(c) Let $f(x, y) = (x^2 + y)/(x + y^2)$. Then the slope at any point on the curve $(x^2 + y)/(x + y^2) = c$ is given by

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y} = -\left(\frac{x^2 - y + 2xy^2}{(x + y^2)^2}\right)\left(\frac{(x + y)^2}{x - 2x^2y + y^2}\right)$$

using (29.6). The family of curves is generated by the differential equation

$$(x - 2x^2y + y^2)\frac{dy}{dx} = -(x^2 - y + 2xy^2).$$

The computed contours are shown in Figure 13.

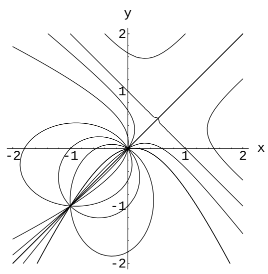

Figure 13: Problem 29.15c

(d) Let $f(x, y) = xy\mathrm{e}^{-x}$. Then the slope at any point on the curve $xy\mathrm{e}^{-x} = c$ is given by

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y} = -\frac{y(1 - x)\mathrm{e}^{-x}}{x\mathrm{e}^{-x}} = -\frac{y(1 - x)}{x}$$

using (29.6). The family of curves is generated by the differential equation

$$x\frac{dy}{dx} = -y(1 - x).$$

The computed contours are shown in the figure.

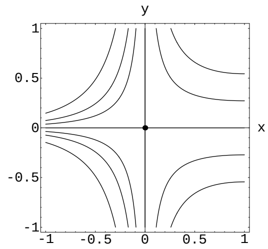

Figure 14: Problem 29.15d

29.16. (See Example 29.12.) (a) $f(x,y) \equiv y^2 - x^2 = c$. The differential equation which generates this family of curves is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\left(\frac{-2x}{2y}\right) = \frac{x}{y}.$$

Since the orthogonal system is everywhere perpendicular to the family above, its differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}.$$

This is a separable equation (see Section 22.3) with solution

$$\int \frac{\mathrm{d}y}{y} = -\int \frac{\mathrm{d}x}{x} + C, \text{ or } \ln|y| = -\ln|x| + C, \text{ or } \ln|xy| = C.$$

All cases are covered by the family of rectangular hyperbolas $xy = A$, which is the orthogonal system.

(b) $f(x,y) \equiv y^3 + x^3 = c$. The differential equation which generates this family of curves is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\left(\frac{3x^2}{3y^2}\right) = -\frac{x^2}{y^2}.$$

Since the orthogonal system is everywhere perpendicular top the family above its differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x^2}.$$

This is a separable equation (see Section 22.3) with solution

$$\int \frac{\mathrm{d}y}{y^2} = \int \frac{\mathrm{d}x}{x^2} + C, \text{ or } -\frac{1}{y} = -\frac{1}{x} + C.$$

The orthogonal system is

$$y = \frac{x}{1 - Cx}.$$

(c) $f(x,y) \equiv y^2/x = c$. The differential equation which generates this family of curves is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\left(-\frac{y^2}{x^2}\right)\Big/\left(\frac{2y}{x}\right) = \frac{y}{2x}.$$

Since the orthogonal system is everywhere perpendicular to the family above its differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{y}.$$

This is a separable equation (see Section 22.3) with solution

$$\int y\mathrm{d}y = -\int 2x\mathrm{d}x + C, \text{ or } \frac{1}{2}y^2 = -x^2 + C,$$

which is the orthogonal system (a family of ellipses).

(d) $f(x,y) \equiv \mathrm{e}^y - \mathrm{e}^x = c$. The differential equation which generates this family of curves is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\left(\frac{-\mathrm{e}^x}{\mathrm{e}^y}\right) = \mathrm{e}^{x-y}.$$

Since the orthogonal system is everywhere perpendicular top the family above its differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{y-x}.$$

This is a separable equation (see Section 22.3) with solution

$$\int \mathrm{e}^{-y}\mathrm{d}y = -\int \mathrm{e}^{-x}\mathrm{d}x + C, \text{ or } -\mathrm{e}^{-y} = \mathrm{e}^{-x} + C,$$

or

$$\mathrm{e}^{-x} + \mathrm{e}^{-y} = -C,$$

which is the orthogonal system.

29.17. Let $z = f(x,y)$, and let $A = \partial f/\partial x$ and $B = \partial f/\partial y$ at the point $P : (a,b)$. The slope on the surface $z = f(x,y)$ in the direction θ at P is

$$\frac{\mathrm{d}z}{\mathrm{d}s} = A\cos\theta + B\sin\theta.$$

Then the direction of steepest ascent occurs when

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right) = -A\sin\theta + B\cos\theta = 0 \text{ and } M = \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right) = -A\cos\theta - B\sin\theta < 0$$

at P by the test (4.2) for a local maximum .

(a) $z = f(x,y) = \frac{1}{2}x^2 + y^2$. Then

$$A = \frac{\partial f}{\partial x} = x = a, \quad B = \frac{\partial f}{\partial y} = 2y = 2b$$

at P. The directions of steepest ascent/descent are given by $\tan\theta = B/A = 2b/a$, that is where $\theta = \alpha_1 = \arctan(2b/a)$ and $\theta = \alpha_2 = \arctan(2b/a) + \pi$. Further, for $\theta = \alpha_1$,

$$M = -a\cos\alpha_1 - b\sin\alpha_1 = -\sqrt{(a^2 + 2b^2)} < 0.$$

Therefore the direction α_1 is the direction of steepest ascent.

(b) $z = f(x,y) = x^3y^3$. Then

$$A = \frac{\partial f}{\partial x} = 3x^2y^3 = 3a^2b^3, \quad B = \frac{\partial f}{\partial y} = 3x^3y^2 = 3a^3b^2$$

at P. The directions of steepest ascent/descent are given by $\tan\theta = B/A = a/b$, that is where $\theta = \alpha_1 = \arctan(a/b)$ and $\theta = \alpha_2 = \arctan(a/b) + \pi$. Further, for $\theta = \alpha_1$,

$$M = -3a^2b^3\cos\alpha_1 - 3a^3b^2\sin\alpha_1 = -3a^2b^2\sqrt{(a^2 + b^2)} < 0.$$

Therefore the direction α_1 is the direction of steepest ascent.

(c) $z = f(x,y) = \frac{1}{2}y^2 - y - x^2$. Then

$$A = \frac{\partial f}{\partial x} = -2x = -2a, \quad B = \frac{\partial f}{\partial y} = y - 1 = b - 1$$

at P. The directions of steepest ascent/descent are given by $\tan\theta = B/A = -(b-1)/(2a)$, that is where $\theta = \alpha_1 = \arctan[-(b-1)/(2a)]$ and $\theta = \alpha_2 = \arctan[-(b-1)/(2a)] + \pi$.
• $b \neq 1$. For $\theta = \alpha_1$, $M(\alpha_1) = -\sqrt{(4a^2 + (b-1)^2)} < 0$. Hence the steepest ascent is in the direction α_1.
•$b = 1$. In this case $A = -2a$ and $B = 0$, and $\alpha_1 = 0$ and $\alpha_2 = \pi$. Hence $M(\alpha_1) = 2a$. Hence α_1 is the direction of steepest ascent if $a < 0$ and α_2 is the direction of steepest ascent if $a > 0$. If $a = 0$, then the point $(0, 1)$ is a minimum of $z = \frac{1}{2}y^2 - x^2$.

29.18. Given $f(x, y) = x^2 + 2xy + y^2 = c$, direct differentiation gives

$$2x + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

as stated in the question. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1.$$

Using (29.6) instead,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{2x + 2y}{2x + 2y} = -1$$

confirming the result.

29.19. The slope at a point P on the curve $f(x, y) = c$ is, by (29.6),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{\partial f}{\partial x}\right)_P \Big/ \left(\frac{\partial f}{\partial y}\right)_P.$$

Hence a normal vector $\mathbf{n}$ to the curve at P is

$$\mathbf{n} = \left[\left(\frac{\partial f}{\partial x}\right)_P, \left(\frac{\partial f}{\partial y}\right)_P\right]$$

(see (29.7)).
(a) Let $f(x, y) \equiv xy = 2$ and $g(x, y) \equiv x^2 - y^2 = -3$. Then normal vectors to the two curves are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x) = (2, 1), \quad \mathbf{n}_2 = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (2x, -2y) = (2, -4)$$

at $P : (1, 2)$. By (10.4), the angle θ between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = 0.$$

Hence the curves meet at right angles at $(1, 2)$.
(b) Let $f(x, y) \equiv y - x^3 = 0$ and $g(x) \equiv x^2 + \frac{1}{2}y^2 = 36$. The normal vectors to the two curves are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-3x^2, 1) = (-12, 1), \quad \mathbf{n}_2 = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (2x, y) = (4, 8)$$

at $P : (2, 8)$. By (10.4), the angle θ between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{-48 + 8}{\sqrt{145}\sqrt{80}} = -\frac{2}{\sqrt{29}}.$$

(c) Let $f(x, y) \equiv x^2 + xy + y^2 = 3$ and $g(x) \equiv x + y = 2$. The normal vectors to the two curves are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x + y, x + 2y) = (3, 3), \quad \mathbf{n}_2 = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (1, 1)$$

at $P:(1,1)$. By (10.4), the angle θ between the $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{3+3}{\sqrt{18}\sqrt{2}} = 1.$$

The surfaces have a common tangent plane at $(1,1)$.
(d) Let $f(x) \equiv ax^2 + 2hxy + by^2 + c = 0$ and

$$g(x) = ax_0x + \tfrac{1}{2}h(x_0+x)(y_0+y) + by_0y + c = 0$$

(note the correction to $g(x)$). The normal vectors to the two curves at (x_0,y_0) are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} = (2ax_0+2hy_0, 2hx_0+2by_0),$$

$$\mathbf{n}_2 = \left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right)_{(x_0,y_0)} = (ax_0+hy_0, hx_0+by_0) = \frac{1}{2}\mathbf{n}_1.$$

The normals $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel and in the same direction at (x_0,y_0), so $\theta=0$. Note that any point (x_0,y_0) on the first curve also lies on the second. In fact the second equation is the equation of the tangent line at (x_0,y_0) (compare Problem 29.12(f)).

29.20. (a) Let $f(x,y) \equiv x^4 - y^4 = 1$. Then, by (29.6),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = \frac{x^3}{y^3}, \text{ or } y^3\frac{\mathrm{d}y}{\mathrm{d}x} - x^3 = 0. \qquad \text{(i)}$$

Differentiate this equation with respect to x treating y as a function of x. Then

$$3y^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x^2 = 0,$$

or

$$3y^2\left(\frac{x^3}{y^3}\right)^2 + y^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x^2 = 0$$

using (i). Hence

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3x^2(y^4-x^4)}{y^7}.$$

(b) Let $f(x,y) \equiv xy = 1$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}, \text{ or } x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Differentiate this equation with respect to x so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Therefore

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x^2} = \frac{2}{x^3}.$$

The result can be checked by differentiating $y=1/x$ directly.
(c) Let $f(x,y) \equiv xy\mathrm{e}^{xy} = 1$. By (29.6)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y\mathrm{e}^{xy} + xy^2\mathrm{e}^{xy}}{x\mathrm{e}^{xy} + x^2y\mathrm{e}^{xy}} = -\frac{y}{x}.$$

(Note that this can also be inferred since $xy =$ constant.) The answer is therefore the same as the previous one:

$$\frac{d^2y}{dx^2} = \frac{2y}{x^2} = \frac{2}{x^3}.$$

29.21. (a) $f(x, y) = 1/(x + y)$ at $(1, -2)$. The gradient is given by

$$\mathbf{grad}\, f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(-\frac{1}{(x+y)^2}, -\frac{1}{(x+y)^2}\right) = (-1, -1)$$

at $(1, -2)$. Further, the direction of the gradient is $-135°$ to the positive x axis, and its magnitude is $|\mathbf{grad}\, f| = \sqrt{(1 + 1)} = \sqrt{2}$.
(b) $f(x, y) = y/x$ at $(2, 0)$. The gradient is given by

$$\mathbf{grad}\, f = \left(-\frac{y}{x^2}, \frac{1}{x}\right) = (0, \tfrac{1}{2})$$

at $(2, 0)$. The direction of the gradient is at $90°$ to the x axis, and its magnitude is $|\mathbf{grad}\, f| = \frac{1}{2}$.
(c) $f(x, y) = y^2 - 3x^2 + 1$ at $(0, 0)$. The gradient is given by

$$\mathbf{grad}\, f = (-6x, 2y) = (0, 0)$$

at $(0, 0)$. Since this is a vector of zero magnitude, we cannot associate a direction with it.
(d) $f(x, y) = 1/x - 1/y$ at $(2, 1)$.The gradient is given by

$$\mathbf{grad}\, f = \left(-\frac{1}{x^2}, \frac{1}{y^2}\right) = (-\tfrac{1}{4}, 1)$$

at $(2, 1)$. Its direction is $30°$ to the x axis., and its magnitude is about $104°$ to the x axis.
(e) $f(x, y) = 1/r = 1/(x^2 + y^2)^{\frac{1}{2}}$ at any point. The gradient is given by

$$\mathbf{grad}\, f = \left(-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}\right) = \left(-\frac{x}{r^3}, -\frac{y}{r^3}\right).$$

A vector with components (x, y) points outward radially. The gradient has components which ar proportional to these. The negative sign means that the gradient always points towards the origin.

29.22. A normal to the curve $f(x, , y) = c$ at any point is $\mathbf{grad}\, f$ and a unit normal is $\mathbf{grad}\, f/|\mathbf{grad}\, f|$.
(a) $f(x, y) \equiv 2x - 3y + 1 = 0$ at any point. A unit normal is given by

$$\mathbf{n} = \frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \frac{(2, -3)}{\sqrt{(2^2 + 3^2)}} = \left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\right).$$

(b) $f(x, y) \equiv x^2 + y^2 = 5$ at $(2, 1)$. A unit normal is given by

$$\mathbf{n} = \frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \frac{(4, 2)}{\sqrt{(4^2 + 2^2)}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

(c) $f(x, y) \equiv x^2 + y^2 = r^2$ at (x_0, y_0). Since this point lies on the circle, $r^2 = x_0^2 + y_0^2$. A unit normal is given by

$$\mathbf{n} = \frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \frac{(2x, 2y)}{\sqrt{(4x^2 + 4y^2)}} = \left(\frac{x_0}{r}, \frac{y_0}{r}\right)$$

at (x_0, y_0).
(d) $f(x, y) \equiv x^2/a^2 + y^2/b^2 = 1$ at (x_0, y_0). Since this point lies on the ellipse, $x_0^2/a^2 + y_0^2/b^2 = 1$. A unit normal is given by

$$\mathbf{n} = \frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \left(\frac{2x}{a^2}, \frac{2y}{b^2}\right)\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{-\frac{1}{2}} = \frac{(x_0 b^2, y_0 a^2)}{\sqrt{(x_0^2 b^4 + y_0^2 a^2)}}$$

at (x_0, y_0).

(e) $f(x,y) \equiv y - 3x^2 = -2$ at $(2,10)$. A unit normal is

$$\mathbf{n} = \frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \frac{(-6x,1)}{\sqrt{(36x^2+1)}} = \frac{(-12,1)}{\sqrt{145}}$$

at $(2,10)$.

29.23. See (29.9) and Problem 29.19.

(a) Let $f(x,y) \equiv y^2 - x^2 = -3$ and $g(x,y) \equiv x^3 - y^3 = 7$ at $(2,1)$. Then normal vectors to the two curves are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-2x, 2y) = (-4, 2),$$

$$\mathbf{n}_2 = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (3x^2, -3y^2) = (12, -3)$$

at $(2,1)$. By (10.4), the angle θ between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{(-4,2)\cdot(12,-3)}{\sqrt{20}\sqrt{153}} = -\frac{9}{\sqrt{85}}.$$

Hence $\theta = 167^\circ$.

(b) Let $f(x,y) \equiv x^2y - xy^2 = 0$ and $g(x,y) = x/y - y/x = 0$ at $(2,2)$. Then normal vectors to the two curves are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2xy - y^2, x^2 - 2xy) = (4, -4),$$

$$\mathbf{n}_2 = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = \left(\frac{1}{y} + \frac{y}{x^2}, -\frac{x}{y^2} - \frac{1}{x}\right) = (1, -1)$$

at $(2,2)$. By (10.4), the angle θ between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{(4,-4)\cdot(1,-1)}{\sqrt{32}\sqrt{2}} = 0.$$

Hence the curves intersect at right angles.

(c) Let $f(x,y) \equiv x^2 + y^2 + 2x - 4y + 4 = 0$ and $g(x,y) \equiv y - x^2 - 2x - 2 = 0$ at $(-1,1)$. Then normal vectors to the two curves are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x+2, 2y-4) = (0, -2),$$

$$\mathbf{n}_2 = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (-2x-2, 1) = (0, 1)$$

at $(-1,1)$. By (10.4), the angle θ between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{(0,-2)\cdot(0,1)}{2\times 1} = -1.$$

Hence $\theta = 180^\circ$, which means that the curves touch at $(-1,1)$ having a common tangent there.

29.24. From (29.12), the directional derivative is given by

$$\frac{\mathrm{d}f}{\mathrm{d}s} = |\mathbf{grad}\, f|\cos\phi$$

where ϕ is the smaller angle between $\mathbf{grad}\,\phi$ and the chosen direction.

(a) Along the contour through a given point f is constant so that $\mathrm{d}f/\mathrm{d}s = 0$. Hence $\cos\phi = 0$ and $\phi = 90°$. The directional derivative $\mathrm{d}f/\mathrm{d}s$ takes its greatest and least values where $\phi = 0$ and $\phi = \pi$ which are both perpendicular to the contour.

(b) The maximum rate of increase of f is equal to $|\mathbf{grad}\, f|$, achieved when $\phi = 0$.

Chapter 30: Chain rules, restricted maxima, coordinate systems

30.1. A list of possible parametrizations is given below: they are not unique.

(a) $x^2 + y^2 = 25$. This is a circle which can be parametrized by $x(t) = 5\cos t$, $y(t) = \sin t$ for $0 \le t < 2\pi$.

(b) $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$. This is an ellipse which can be described by $x(t) = 2\cos t$, $y(t) = 3\sin t$ for $0 \le t < 2\pi$.

(c) $xy = 4$. this is a rectangular hyperbola which can be described by $x(t) = 2t$, $y(t) = 2/t$ for $0 < t < \infty$ and $-\infty < t < 0$.

(d) $x^2 - y^2 = 1$. This curve is a hyperbola which can be represented parametrically by $x(t) = \sec t$, $y(t) = \tan t$ for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$.

(e) $\frac{1}{4}x^2 - \frac{1}{9}y^2 = 1$. This curve is a hyperbola which can be represented parametrically by $x = 2\sec t$, $y = 3\tan t$ for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$, for the branch in $x > 0$.

(f) $y^2 = 4ax$. This curve is a parabola which can be represented parametrically by $x = at^2$, $y = 2at$ for $-\infty < t < \infty$.

(g) $(x-1)^2 + (y-2)^2 = 9$. This curve is a circle centred at $(1, 2)$ which can be represented parametrically by $x = 1 + 3cost$, $y = 2 + 3\sin t$ for $0 \le t < 2\pi$

(h) $2x - 5y + 2 = 0$. This is a straight line which can be represented in many ways parametrically including $x = t$, $y = 2(t+1)/5$ for $-\infty < t < \infty$.

30.2. Given $f(x, y)$, $x = x(t)$ and $y = y(t)$ the chain rule (30.1) states that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

(a) $f(x, y) = x^2 + y^2$, $x(t) = t$, $y(t) = 1/t$. Then

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{t^2}.$$

Therefore

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = 2x.1 - 2y.\frac{1}{t^2} = 2t - \frac{2}{t^3}.$$

(b) $f(x, y) = x^2 - y^2$, $x(t) = \cos t$ and $y(t) = \sin t$. Then

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos t.$$

Therefore

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2x(-\sin t) - 2y.\cos t = -4\sin t\cos t = -2\sin 2t.$$

(c) $f(x, y) = xy$, $x(t) = 2\cos t$ and $y(t) = \sin t$. Then

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -2\sin t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos t.$$

Therefore

$$\frac{\mathrm{d}f}{\mathrm{d}t} = y(-2\sin t) + x.\cos t = -2\sin^2 t + 2\cos^2 t = 2\cos 2t.$$

(d) $f(x,y) = x\sin y$, $x(t) = 2t$ and $y(t) = t^2$. Then

$$\frac{\partial f}{\partial x} = \sin y, \quad \frac{\partial f}{\partial y} = x\cos y, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2t.$$

Therefore

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2\sin y + x\cos y.(2t) = 2\sin(t^2) + 4t^2\cos(t^2).$$

(e) $f(x,y) = 4x^2 + 9y^2$, $x(t) = \frac{1}{2}\cos t$ and $y(t) = \frac{1}{3}\sin t$. Then

$$\frac{\partial f}{\partial x} = 8x, \quad \frac{\partial f}{\partial y} = 18y, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{2}\sin t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{3}\cos t.$$

Therefore

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 8x(-\tfrac{1}{2}\sin t) + 18y(\tfrac{1}{3}\cos t) = 0.$$

30.3. Assume that $R > r$ (for the sake of description although it is not a requirement in the solution), that the origin is at the centre of the tracks, that the coordinates of the athlete on the inner track are (x,y) and that of the athlete on the outer track are (X,Y). Assume also that the athletes start with $y = Y = 0$ at time $t = 0$. The athlete on the inner track makes one circuit in time $2\pi r/v$, so that the athlete's subsequent location is given parametrically by

$$x = r\cos(vt/r) \quad y = r\sin(vt/r).$$

Similarly, the location of the other athlete is given by

$$X = R\cos(Vt/R), \quad Y = R\sin(Vt/R).$$

The distance D between the runners is therefore

$$\begin{aligned} D &= \sqrt{[(x-X)^2 + y - Y)^2]} \\ &= \sqrt{[\{r\cos(vt/r) - R\cos(Vt/R)\}^2 + \{r\sin(vt/r) - R\sin(Vt/R)\}^2]}. \end{aligned}$$

The rate of change of the distance D with time is

$$\begin{aligned} \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{D}[(r\cos(vt/r) - R\cos(Vt/R))(-v\sin(vt/r) + V\sin(Vt/R)) \\ &\qquad +(r\sin(vt/r) - R\sin(Vt/R))(v\cos(vt/r) - V\cos(Vt/R))] \\ &= \frac{vR - rV}{D}[\cos(Vt/R)\sin(vt/r) - \sin(Vt/R)\cos(vt/r)] \\ &= \frac{vR - rV}{D}\sin\left(\frac{v}{r} - \frac{V}{R}\right)t. \end{aligned}$$

The distance is stationary when $\mathrm{d}D/\mathrm{d}t = 0$, which occurs when

$$\sin\left(\frac{v}{r} - \frac{V}{R}\right)t = 0,$$

assuming that $vR \neq Vr$. These stationary values occur when

$$t = \frac{rRn\pi}{vR - Vr}, \quad (n = 0, 1, 2, \ldots).$$

The minimum distances occur for n even and the maximum for n odd. The two athletes orbit at different rates (just like two planets): they are at their closest when they lie on the same radius and at their furthest when they lie on two directly opposite radii.

30.4. (a) Let the edges of the rectangle have lengths x and y. Then the area $A(x, y) = xy$ and the perimeter $P(x, y) = 2x + 2y$. We wish to find the maximum area subject to the restriction $P(x, y) = 10$. Using (30.4), we must solve

$$x + y = 5, \tag{i}$$

$$\frac{\partial A}{\partial x} - \lambda \frac{\partial P}{\partial x} = y - 2\lambda = 0, \tag{ii}$$

$$\frac{\partial A}{\partial y} - \lambda \frac{\partial P}{\partial y} = x - 2\lambda = 0. \tag{iii}$$

From (i) and (ii), $x = y$, so that from (i) $x = y = \frac{5}{2}$. Hence the maximum area is $\frac{25}{4}$.

(b) Let the edges of the rectangle have lengths x and y. The perimeter $P(x, y) = 2x + 2y$ and the area $A(x, y) = xy = 9$. By (30.4), solve the equations

$$xy = 9, \tag{i}$$

$$\frac{\partial P}{\partial x} - \lambda \frac{\partial A}{\partial x} = 2 - y\lambda = 0, \tag{ii}$$

$$\frac{\partial P}{\partial y} - \lambda \frac{\partial A}{\partial y} = 2 - x\lambda = 0. \tag{iii}$$

From (ii) and (iii), $x = y$, so that from (i) $x = y = 3$. The maximum value of the perimeter is 12.

(c) Let $f(x, y) = x^2 + 2y^2$ and $g(x, y) \equiv x^2 + y^2 = 1$. By the Lagrange-multiplier method x, y and λ satisfy

$$x^2 + y^2 = 1, \tag{i}$$

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2x - 2x\lambda = 2x(1 - \lambda) = 0, \tag{ii}$$

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 4y - 2y\lambda = 2y(2 - \lambda) = 0. \tag{iii}$$

From (iii), either $y = 0$ or $\lambda = 2$. If $y = 0$, then $\lambda = 1$ from (ii) ($x = 0$ is not a possibility since (i) would not be satisfied) and from (i) $y = \pm 1$. If $\lambda = 1$, then $x = 0$ and $y = \pm 1$ from (i). There are four solutions:

$\lambda = 1$, $x = \pm 1$, $y = 0$;

$\lambda = 2$, $x = 0$, $y = \pm 1$.

(d) Let x and y be the lengths of the sides of the rectangle parallel to the x and y axes. Then the area of the rectangle is $A(x, y) = xy$ and the restriction is $g(x, y) \equiv 2x + y = 1$. The Lagrange-multiplier method (30.4) gives the equations

$$2x + y = 1, \tag{i}$$

$$\frac{\partial A}{\partial x} - \lambda \frac{\partial g}{\partial x} = y - 2\lambda = 0, \tag{ii}$$

$$\frac{\partial A}{\partial y} - \lambda \frac{\partial g}{\partial y} = x - \lambda = 0. \tag{iii}$$

From (ii) and (iii), $y = 2x$ so that, from (i), $x = \frac{1}{4}$ and $y = \frac{1}{2}$. There is one stationary value at $(x, y) = (\frac{1}{4}, \frac{1}{2})$, where $A = xy = \frac{1}{8}$, which a sketch shows to be a maximum.

(e) For any point (x, y) on the line $x + 2y = 1$, the square of the distance from this point to $(1, 1)$ is $f(x, y) = (x - 1)^2 + (y - 1)^2$ with the restriction $g(x, y) \equiv x + 2y = 1$. By (30.4) x, y and the parameter λ satisfy

$$x + 2yh = 1, \tag{i}$$

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2(x - 1) - \lambda = 0, \tag{ii}$$

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 2(y-1) - 2\lambda = 0, \qquad \text{(iii)}$$

From (ii) and (iii), $y = 2x - 1$, so that, from (i), $x = \frac{3}{5}$. Hence there is one stationary point at $(\frac{3}{5}, \frac{1}{5})$. The minimum distance is therefore $2/\sqrt{5}$.

(f) The square of the distance from the origin of a point (x, y) given by $f(x, y) = x^2 + y^2$, and (x, y) is restricted to the curve $g(x, y) \equiv x^2 + 8xy + 7y^2 = 225$. Using the Lagrange-multiplier method of (30.4), x, y and the parameter λ satisfy

$$x^2 + 8xy + 7y^2 = 225, \qquad \text{(i)}$$

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2x - (2x + 8y)\lambda = 0, \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 2y - (8x + 14y)\lambda = 0. \qquad \text{(iii)}$$

Eliminate λ between (ii) and (iii):

$$x(4x + 7y) = y(x + 4y), \text{ or } 2x^2 - 3xy - 2y^2 = 0, \text{ or } (2x + y)(x - 2y) = 0.$$

Hence $y = -2x$ or $y = \frac{1}{2}x$. Substitute $y = -2x$ into (i) so that

$$x^2 - 16x^2 + 28x^2 = 225, \text{ or } x^2 = \tfrac{225}{13} \text{ or } x = \pm\tfrac{15}{\sqrt{13}}.$$

Substitute $y = \frac{1}{2}x$ into (i) so that

$$27x^2 = 4 \times 225, \text{ or } x = \pm\tfrac{10}{\sqrt{3}}.$$

There are two points on the curve, namely $(10/\sqrt{3}, 5/\sqrt{3})$ and $(-10\sqrt{3}, -5/\sqrt{3})$ which give minimum distances (the curve $g(x, y) = 225$ is a hyperbola with two branches). The minimum distance is $5\sqrt{5}/\sqrt{3}$.

(g) (*Note*: 'minimum' should be replaced by 'maximum' in the question.)
Referring to Example 30.8 and Fig. 30.4, the perimeter of the rectangle is $P(x, y) = 4x + 4y$ subject to the restriction $g(x, y) \equiv x^2 + 4y^2 = 1$. The Lagrange-multiplier equations are

$$x^2 + 4y^2 = 1, \qquad \text{(i)}$$

$$\frac{\partial P}{\partial x} - \lambda \frac{\partial g}{\partial x} = 4 - 2x\lambda = 0, \qquad \text{(ii)}$$

$$\frac{\partial P}{\partial y} - \lambda \frac{\partial g}{\partial y} = 4 - 8y\lambda = 0. \qquad \text{(iii)}$$

From (ii) and (iii), $\lambda = 2/x = 1/(2y)$. Therefore $x = 4y$, and from (i), $y = 1/(2\sqrt{5})$, $x = 2/\sqrt{5}$ (assuming positive values for x and y) . The maximum perimeter is $2\sqrt{5}$.

(h) In this problem $f(x, y) = (x - y + 1)^2$ and $g(x, y) \equiv y - x^2 = 0$. The Lagrange-multiplier equations (30.4) become

$$y - x^2 = 0, \qquad \text{(i)}$$

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2(x - y + 1) + 2x\lambda = 0, \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = -2(x - y + 1) - \lambda = 0. \qquad \text{(iii)}$$

Eliminate λ between (ii) and (iii) to obtain $(x - y + 1)(1 + 2x) = 0$. Hence either $x - y + 1 = 0$ or $1 + 2x = 0$. If $x - y + 1 = 0$, then elimination of y in (i) leads to $x^2 - x - 1 = 0$ with solutions $x = \frac{1}{2}[1 \pm \sqrt{5}]$ with corresponding $y = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$. If $x = -\frac{1}{2}$, then, from (i), $y = \frac{1}{4}$. Hence there are three stationary points at:

$$(\tfrac{1}{2}[1 + \sqrt{5}], \tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}), \quad (\tfrac{1}{2}[1 - \sqrt{5}], \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}), \quad (-\tfrac{1}{2}, \tfrac{1}{4}).$$

(i) Without loss we can choose the parabola to have the equation $y^2 = 2x$. Let $f(x,y)$ be the square of the length of a straight line which joins the point (a,b) ($a > 0$ and $-\sqrt{2a} < b < \sqrt{2a}$), which lies inside the parabola, and a point (x,y) on the parabola. At points on the parabola where $f(x,y)$ is stationary, the line will be normal to the parabola. Hence we require the points where

$$f(x,y) = (x-a)^2 + (y-b)^2 \text{is stationary subject to } g(x,y) \equiv y^2 - 2x = 0.$$

The Lagrange multiplier equations are

$$y^2 - 2x = 0, \tag{i}$$

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2(x-a) + 2\lambda = 0, \tag{ii}$$

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 2(y-b) - 2y\lambda = 0. \tag{iii}$$

Eliminate λ between (ii) and (iii) so that

$$y - b + y(x-a) = 0, \text{ or } y^3 + 2y(1-a) - 2b = 0.$$

This cubic equation always has at least one real solution (consider how the left-hand side behaves as $\to \pm\infty$) and at most three real solutions. Let

$$h(y) = y^3 + 2y(1-a) - 2b.$$

Then

$$\frac{\mathrm{d}h(y)}{\mathrm{d}y} = 3y^2 + 2(1-a).$$

The curve $z = h(y)$ has distinct stationary values where $y = \pm\sqrt{[2(a-1)/3}$ provided $a > 1$. These lie above and below the y axis (in the (y,z) plane) which implies that the curve $z = h(y)$ cuts the y axis three times. For $0 < a < 1$ there is just one real solution.

30.5. (a) $f(x,y) = x^2 + y^2$ on $g(x,y) \equiv xy = 1$.
(i) The curve $xy = 1$ can be parametrized by putting $x = t$, $y = 1/t$ for $-\infty < t < 0$ and $0 < t < \infty$. On this curve

$$f(x,y) = t^2 + \frac{1}{t^2}.$$

Stationary points occur where

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2t - \frac{2}{t^3} = 0,$$

that is, where $t^4 = 1$ or $t = \pm 1$. The corresponding coordinates of the stationary points are $(1,1)$ and $(-1,-1)$.

(ii) The equations (30.4) in the Lagrange-multiplier method are

$$xy = 1, \tag{i}$$

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 2x - y\lambda = 0, \tag{ii}$$

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 2y - x\lambda = 0. \tag{iii}$$

From (ii) and (iii), $y^2 = x^2$ so that $y = \pm x$ but only the plus sign leads to real solutions. Hence (i) gives $x^2 = 1$ or $x = \pm 1$. As in the parametric method the stationary points occur at $(1,1)$ and $(-1,-1)$.

(b) $f(x,y) = x^2 + y^2$ on $g(x,y) \equiv (x-1)^2 + y^2 = 1$.
(i) The circle $(x-1)^2 + y^2 = 1$ can be parametrized by $x = 1 + \cos t$, $y = \sin t$ for $0 \le t < 2\pi$. On this circle

$$f(x,y) = (1 + \cos t)^2 + \sin^2 t = 2 + 2\cos t.$$

Stationary points occur where

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -2\sin t = 0,$$

that is, at $t = 0$ and $t = \pi$ for the given interval of t. The corresponding stationary points are at $(2, 0)$ and $(0, 0)$.

(ii) The equations using the Lagrange-multiplier method are

$$(x-1)^2 + y^2 = 1, \tag{i}$$

$$2x - 2(x-1)\lambda = 0, \tag{ii}$$

$$2y - 2y\lambda = 0. \tag{iii}$$

From (ii) either $y = 0$ or $\lambda = 1$. If $y = 0$, then $x = 2$ or 0 from (i). The case $\lambda = 1$ is not consistent with (ii). This confirms that $f(x, y)$ is stationary at $(2, 0)$ and $(0, 0)$.

(c) $f(x, y) = x^2 + 4y^2$ on $g(x, y) \equiv x^2 + y^2 = 1$.

(i) The circle $x^2 + y^2 = 1$ can be parametrized by $x = \cos t$, $y = \sin t$ for $0 \le t < 2\pi$. On this circle

$$f(x, y) = \cos^2 t + 4\sin^2 t.$$

Stationary points occur where

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -2\cos t\sin t + 8\sin t\cos t = 6\sin t\cos t = 3\sin 2t = 0,$$

that is, at $t = 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$. The corresponding stationary points are $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.

(ii) The equations using the Lagrange-multiplier method are

$$x^2 + y^2 = 1, \tag{i}$$

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 2x - 2x\lambda = 0, \tag{ii}$$

$$\frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 8y - 2y\lambda = 0. \tag{iii}$$

From (iii), either $y = 0$ or $\lambda = 4$. If $y = 0$, then from (i) $x = \pm 1$ and from (ii) $\lambda = 1$. If $\lambda = 4$, then from (ii) $x = 0$ and from (i) $y = \pm 1$. The stationary points are therefore $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ as above.

(d) $f(x, y) = 3x - 2y$ on $g(x, y) \equiv x^2 - y^2 = 4$.

(i) The curve $x^2 - y^2 = 4$ can be parametrized by $x = 2\sec t$, $y = 2\tan t$ for $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < t < \frac{3}{2}\pi$. On this hyperbola

$$f(x, y) = 6\sec t - 4\tan t.$$

Stationary points occur where

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 6\sec t\tan t - 4\sec^2 t = 0,$$

that is, where $\sin t = 2/3$. For $-\frac{1}{2} < t < \frac{1}{2}$, $x = 6/\sqrt{5}$, $y = 4/\sqrt{5}$ and for $\frac{1}{2}\pi < t < \frac{3}{2}\pi$, $x = -6/\sqrt{5}$, $y = -4/\sqrt{5}$.

(ii) The equations using the Lagrange-multiplier method are

$$x^2 - y^2 = 4, \tag{i}$$

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 3 - 2x\lambda = 0, \tag{ii}$$

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = -2 + 2y\lambda = 0. \qquad \text{(iii)}$$

Eliminating λ between (ii) and (iii), $2x = 3y$. Elimination of y in (i) leads to $x = \pm 6/\sqrt{5}$. The stationary points are

$$\left(\frac{6}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), \quad \left(-\frac{6}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right).$$

(e) $f(x, y) = xy$ on $g(x, y) \equiv x^2 + y^2 = 1$.
(i) The circle $x^2 + y^2 = 1$ can be parametrized by $x = \cos t$, $y = \sin t$ for $0 \le t < 2\pi$. On this circle

$$f(x, y) = \cos t \sin t = \tfrac{1}{2} \sin 2t.$$

Stationary points occur where

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \cos 2t = 0,$$

that is, where $t = \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$. The stationary points are therefore

$$(1/\sqrt{2}, 1/\sqrt{2}), \quad (-1/\sqrt{2}, 1/\sqrt{2}), \quad (-1/\sqrt{2}, -1/\sqrt{2}), \quad (1/\sqrt{2}, 1/\sqrt{2}).$$

(ii) The equations using the Lagrange-multiplier method are

$$x^2 + y^2 = 1, \qquad \text{(i)}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} - \lambda \frac{\mathrm{d}g}{\mathrm{d}x} = y - 2x\lambda = 0, \qquad \text{(ii)}$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} - \lambda \frac{\mathrm{d}g}{\mathrm{d}y} = x - 2y\lambda = 0. \qquad \text{(iii)}$$

From (ii) and (iii), $y^2 = x^2$ so that $y = \pm x$ and from (i) $x = \pm 1/\sqrt{2}$. Hence there are four stationary points which agree with those listed above using the parametric method.

30.6. Figure 15 shows the curve $g(x, y) = c$ and the family of curves $f(x, y) =$ constant. Assume, for the sake of discussion, that the values of $f(x, y)$ are greater above the curve $f(x, y) =$ constant which just touches the curve $g(x, y) = c$. Then if we track the values of f along $g(x, y) = c$, they will pass through a minimum at the point of contact.

The normals at the point of contact P are

$$\mathbf{n}_1 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)_P, \quad \mathbf{n}_2 = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)_P.$$

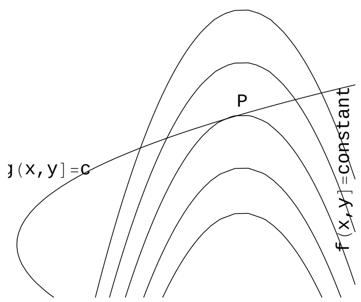

Figure 15: Problem 30.6

At the point of contact $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel so that $\mathbf{n}_1 = \lambda \mathbf{n}_2$, from which the Lagrange-multiplier equations (30.4) follow.

The non-tangential case can be illustrated by an example. Let $f(x,y) = x^2y$ subject to $x^2+y^2 = 1$. The Lagrange equations (30.4) become

$$x^2 + y^2 = 1, \tag{i}$$

$$\frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 2xy - 2x\lambda = 2x(y-\lambda) = 0, \tag{ii}$$

$$\frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = x^2 - 2y\lambda = 0. \tag{iii}$$

From (ii) either $x = 0$ or $\lambda = y$. If $x = 0$, then $\lambda = 0$ from (iii) (y cannot be zero) which means that $y = \pm 1$. If $\lambda = y$, then, from (iii), $x^2 = 2y^2$. Finally from (i) $3y^2 = 1$ so that $y = \pm 1/\sqrt{3}$. We can now list the stationary points:

$$(0,1),\ (0,-1),\ \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),\ \left(\frac{-\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),\ \left(\frac{-\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right),\quad \left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

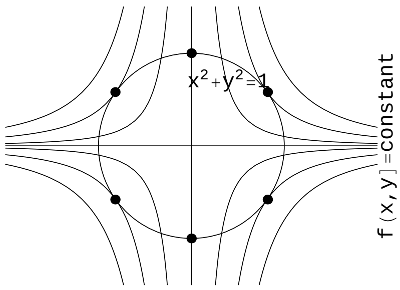

Figure 16: Problem 30.6: the dots show the stationary points.

The family of curves $x^2y =$ constant are shown in the figure together with the locations of the stationary points on the circle $x^2 + y^2 = 1$. Contours of $f(x,y)$ are tangential to the circle at the four stationary points which are not on the axes. However, at the points at $(0,1)$ and $(0,-1)$ the contour is not tangential. At these points $\lambda = 0$, which means that $\partial f/\partial x = \partial f/\partial y = 0$: they are unrestricted stationary points of $f(x,y)$. It is not possible to define a normal vector to the curve $f(x,y) = c$ at points where $\partial f/\partial x = \partial f/\partial y = 0$: the method described in the first part of this problem fails in this case.

30.7. The orthogonality conditions are given by (30.5).
(a) $u = 2x + 3y$, $v = -3x + 2y$. Then

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = (2)(-3) + (3)(2) = 0.$$

(b) $u = xy$, $v = x^2 - y^2$. Then

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = (y)(2x) + (x)(-2y) = 0.$$

(c) $u = x^2 + 2y^2$, $v = y/x^2$.

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = (2x)\left(-\frac{2y}{x^3}\right) + (4y)\left(\frac{1}{x^2}\right) = 0.$$

(d) $u = xy^2$, $v = y^2 - 2x^2$. Then

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = (y^2)(-4x) + (2xy)(2y) = 0.$$

(e) $u = x + 1/x + y^2/x$, $v = y - 1/y + x^2/y$. Then

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \left(1 - \frac{1}{x^2} - \frac{y^2}{x^2}\right)\left(\frac{2x}{y}\right) + \left(\frac{2y}{x}\right)\left(1 + \frac{1}{y^2} - \frac{x^2}{y^2}\right) = 0.$$

(f) $x = 2u - v$, $y = u + 2v$. Then

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = 2 \times (-1) + 1 \times 2 = 0.$$

(g) $x = u^2 - v^2$, $y = 2uv$. Then

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = (2u)(-2v) + (2v)(2u) = 0.$$

(h) $x = u/(u^2 + v^2)$, $y = v/(u^2 + v^2)$.

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = \frac{v^2 - u^2}{(u^2+v^2)^2} \times \frac{(-2uv)}{(u^2+v^2)^2} + \frac{(-2uv)}{(u^2+v^2)^2} \times \frac{u^2 - v^2}{(u^2+v^2)^2} = 0.$$

(i) $x = u^2 - v^2$, $y = -2uv$.

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = (2u)(-2v) + (-2v)(-2u) = 0.$$

30.8. In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.
(a) By the chain rule (30.1)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial}{\partial r}(r\cos\theta)\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial}{\partial \theta}(r\cos\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \cos\theta\frac{\mathrm{d}r}{\mathrm{d}t} - r\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial}{\partial r}(r\sin\theta)\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial}{\partial \theta}(r\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sin\theta\frac{\mathrm{d}r}{\mathrm{d}t} + r\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

(b) Differentiate $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}y/\mathrm{d}t$ in (a) with respect to t:

$$\begin{aligned}
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta)\frac{\mathrm{d}r}{\mathrm{d}t} + \cos\theta\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - \frac{\mathrm{d}}{\mathrm{d}t}(r\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} - r\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \\
&= -\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} + \cos\theta\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - \sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} - r\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - r\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \\
&= -2\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} + \cos\theta\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\cos\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - r\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}.
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta)\frac{\mathrm{d}r}{\mathrm{d}t} + \sin\theta\frac{\mathrm{d}^2r}{\mathrm{d}t^2} + \frac{\mathrm{d}}{\mathrm{d}t}(r\cos\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} + r\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \\
&= 2\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} + \sin\theta\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + r\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}
\end{aligned}$$

(c) The result

$$\cos\theta\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \sin\theta\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$$

can be verified using (b). The second result follows since, from (b),

$$\cos\theta\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \sin\theta\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}t} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right).$$

30.9. (a) $f(x,y) = 2x - y$, $x = uv$, $y = u^2 - v^2$. By the chain rule (30.6)

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2v + (-1)(2u) = 2v - 2u.$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 2u + 2v.$$

(b) $f(x,y) = y/x$, $x = u + v$, $y = u - v$. By the chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = -\frac{y}{x^2} + \frac{1}{x} = -\frac{u-v}{(u+v)^2} + \frac{1}{u+v} = \frac{2v}{(u+v)^2}.$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -\frac{y}{x^2} - \frac{1}{x} = -\frac{u-v}{(u+v)^2} - \frac{1}{u+v} = \frac{-2u}{(u+v)^2}.$$

(c) $f(x,y) = y^2$, $x = u^2 + v^2$, $y = v/u$. By the chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 0 + 2y\left(-\frac{v}{u^2}\right) = -\frac{2v^2}{u^3}.$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 0 + 2y\frac{1}{u} = \frac{2v}{u^2}.$$

(d) $f(x,y) = (x-y)/(x+y)$, $x = v$, $y = u - v$. By the chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{2y}{(x+y)^2} \times 0 + \frac{2x}{(x+y)^2} \times 1 = \frac{2x}{(x+y)^2} = \frac{2v}{u^2}.$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = \frac{2y}{(x+y)^2} \times 1 + \frac{2x}{(x+y)^2} \times (-1) = \frac{2u - 4v}{u^2}.$$

30.10 Use the chain rule (30.6) twice:

$$\begin{aligned}\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right] \\ &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial u} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u^2}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right) \\ &= \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial v^2} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial v^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial u \partial v} &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right) \\ &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u \partial v} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u \partial v}.\end{aligned}$$

(a) $f(x,y) = y/x$, $x = u + v$, $y = u - v$. We require the following first and second derivatives

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2} = -\frac{u-v}{(u+v)^2}, \quad \frac{\partial f}{\partial y} = \frac{1}{x} = \frac{1}{u+v},$$

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = 1, \quad \frac{\partial y}{\partial v} = -1,$$

$$\frac{\partial^2 x}{\partial u^2} = 0, \quad \frac{\partial^2 x}{\partial v^2} = 0, \quad \frac{\partial^2 x}{\partial u \partial v} = 0, \quad \frac{\partial^2 y}{\partial u^2} = 0, \quad \frac{\partial^2 y}{\partial v^2} = 0, \quad \frac{\partial^2 y}{\partial u \partial v} = 0,$$

Using the formulae above

$$\begin{aligned}
\frac{\partial^2 f}{\partial u^2} &= \left[\frac{\partial}{\partial u}\left(-\frac{u-v}{(u+v)^2}\right) \times 1\right] - \left[\frac{u-v}{(u+v)^2} \times 0\right] \\
&\quad + \left[\frac{\partial}{\partial u}\left(\frac{1}{u+v}\right) \times 1\right] + \left[\frac{1}{u+v} \times 0\right] \\
&= \frac{u-3v}{(u+v)^3} - \frac{1}{(u+v)^2} = -\frac{4v}{(u+v)^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial v^2} &= \left[\frac{\partial}{\partial v}\left(-\frac{u-v}{(u+v)^2}\right) \times 1\right] - \left[\frac{u-v}{(u+v)^2} \times 0\right] \\
&\quad + \left[\frac{\partial}{\partial v}\left(\frac{1}{u+v}\right) \times (-1)\right] + \left[\frac{1}{u+v} \times 0\right] \\
&= \frac{3u-v}{(u+v)^3} + \frac{1}{(u+v)^2} = \frac{4u}{(u+v)^3}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial u \partial v} &= \left[\frac{\partial}{\partial u}\left(-\frac{u-v}{(u+v)^2}\right) \times 1\right] - \left[\frac{u-v}{(u+v)^2} \times 0\right] \\
&\quad + \frac{\partial}{\partial u}\left(\frac{1}{u+v}\right) \times (-1) + \frac{1}{u+v} \times 0 \\
&= \frac{u-3v}{(u+v)^3} + \frac{1}{(u+v)^2} = \frac{2u-2v}{(u+v)^3}
\end{aligned}$$

(b) $f(x,y) = x^2 + y^2$, $x = uv$, $y = u^2 - v^2$. We require the following first and second derivatives

$$\frac{\partial f}{\partial x} = 2x = 2uv, \quad \frac{\partial f}{\partial y} = 2y = 2u^2 - 2v^2,$$

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u, \quad \frac{\partial y}{\partial u} = 2u, \quad \frac{\partial y}{\partial v} = -2v,$$

$$\frac{\partial^2 x}{\partial u^2} = 0, \quad \frac{\partial^2 x}{\partial v^2} = 0, \quad \frac{\partial^2 x}{\partial u \partial v} = 1, \quad \frac{\partial^2 y}{\partial u^2} = 2, \quad \frac{\partial^2 y}{\partial v^2} = -2, \quad \frac{\partial^2 y}{\partial u \partial v} = 0,$$

Using the formulae above

$$\begin{aligned}
\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u}(2uv) \times v - (2uv \times 0) + \frac{\partial}{\partial u}(2u^2 - 2v^2)(2u) + [(2u^2 - 2v^2) \times 2] \\
&= 12u^2 - 2v^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v}(2uv)u - (2uv \times 0) + \frac{\partial}{\partial v}(2u^2 - 2v^2))(-2v) + [(2u^2 - 2v^2) \times (-2)] \\
&= -2u^2 + 12v^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial u \partial v} &= \frac{\partial}{\partial u}(2uv)u + (2uv \times 1) + \frac{\partial}{\partial u}(2u^2 - 2v^2)(-2v) + [(2u^2 - 2v^2) \times 0] \\
&= -4uv.
\end{aligned}$$

(c) $f(x,y)=y^2$, $x=uv$, $y=v$. We require the following first and second derivatives

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=2y=2v,$$

$$\frac{\partial x}{\partial u}=v,\quad \frac{\partial x}{\partial v}=u,\quad \frac{\partial y}{\partial u}=0,\quad \frac{\partial y}{\partial v}=1,$$

$$\frac{\partial^2 x}{\partial u^2}=0,\quad \frac{\partial^2 x}{\partial v^2}=0,\quad \frac{\partial^2 x}{\partial u\partial v}=1,\quad \frac{\partial^2 y}{\partial u^2}=0,\quad \frac{\partial^2 y}{\partial v^2}=0,\quad \frac{\partial^2 y}{\partial u\partial v}=0,$$

Using the formulae above

$$\frac{\partial^2 f}{\partial u^2}=\frac{\partial}{\partial u}(0)v+(0)(0)+\frac{\partial}{\partial u}(2v)(0)+(0)(0)=0,$$

$$\frac{\partial^2 f}{\partial v^2}=\frac{\partial}{\partial v}(0)u+(0)(0)+\frac{\partial}{\partial v}(2v)(1)+2v(0)=2,$$

$$\frac{\partial^2 f}{\partial u\partial v}=\frac{\partial}{\partial u}(0)u+(0)(1)+\frac{\partial}{\partial u}(2v)(1)+2v(0)=0.$$

(*Comment*: whilst the second-order chain rule is important for theoretical reasons, it is generally not the simplest method of obtaining the u and v derivatives. Direct differentiation is often much quicker in explicit cases. For example in (b),

$$f(x,y)=x^2+y^2=u^2v^2+(u^2-v^2)^2.$$

It can be shown easily that

$$\frac{\partial f}{\partial u}=2uv^2+4u(u^2-v^2),\quad \frac{\partial f}{\partial v}=2u^2v-4v(u^2-v^2),$$

$$\frac{\partial^2 f}{\partial u^2}=12u^2-2v^2,\quad \frac{\partial^2 f}{\partial v^2}=-2u^2+12v^2,\quad \frac{\partial^2 f}{\partial u\partial v}=-4uv.)$$

30.11. $f(x,y)=g(x^2-y^2)$, $x=u+v$, $y=u-v$. Using the chain rule (30.6),

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\\ &= g'(x^2-y^2)(2x)(1)+g'(x^2-y^2)(-2y)(1)\\ &= g'(4uv)2(u+v)+g'(4uv)(-2u+2v)=4vg'(4uv),\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\\ &= g'(x^2-y^2)(2x)(1)+g'(x^2-y^2)(-2y)(-1)=4ug'(4uv),\end{aligned}$$

$$\frac{\partial^2 f}{\partial u^2}=\frac{\partial}{\partial u}[4vg'(4uv)]=16v^2g''(4v),\quad \frac{\partial^2 f}{\partial v^2}=\frac{\partial}{\partial v}[4ug'(4uv)]=16u^2g''(4uv),$$

$$\frac{\partial^2 f}{\partial u\partial v}=\frac{\partial}{\partial u}[4ug'(4uv)]=4g'(4uv)+16uvg''(uv).$$

30.12. Given

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},\tag{i}$$

differentiate the first equation in (i) with respect to x and the second equation with respect to y so that

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y},\quad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x}.\tag{ii}$$

Since the mixed derivatives

$$\frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 v}{\partial y \partial x}$$

are identical, elimination of them in (ii) leads to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{iii}$$

The other equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{iv}$$

can be obtained similarly by differentiating the first equation in (i) with respect to y and the second equation with respect to x, and eliminating the mixed derivatives.

By (30.6),

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}, \tag{v}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}. \tag{vi}$$

Differentiate (v) with respect to x and (vi) with respect to y:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right)\frac{\partial v}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

Add these equations noting the results (i), (iii) and (iv):

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right)\right]\frac{\partial u}{\partial x} + \left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right) - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)\right]\frac{\partial u}{\partial y}.$$

Apply the chain rule (30.6) to the terms in the square brackets in the previous equation:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right) &= \frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial x} + \frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y} \\
&= \frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}\right) - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right) &= \frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial y} - \frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial x} - \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x} \\
&= \frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} - \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x},
\end{aligned}$$

using (i). Hence

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= \frac{\partial^2 w}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2 w}{\partial v^2}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \frac{\partial^2 w}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 \\
&\quad - \frac{\partial^2 w}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) \\
&= \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\left[\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right].
\end{aligned}$$

using (i) again. (Note that the right-hand side in the text is incorrect.)

30.13. Given $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$.
(a) Using the chain rule (30.6)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta,$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta.$$

Then

$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \left(-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right)^2 \\ &= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\end{aligned}$$

as required.
(b) By using the chain rule:

$$\frac{\partial z}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial z}{\partial \theta},$$
$$\frac{\partial z}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial z}{\partial \theta},$$

Hence

$$\frac{\partial^2 z}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial z}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial z}{\partial \theta}\right), \qquad \text{(i)}$$

and

$$\frac{\partial^2 z}{\partial y^2} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial z}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial z}{\partial \theta}\right). \qquad \text{(ii)}$$

The result

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}$$

follows by evaluating the derivatives $\partial/\partial r$ and $\partial/\partial\theta$ on the right-hand sides of (i) and (ii) and adding the results.

Chapter 31: Functions of any number of variables

31.1. The incremental formula for $f(x, y, z, \ldots)$ is (see (31.1))

$$\delta f \approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z.$$

(a) $f(x, y, z) = 2x + 3y^2 + 4z^2 - 3$. The incremental approximation is

$$\delta f \approx 2\delta x + 6y\delta y + 8z\delta z.$$

(b) $f(x, y, t) = (x^2 + y^2)^{-\frac{1}{2}}\mathrm{e}^{-t}$, The incremental approximation is

$$\delta f \approx -x(x^2 + y^2)^{-\frac{3}{2}}\mathrm{e}^{-t}\delta x - y(x^2 + y^2)^{-\frac{3}{2}}\mathrm{e}^{-t}\delta y - (x^2 + y^2)^{-\frac{1}{2}}\mathrm{e}^{-t}\delta t.$$

(c) $f(r, \theta, t) = \mathrm{e}^{-t}r\cos\theta$. The incremental approximation is

$$\delta f \approx \mathrm{e}^{-t}\cos\theta\delta r - \mathrm{e}^{-t}r\sin\theta\delta\theta - \mathrm{e}^{-t}r\cos\theta\delta t.$$

(d) $f(x, y, z, t) = x^2 + y^2 + z^2 - t^2$. The incremental approximation is

$$\delta f \approx 2x\delta x + 2y\delta y + 2z\delta z - 2t\delta t.$$

(e) $f(x_1, y_1, x_2, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$. The incremental approximation is

$$\delta f \approx 2(x_1 - x_2)\delta x_1 + 2(y_1 - y_2)\delta y_1 - (x_1 - x_2)\delta x_2 - 2(y_1 - y_2)\delta y_2.$$

(f) $f(x, y, t) = (1/r)\mathrm{e}^{-(x^2+y^2)/t}$, $r = (x^2 + y^2)^{\frac{1}{2}}$. The incremental approximation can be expressed as

$$\delta f \approx \mathrm{e}^{-(x^2+y^2)/t}\left[\frac{t + 2x^2 + 2y^2}{t(x^2 + y^2)^{\frac{3}{2}}}(-x\delta x - y\delta y) + \frac{(x^2 + y^2)^{\frac{1}{2}}}{t^2}\delta t\right].$$

If $g(r, t) = (1/r)\mathrm{e}^{-r^2/t}$, then

$$\delta g \approx \left[-\frac{t + 2r^2}{tr^2}\right]\mathrm{e}^{-r^2/t}\delta r + \frac{r}{t^2}\mathrm{e}^{r^2/t}\delta t.$$

The increments δf and δg can be compared noting that $r\delta r = x\delta x + y\delta y$.

31.2 The distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d(x_1, y_1, z_1, x_2, y_2, z_2) = \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]},$$

which is a function of six variables. The first derivatives of d are

$$\frac{\partial d}{\partial x_1} = \frac{x_1 - x_2}{d}, \quad \frac{\partial d}{\partial y_1} = \frac{y_1 - y_2}{d}, \quad \frac{\partial d}{\partial z_1} = \frac{z_1 - z_2}{d},$$
$$\frac{\partial d}{\partial x_2} = -\frac{x_1 - x_2}{d}, \quad \frac{\partial d}{\partial y_2} = -\frac{y_1 - y_2}{d}, \quad \frac{\partial d}{\partial z_2} = -\frac{z_1 - z_2}{d},$$

From (31.1),

$$\begin{aligned}\delta d &\approx \frac{\partial d}{\partial x_1}\delta x_1 + \frac{\partial d}{\partial y_1}\delta y_1 + \frac{\partial d}{\partial z_1}\delta z_1 + \frac{\partial d}{\partial x_2}\delta x_2 + \frac{\partial d}{\partial y_2}\delta y_2 + \frac{\partial d}{\partial z_2}\delta z_2 \\ &= \frac{1}{d}[(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) + (z_1 - z_2)(\delta z_1 - \delta z_2)]\end{aligned}$$

Using the data

$$(x_1, y_1, z_1) = (1, 1, 2), \quad (x_2, y_2, z_2) = (1, 2, 1),$$
$$(\delta x_1, \delta y_1, \delta z_1) = (0.1, -0.1, -0.2), \quad (\delta x_2, \delta y_2, \delta z_2) = (-0.1, 0.1, 0.1),$$

then $d = \sqrt{2}$ and

$$\delta d \approx \frac{1}{\sqrt{2}}[0 \times (0.1 + 0.1) + (-1) \times (-0.1 - 0.1) + 1 \times (-0.2 - 0.1)] \approx -0.07.$$

31.3. The resistance R is given by

$$\frac{1}{R} = \frac{1}{R_4} + \frac{R_1 + R_2}{R_1R_2 + R_2R_3 + R_3R_1}.$$

By (31.1), the incremental change in R due to changes in R_1, R_2, R_3 is given by

$$\begin{aligned}-\frac{1}{R^2}\delta R \approx\ & \frac{\delta R_1 + \delta R_2}{R_1R_2 + R_2R_3 + R_3R_1} \\ & -\frac{(R_1 + R_2)(R_2\delta R_1 + R_1\delta R_2 + R_3\delta R_2 + R_2\delta R_3 + R_1\delta R_3 + R_3\delta R_1)}{(R_1R_2 + R_2R_3 + R_3R_1)^2}\end{aligned}$$

The given data are $R_1 = 3$, $R_2 = 10$, $R_3 = 5$ and $R_4 = 10$, and increments $\delta R_1 = 0.2$, $\delta R_2 = -0.2$: R_4 does not change. We have to find the value of δR_3 which causes $\delta R = 0$. The formula simplifies since $\delta R_1 + \delta R_2 = 0$. Hence δR_3 is given by (we need not write down the denominator nor the factor $(R_1 + R_2)$)

$$10 \times 0.2 + 3 \times (-0.2) + 5 \times (-0.2) + 10\delta R_3 + 3\delta R_3 + 5 \times 0.2 = 1.4 + 13R_3 = 0$$

if $R_3 = -1.4/13 \approx -0.11$.

31.4. Consider the general function

$$f(a, b, c, x) = ax^3 - bx - c.$$

Then the incremental change in f due to changes in the coefficients is

$$\delta f \approx \frac{\partial f}{\partial a}\delta a + \frac{\partial f}{\partial b}\delta b + \frac{\partial f}{\partial c}\delta c + \frac{\partial f}{\partial x}\delta x = x^3\delta a - x\delta b - \delta c + (3ax^2 - b)\delta x.$$

Now use the data: $a = 2$, $b = 3$, $c = 45$, $x = 3$, $\delta a = 0.1$, $\delta b = -0.1$, $\delta c = 2$. We put $\delta f = 0$ and calculate δx. Hence

$$0 = 27 \times 0.1 - 3 \times (-0.1) - 2 + (3 \times 2 \times 9 - 3)\delta x.$$

Solving this equation $\delta x = -1/51 = -0.0196\ldots$. The approximate solution is $x = 2.98$ to 2 decimal places.

31.5. The small-error formula (31.3) for $w = f(x, y, z, \ldots)$ is

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z + \cdots.$$

In this question Δ stands for
(central value) -(exact value).

(a) $w = yz + zx + xy$, $x = 2(\pm 0.1)$, $y = 3(\pm 0.2)$, $z = 1(\pm 0.1)$. Then, at $(2, 3, 1)$

$$\Delta w \approx (z + y)\Delta x + (z + x)\Delta y + (y + x)\Delta z = 4\Delta x + 3\Delta y + 5\Delta z.$$

The maximum value of the magnitude of Δw occurs if $\Delta x = \pm 0.1$, $\Delta y = \pm 0.2$, $\Delta z = \pm 0.1$. Hence the maximum value of $|\Delta w|$ is given by

$$|\Delta w| \approx (4 \times 0.1) + (3 \times 0.2) + (5 \times 0.1) = 1.5.$$

The central estimate value is $w = 3 + 2 + 6 = 11$ giving a maximum error in w of about 14%.

(b) $w = (x - y)(y - z)(z - x)$, $x = 1(\pm 0.1)$, $y = 2(\pm 0.1)$, $z = 3(\pm 0.1)$. The first derivatives of w are

$$\frac{\partial w}{\partial x} = (y - z)(-2x + y + z), \quad \frac{\partial w}{\partial y} = (z - x)(x - 2y + z),$$

$$\frac{\partial w}{\partial z} = (x - y)(x + y - 2z).$$

Then, at $(1, 2, 3)$,

$$\begin{aligned} \Delta w &\approx (y - z)(-2x + y + z)\Delta x + (z - x)(x - 2y + z)\Delta y \\ &\quad +(x - y)(x + y - 2z)\Delta z \\ &= -3\Delta x + 0 \times \Delta y + 3\Delta z = -3\Delta x + 3\Delta z. \end{aligned}$$

(The contribution of Δy is of order $(\Delta y)^2$ in this case.) The maximum value of $|\Delta w|$ occurs if $\Delta x = \mp 0.1$, $\Delta y = \pm 0.1$, $\Delta z = \pm 0.1$ so that

$$|\Delta w| \approx 0.3 + 0.3 = 0.6.$$

The estimated value is $w = 2$ which means a maximum error in w of about 30%.

(c) $w = (x+y+z-t)^{-1}$, $x = 1.2$, $y = 2.9$, $z = 1.9$, $t = 2.1$ after rounding to 1 decimal place. In each case the rounding implies that $x = 1.2(\pm 0.05)$, $y = 2.9(\pm 0.05)$, $z = 1.9(\pm 0.05)$, $t = 2.1(\pm 0.05)$. The first derivatives of w are

$$\frac{\partial w}{\partial x} = -w^2, \quad \frac{\partial w}{\partial y} = -w^2, \quad \frac{\partial w}{\partial z} = -w^2, \quad \frac{\partial w}{\partial t} = w^2.$$

Hence

$$\Delta w = w^2(-\Delta x - \Delta y - \Delta z + \Delta t) = (3.9)^{-2}(-\Delta x - \Delta y - \Delta z + \Delta t).$$

The maximum value of $|\Delta w|$ occurs where $\Delta x = \Delta y = \Delta z = \mp 0.05$ and $\Delta t = \pm 0.05$, so that $|\Delta w| = 0.013$ since $w = 1/3.9 \approx 0.256$. The percentage error is about 5%.

31.6. (a) The cosine rule for a triangle ABC is

$$c^2 = a^2 + b^2 - 2ab\cos A$$

with $a = 2(\pm 0.1)$, $b = 4(\pm 0.1)$, $A = 135°$ or $\frac{3}{4}\pi(\pm 0.035)$ radians. The incremental formula (31.3) is

$$\Delta(c^2) \approx 2c\Delta c \approx (2a - 2b\cos A)\Delta a + (2b - 2a\cos A)\Delta b + 2ab\sin A\Delta A.$$

For the measured values $c = 2\sqrt{(5 + 2\sqrt{2})} = 5.60$. Hence

$$\begin{aligned}\Delta c &\approx \frac{1}{c}(a - b\cos A)\Delta a + (b - a\cos A)\Delta b + (ab\sin A)\Delta A \\ &= \left(\frac{1+\sqrt{2}}{\sqrt{(5+2\sqrt{2})}}\right)\Delta a + \left(\frac{4+\sqrt{2}}{2\sqrt{(5+2\sqrt{2})}}\right)\Delta b + \left(\frac{2\sqrt{2}}{\sqrt{(5+\sqrt{2})}}\right)\Delta A \\ &\approx 0.43\Delta a + 0.48\Delta b + 0.51\Delta A.\end{aligned}$$

The maximum error in $|\Delta c|$ occurs where $\Delta a = \pm 0.1$, $\Delta b = \pm 0.1$ and $\Delta A = \pm 0.035$. Hence the maximum value of $|\Delta c|$ is 0.11. The maximum percentage error is about 2% .

(b) $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$, $(x_1, y_1, z_1) = (1, 2, 1)$, $(x_2, y_2, z_2) = (2, 1, 1)$ rounded to 1 significant figure. Hence all coordinates have maximum errors (± 0.5). The incremental formula is

$$\begin{aligned}\Delta(d^2) = 2d\Delta d &\approx 2(x_1 - x_2)\Delta x_1 + 2(y_1 - y_2)\Delta y_1 + 2(z_1 - z_2)\Delta z_1 \\ &\quad -2(x_1 - x_2)\Delta x_2 - 2(y_1 - y_2)\Delta y_2 - 2(z_1 - z_2)\Delta z_2.\end{aligned}$$

At the rounded values, $d = \sqrt{2} = D$, say. Hence

$$\Delta d = \frac{1}{\sqrt{2}}[-\Delta x_1 + \Delta y_1 + \Delta x_2 - \Delta y_1].$$

Δd takes its maximum/minimum values when $\Delta x_1 = \mp 0.5$, $\Delta y_1 = \pm 0.5$, $\Delta x_2 = \pm 0.5$, $\Delta y_2 = \mp 0.5$, giving $\Delta d = \pm\sqrt{2}$. The range of possible values for d is given approximately by $D - \sqrt{2} = 0 \leq d \leq D + \sqrt{2}$, or $0 \leq d \leq 2.83$. (The *exact* range is $0 \leq d \leq 3$, so approximate calculation is quite good.) The greatest percentage error is $\pm 100\%$.

(c) Area $A = [s(s-a)(s-b)(s-c)]^{\frac{1}{2}}$, where $s = \frac{1}{2}(a+b+c)$, $a = 2(\pm 0.1)$, $b = 4(\pm 0.1)$, $c = 3(\pm 0.1)$. For the central values

$$A = A_0 = \sqrt{\frac{9}{2}\frac{5}{2}\frac{1}{2}\frac{3}{2}} = \frac{3}{4}\sqrt{15} = 2.9.$$

The incremental formulae are

$$\begin{aligned}2A\Delta A \approx\ & [(s-a)(s-b)(s-c) + s(s-b)(s-c) + s(s-a)(s-c) \\ & +s(s-a)(s-b)]\Delta s \\ & -s(s-b)(s-c)\Delta a - s(s-a)(s-c)\Delta b - s(s-a)(s-b)\Delta c\end{aligned}$$

$$\Delta s \approx \tfrac{1}{2}(\Delta a + \Delta b + \Delta c).$$

Hence

$$\begin{aligned}
\frac{3}{2}\sqrt{15}\Delta A &\approx \left[\frac{5}{2}\frac{1}{2}\frac{3}{2} + \frac{9}{2}\frac{1}{2}\frac{3}{2} + \frac{9}{2}\frac{5}{2}\frac{3}{2} + \frac{9}{2}\frac{5}{2}\frac{1}{2}\right]\frac{1}{2}(\Delta a + \Delta b + \Delta c) \\
&\quad -\frac{9}{2}\frac{1}{2}\frac{3}{2}\Delta a - \frac{9}{2}\frac{5}{2}\frac{3}{2}\Delta b - \frac{9}{2}\frac{5}{2}\frac{1}{2}\Delta c \\
&= \frac{111}{8}(\Delta a + \Delta b + \Delta c) - \frac{27}{8}\Delta a - \frac{135}{8}\Delta b - \frac{45}{8}\Delta c \\
&= \frac{3}{8}(28\Delta a - 8\Delta b + 22\Delta c)
\end{aligned}$$

Hence

$$\Delta A \approx \frac{1}{4\sqrt{15}}(28\Delta a - 8\Delta b + 22\Delta c).$$

The maximum value of $|\Delta A|$ occurs where $\Delta a = \Delta b = \Delta c = \pm 0.1$. Hence

$$\max|\Delta A| \approx \frac{1}{4\sqrt{15}} \times 0.1(28 + 8 + 22) = 0.374,$$

with $A_0 = 2.9$, giving the percentage error 13%.

31.7. Implicit differentiation: if $f(x, y, z, \ldots) = 0$ then

$$\frac{\partial y}{\partial x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y},$$

and similarly for any two other variables (see (31.6).

In this problem $f(x, y, z, w) - c = 0$ (the constant c does not affect the results).

(a) By the result above

$$\frac{\partial y}{\partial x} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}, \quad \frac{\partial x}{\partial y} = -\frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial x}.$$

In the product of these two derivatives the derivatives of f cancel leaving

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1.$$

(b) Using the result above three times,

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \left(-\frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial x}\right)\left(-\frac{\partial f}{\partial z}\Big/\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial z}\Big/\frac{\partial f}{\partial x} = -\frac{\partial x}{\partial z}.$$

(c) Using (b) and (a)

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial w}\frac{\partial w}{\partial x} = \left(-\frac{\partial x}{\partial z}\right)\left(-\frac{\partial z}{\partial x}\right) = 1.$$

(i) $x + 2y + 3z + 4w - 5 = 0$.

(a) Treating x as a function of the remaining variables, and y as a function of the remaining variables,

$$\frac{\partial x}{\partial y} = -2, \quad \frac{\partial y}{\partial x} = -\frac{1}{2},$$

confirming that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1.$$

(b) Similarly

$$\frac{\partial x}{\partial y} = -2, \quad \frac{\partial y}{\partial z} = -\frac{3}{2}, \quad \frac{\partial x}{\partial z} = -3$$

confirming the result.

(c) It follows that

$$\frac{\partial x}{\partial y} = -2, \quad \frac{\partial y}{\partial z} = -\frac{3}{2}, \quad \frac{\partial z}{\partial w} = -\frac{4}{3}, \quad \frac{\partial w}{\partial x} = -\frac{1}{4},$$

which confirms (c) above.

(ii) $xy^2z^3w - 1 = 0$.

(a) Differentiate the equation with respect to y treating x as a function of y, z and w, and then with respect to x treating y as a function of x, z and w

$$\frac{\partial x}{\partial y}y^2z^3w + x(2y)z^3w = 0, \quad y^2z^3w + x(2y)\frac{\partial y}{\partial x}z^3w = 0.$$

Hence

$$\frac{\partial x}{\partial y} = -\frac{2x}{y}, \quad \frac{\partial y}{\partial x} = -\frac{y}{x},$$

and result (a) follows.

(b) The partial derivatives are given by

$$\frac{\partial x}{\partial y} = -\frac{2x}{y}, \quad \frac{\partial y}{\partial z} = -\frac{3y}{2z}, \quad \frac{\partial x}{\partial z} = -\frac{3x}{z},$$

from which result (b) follows.

(c) The partial derivatives are given by

$$\frac{\partial x}{\partial y} = -\frac{2x}{y}, \quad \frac{\partial y}{\partial z} = -\frac{3y}{2z}, \quad \frac{\partial z}{\partial w} = -\frac{z}{3w}, \quad \frac{\partial w}{\partial x} = -\frac{w}{x}.$$

Hence

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial w}\frac{\partial w}{\partial x} = \left(-\frac{2x}{y}\right)\left(-\frac{3y}{2z}\right)\left(-\frac{z}{3w}\right)\left(-\frac{w}{x}\right) = 1.$$

31.8. The required partial derivatives can be deduced from the incremental formula

$$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

(a) $2x - 3y + 4z = 1$. This is a linear relation so that

$$2\delta x - 3\delta y + 4\delta z = 0, \text{ or } \delta z = -\tfrac{1}{2}\delta x + \tfrac{3}{4}\delta y.$$

Therefore, putting $\delta y = 0$ and $\delta x = 0$ successively,

$$\frac{\partial z}{\partial x} = -\frac{1}{2}, \quad \frac{\partial z}{\partial y} = \frac{3}{4}.$$

(b) $x^2 + y^2 + z^2 = 14$ at $(1, 2, -3)$. The incremental formula gives

$$2x\delta x + 2y\delta y + 2z\delta z = 0.$$

Therefore

$$\delta z = -\frac{x}{z}\delta x - \frac{y}{z}\delta y.$$

By putting $\delta y = 0$ and $\delta x = 0$ we obtain respectively:

$$\frac{\partial z}{\partial x} = -\frac{x}{z} = \frac{1}{3}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z} = \frac{2}{3}$$

at $(1, 2, -3)$.

(c) $4x^3 + y^4 + 9z^3 - xyz^2 = 13$ at $(1, 1, 1)$. The incremental formula is

$$12x^2\delta x + 4y^3\delta y + 27z^2\delta z - yz^2\delta x - xz^2\delta y - 2xyz\delta z = 0.$$

Solving for δz:

$$\delta z = \frac{-(12x^2 - yz^2)\delta x - (4y^3 - xz^2)\delta y}{27z^2 - 2xyz}.$$

As above

$$\frac{\partial z}{\partial x} = \frac{yz^2 - 12x^2}{27z^2 - 2xyz} = -\frac{11}{25}, \quad \frac{\partial z}{\partial y} = \frac{xz^2 - 4y^3}{27z^2 - 2xyz} = -\frac{3}{25}$$

at $(1, 1, 1)$.

(d) $x^2 - z^2 = 9$ at $x = 5$, $y = y_0$, $z = 4$. The incremental formula is

$$2x\delta x - 2z\delta z = 0.$$

Therefore $\delta z = (x/z)\delta x$, and

$$\frac{\partial z}{\partial x} = \frac{x}{z}, \quad \frac{\partial z}{\partial y} = 0.$$

31.9. (a) Given $f(x, y, z) = xy/z$ and $x = t$, $y = 4t$, $z = 2t$

Method 1: chain rule. Then, for $x = t$, $y = 4t$, $z = 2t$,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \left(\frac{y}{z} \times 1\right) + \left(\frac{x}{z} \times 4\right) - \left(\frac{xy}{z^2} \times 2\right) \\ &= \frac{4t}{2t} + 4\frac{t}{2t} - 2\frac{4t^2}{4t^2} = 2\end{aligned}$$

Method 2: direct substitution. With $x = t$, $y = 4t$, $z = 2t$, $f(x, y, z) = xy/z = 2t$. Therefore

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2,$$

agreeing with Method 1.

(b) $f(x, y, z) = \sin(xy/z)$.

Method 1: chain rule. For $x = t$, $y = 4t$, $z = 2t$,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \left(\frac{y}{z}\cos(xy/z) \times 1\right) + \left(\frac{x}{z}\cos(xy/z) \times 4\right) - \left(\frac{xy}{z^2}\cos(xy/z) \times 2\right) \\ &= 2\cos 2t + 2\cos 2t - 2\cos 2t = 2\cos 2t\end{aligned}$$

Method 2: direct substitution. With $x = t$, $y = 4t$, $z = 2t$, $f(x, y, z) = \sin(xy/z) = \sin 2t$. Therefore

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2\cos 2t,$$

which agrees with Method 1.

(c) $f(x, y, z) = g(xy/z)$. With $x = t$, $y = 4t$, $z = 2t$, $f(x, y, z) = g(2t)$. Therefore, by direct substitution,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 2g'(2t).$$

In case (b), on the path $g(xy/z) = \sin(xy/z) = \sin 2t$. Hence $g'(t) = 2\cos 2t$, which is the answer given in (b).

31.10. For cylindrical coordinates (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

(a) Using the chain rule (31.8),

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta, \qquad \text{(i)}$$

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}=-\frac{\partial f}{\partial x}r\sin\theta+\frac{\partial f}{\partial y}r\cos\theta, \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial z}=\frac{\partial f}{\partial z}.$$

(b) The solution of (i) and (ii) by elimination leads to

$$\frac{\partial f}{\partial x}=\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial \theta},$$

$$\frac{\partial f}{\partial y}=\sin\theta\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}.$$

(c) Using the operators

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta},$$

twice, it follows that

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)f\\ &= \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta}+\cos^2\theta\frac{\partial^2 f}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)f\\ &= \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta}+\sin^2\theta\frac{\partial^2 f}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial\theta\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}.\end{aligned}$$

The addition of these two equations gives the required answer:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

31.11. The vector function **grad** f is defined as (see (31.9))

$$\mathbf{grad}\, f=\hat{\mathbf{i}}\frac{\partial f}{\partial x}+\hat{\mathbf{j}}\frac{\partial f}{\partial y}+\hat{\mathbf{k}}\frac{\partial f}{\partial z}.$$

(a) $f(x,y,z)=x+y+z$. Then

$$\mathbf{grad}\, f=\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}.$$

(b) $f(x,y,z)=2x-3y+5z-6$. Then

$$\mathbf{grad}\, f=2\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}}.$$

(c) $f(x,y,z)=x^2+y^2+z^2$. Then

$$\mathbf{grad}\, f=2x\hat{\mathbf{i}}+2y\hat{\mathbf{j}}+2z\hat{\mathbf{k}}.$$

(d) $f(x,y,z)=x^3+3z^3-1$. Then

$$\mathbf{grad}\, f=3x^2\hat{\mathbf{i}}+9z^2\hat{\mathbf{k}}.$$

(e) $f(x,y,z) = x^2 - \frac{1}{4}y^2 + \frac{1}{9}z^2$. Then

$$\mathbf{grad}\, f = 2x\hat{\mathbf{i}} - \frac{1}{2}y * \hat{\mathbf{j}} + \frac{2}{9}z\hat{\mathbf{k}}.$$

(f) $f(x,y,z) = 1/r = (x^2+y^2+z^2)^{-\frac{1}{2}}$. Then

$$\mathbf{grad}\, f = -\frac{x}{r^3}\hat{\mathbf{i}} - \frac{y}{r^3}\hat{\mathbf{j}} - \frac{z}{r^3}\hat{\mathbf{k}},$$

which is a multiple $(1/r^3)$ of the position vector $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

31.12. Given a point P on the surface $f(x,y,z) = k$, then $\mathbf{n} = \mathbf{grad}\, f$ evaluated at P is normal to the surface at P (see Section 31.5). A unit vector normal to the surface at P is $\hat{\mathbf{n}} = \mathbf{grad}\, f/|\mathbf{grad}\, f|$.

(a)$f(x,y,z) \equiv x - 2y + z = 0$ at any point. A normal is $\mathbf{n} = \mathbf{grad}\, f = (1,-2,1)$ and a unit normal vector is

$$\hat{\mathbf{n}} = \frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \frac{(1,-2,1)}{\sqrt{(1+4+1)}} = \frac{1}{\sqrt{6}}(1,-2,1).$$

The surface in this case is a plane so that normals given by $\mathbf{grad}\, f$ at all points on the plane are parallel.

(b) $f(x,y,z) \equiv y^2 + z^2 = 2$ at any point. A normal is

$$\mathbf{n} = \mathbf{grad}\, f = (0, 2y, 2z),$$

and a unit normal is

$$\hat{\mathbf{n}} = \frac{1}{2\sqrt{(y^2+z^2)}}(0,2y,2z) = \frac{1}{\sqrt{2}}(0,y,z),$$

since $y^2 + z^2 = 2$ on the surface.

(c) $f(x,y,z) \equiv x^2 + y^2 + z^2 = 9$ at $(2,1,-2)$. A normal to the surface is

$$\mathbf{n} = \mathbf{grad}\, f = (2x, 2y, 2z) = (4, 2, -4),$$

and a unit normal is

$$\hat{\mathbf{n}} = \frac{1}{2\sqrt{(x^2+y^2+z^2)}}(2x,2y,2z) = \frac{1}{3}(2,1,-2)$$

at $(2,1,-2)$.

(d) $f(x,y,z) \equiv \frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{16}z^2 = 1$ at $(2,3,4)$. A normal is

$$\mathbf{n} = \mathbf{grad}\, f = (\tfrac{1}{2}x, \tfrac{2}{9}y, \tfrac{1}{8}z) = (1, \tfrac{2}{3}, \tfrac{1}{2}),$$

and a unit normal vector is

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{61}}(6,4,3)$$

at $(2,3,4)$.

(e) $f(x,y,z) \equiv x^3y + zx^3 = 5$ at $(1,2,3)$. A normal vector is

$$\mathbf{n} = \mathbf{grad}\, f = (3x^2y + 3zx^2, x^3, x^3) = (15,1,1),$$

and a unit normal is

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{227}}(15,1,1)$$

at $(2,3,4)$.

(f) $f(x,y,z) \equiv (1/x) + (1/y) + (1/z) = 1$ at $(2,3,6)$. A normal vector is

$$\mathbf{n} = \mathbf{grad}\, f = \left(-\frac{1}{x^2}, -\frac{1}{y^2}, -\frac{1}{z^2}\right) = (-\tfrac{1}{4}, -\tfrac{1}{9}, -\tfrac{1}{36})$$

and a unit normal vector is

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{\left[\frac{1}{16} + \frac{1}{81} + \frac{1}{1296}\right]}} \left(-\frac{1}{4}, -\frac{1}{9}, -\frac{1}{36}\right) = -\frac{\sqrt{2}}{14}(9, 4, 1)$$

at $(2, 3, 6)$.

(g) $f(x, y, z) \equiv (x^2 + 4y^2 - z^2)^{-1} = \frac{1}{16}$ at $(4, 1, 2)$. A normal vector is

$$\mathbf{n} = \mathbf{grad}\, f = \frac{1}{(x^2 + 4y^2 - z^2)^2}(-2x, -8y, 2z) = \frac{1}{64}(-2, -2, 1),$$

and a unit normal vector is

$$\hat{\mathbf{n}} = \left(-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$$

at $(4, 1, 2)$.

31.13. Given surfaces $f(x, y, z) = \alpha$ and $g(x, y, z) = \beta$, and a common point $P : (a, b, c)$, normals to the surfaces at the point are $\mathbf{n} = \mathbf{grad}\, f$ and $\mathbf{p} = \mathbf{grad}\, g$. The angle θ between the surfaces at P is the angle between the normal vectors. By (10.4),

$$\cos\theta = \frac{\mathbf{n} \cdot \mathbf{p}}{|\mathbf{n}||\mathbf{p}|}.$$

(a) $f(x, y, z) \equiv x^2 + y^2 + z^2 = 9$ and $g(x, y, z) \equiv x^2 - z^2 = (x - z)(x + z) = 0$ at $(2, 1, 2)$. The normal vectors are therefore

$$\mathbf{n} = \mathbf{grad}\, f = (2x, 2y, 2z) = (4, 2, 4), \quad \mathbf{p} = \mathbf{grad}\,(x^2 - z^2) = (4, 0, -4).$$

at $(2, 1, 2)$. Hence

$$\cos\theta = \frac{(4, 2, 4) \cdot (4, 0, -4)}{24\sqrt{2}} = 0,$$

which means that the surfaces meet at right angles.

(b) $f(x, y, z) \equiv x^2 - y^2 + z^2 = 1$, and $g(x, y, z) \equiv 2x - 3y + z = -1$ at $(2, 2, 1)$. The normal vectors are

$$\mathbf{n} = \mathbf{grad}\, f = (2x, -2y, 2z) = (4, -4, 2), \quad \mathbf{p} = \mathbf{grad}\, g = (2, -3, 1)$$

at $(2, 2, 1)$. The angle θ is given by

$$\cos\theta = \frac{(4, -4, 2) \cdot (2, -3, 1)}{6\sqrt{14}} = \frac{11}{3\sqrt{14}}.$$

Hence $\theta = 11.5°$.

(c) $f(x, y, z) = x^2 + y^2 - z^2 = 0$ and $g(x, y, z) = 3x + 4y + 5z = 50$ at $(3, 4, 5)$. The normal vectors are

$$\mathbf{n} = \mathbf{grad}\, f = (2x, 2y, -2z) = (6, 8, -10), \quad \mathbf{p} = \mathbf{grad}\, g = (3, 4, 5)$$

at $(3, 4, 5)$. The angle θ is given by

$$\cos\theta = \frac{(6, 8, 10) \cdot (3, 4, 5)}{\sqrt{200}\sqrt{50}} = 1.$$

Hence $\theta = 0°$, which means that the surfaces touch at $(3, 4, 5)$. The second surface is a plane and is therefore a tangent plane to the surface $x^2 + y^2 - z^2 = 0$.

31.14. (a) $f(x, y, z) = A\mathrm{e}^{\alpha(2x^2+4y^2+z^2)^{\frac{1}{2}}}$. Then

$$\mathbf{grad}\, f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{A\alpha\mathrm{e}^{\alpha(2x^2+4y^2+z^2)^{\frac{1}{2}}}}{(2x^2 + 4y^2 + z^2)^{\frac{1}{2}}}(2x, 4y, z).$$

The common factor does not affect the direction of the vector, which means that $(2x, 4y, z)$ is in the direction $\mathbf{grad}\, f$.

(b) Let $f(x, y, z) = g[u(x, y, z)]$. By the chain rule (28.4),

$$\begin{aligned}
\mathbf{grad}\, f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial g[u(x,y,z)]}{\partial x}, \frac{\partial g[u(x,y,z)]}{\partial y}, \frac{\partial g[u(x,y,z)]}{\partial z}\right) \\
&= \left(g'(u)\frac{\partial u}{\partial x}, g'(u)\frac{\partial u}{\partial y}, g'(u)\frac{\partial u}{\partial z}\right) \\
&= g'(u)\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \\
&= g'(u)\mathbf{grad}\, u
\end{aligned}$$

Hence $\mathbf{grad}\, u$ is in the same direction as $\mathbf{grad}\, f$ if $g'(u) > 0$, and in the opposite direction if $g'(u) < 0$.

31.15. The directional derivative of $f(x, y, z)$ in the direction $\hat{\mathbf{s}}$ is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \mathbf{grad}\, f.$$

(a) $f(x, y, z) = x + 2y + 3z$. The directional derivative is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot (1, 2, 3).$$

(b) $f(x, y, z) = x^2 - y^2 - 3z$. The directional derivative is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot (2x, -2y, -3).$$

(c) $f(x, y, z) = (x-1)^3 + y^3 + z^3$. The directional derivative is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot (3(x-1)^2, 3y^2, 3z^2).$$

31.16. The directional derivative of $f(x, y, z)$ in the direction $\hat{\mathbf{s}}$ is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \mathbf{grad}\, f.$$

In all problems $\hat{\mathbf{s}} = (\frac{1}{4}\sqrt{2}, \frac{1}{4}\sqrt{2}, \frac{1}{2}\sqrt{3})$ and the point is $(2, 3, 2)$. (a) $f(x, y, z) = x - y + 2z$. The gradient of f is

$$\mathbf{grad}\, f = (1, -1, 2)$$

at all points. Hence

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \mathbf{grad}\, f = (\tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}\sqrt{3}) \cdot (1, -1, 2) = \sqrt{3}.$$

(b) $f(x, y, z) = xy + yz + zx$. The gradient of f is

$$\mathbf{grad}\, f = (y + z, x + z, y + x) = (5, 4, 5)$$

at $(2, 3, 2)$. Hence

$$\frac{\mathrm{d}f}{\mathrm{d}s} = (\tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}\sqrt{3}) \cdot (5, 4, 5) = \tfrac{9\sqrt{2}}{4} + \tfrac{5\sqrt{3}}{2}.$$

(c) $f(x, y, z) = (xy + yz + zx)^2$. The gradient of f is

$$\mathbf{grad}\, f = 2(xy + yz + zx)(y + z, x + z, y + x) = (160, 128, 160)$$

at $(2,3,2)$. Hence

$$\frac{\mathrm{d}f}{\mathrm{d}s} = (\tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}\sqrt{3}) \cdot (160, 128, 160) = 72\sqrt{2} + 80\sqrt{3}.$$

(d) $f(x,y,z) = x^2 - y^2 + 5$. The gradient of f is

$$\mathbf{grad}\, f = (2x, -2y, 0) = (4, -6, 0)$$

at $(2,3,2)$. Hence

$$\frac{\mathrm{d}f}{\mathrm{d}s} = (\tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}\sqrt{3}) \cdot (4, -6, 0) = -\tfrac{1}{2}\sqrt{2}.$$

31.17. The normals to the two surfaces $f(x,y,z) = a$ and $g(x,y,z) = b$ are respectively $\mathbf{n} = \mathbf{grad}\, f$ and $\mathbf{p} = \mathbf{grad}\, g$. Consequently they are also respectively normal to any curves on the two surfaces. Hence, at any point P on the curve of intersection C, $\mathbf{n}$ and $\mathbf{p}$ are each perpendicular to C. By the property of the vector product (see Section 11.2), $\mathbf{n} \times \mathbf{p}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{p}$ and is therefore in the direction of the curve C. A unit vector $\hat{\mathbf{s}}$ in the direction of C is

$$\hat{\mathbf{s}} = \frac{\mathbf{n} \times \mathbf{p}}{|\mathbf{n} \times \mathbf{p}|}.$$

(a) $f(x,y,z) \equiv 2x + 3y - z = 1$, $g(x,y,z) \equiv x - y - z = 0$; these represent two planes and the intersection will be a straight line. The normal vectors are

$$\mathbf{n} = \mathbf{grad}\, f = (2, 3, -1), \quad \mathbf{p} = \mathbf{grad}\, g = (1, -1, -1).$$

Hence

$$\hat{\mathbf{s}} = \frac{\mathbf{n} \times \mathbf{p}}{|\mathbf{n} \times \mathbf{p}|} = \frac{(2, 3-1) \times (1,-1,-1)}{|(2, 3-1) \times (1,-1,-1)|} = \frac{1}{\sqrt{42}}(-4, 1, -5).$$

(b) $f(x,y,z) \equiv x + y = 0$, $g(x,y,z) \equiv x - z = 0$; these represent two planes and the line of intersection will be a straight line through the origin. The normal vectors are

$$\mathbf{n} = (1, 1, 0), \quad \mathbf{p} = (1, 0, -1).$$

Hence

$$\hat{\mathbf{s}} = \frac{\mathbf{n} \times \mathbf{p}}{|\mathbf{n} \times \mathbf{p}|} = \frac{(1,1,0) \times (1,0,-1)}{|(1,1,0) \times (1,0,-1)|} = \frac{1}{\sqrt{3}}(-1, 1, -1).$$

(c) $f(x,y,z) \equiv x^2 + y^2 + z^2 = 6$, $g(x,y,z) \equiv x - y + z = 0$ at $(1,2,1)$. The normal vectors are

$$\mathbf{n} = (2x, 2y, 2z) = (2, 4, 2), \quad \mathbf{p} = (1, -1, 1)$$

at $(1,2,1)$. Hence

$$\hat{\mathbf{s}} = \frac{\mathbf{n} \times \mathbf{p}}{|\mathbf{n} \times \mathbf{p}|} = \frac{(2,4,2) \times (1,-1,1)}{|(2,4,2) \times (1,-1,1)|} = \frac{1}{\sqrt{2}}(1, 0, -1).$$

(d) $f(x,y,z) \equiv x^2 + (y-1)^2 = 1$, $g(x,y,z) \equiv x^2 + (y-2)^2 = 4$ at $x = 0$, $y = 0$ and any value of z. The normal vectors are

$$\mathbf{n} = (2x, 2(y-1), 0) = (0, -2, 0), \quad \mathbf{p} = (2x, 2(y-2), 0) = (0, -4, 0)$$

at the given point. Hence

$$\hat{\mathbf{s}} = \frac{\mathbf{n} \times \mathbf{p}}{|\mathbf{n} \times \mathbf{p}|} = \frac{(0,-2,0) \times (0,-4,0)}{|(0,-2,0) \times (0,-4,0)|} = (0, 0, 0).$$

This is a case in which the method fails. The two surfaces are circular cylinders parallel to the z axis which touch along the line $x = 0$, $y = 0$. As we have seen the two normals are coincident along this line which makes the vector product zero. A unit vector in the direction of C is $\hat{\mathbf{s}} = (0, 0, 1)$.

(e) $f(x,y,z) \equiv xy + yz + zx = 3$, $g(x,y,z) \equiv x + y + z = 3$ at $(1,1,1)$. The normal vectors are

$$\mathbf{n} = (y+z, x+z, y+x) = (2,2,2), \quad \mathbf{p} = (1,1,1)$$

at $(1,1,1)$. These two vectors are parallel, so that we could have the case discussed in (d). We still need to find the direction of the curve of intersection if it exists. Any point on the plane can be represented by the two-parameter formula $x = 1+\alpha$, $y = 1+\beta$, $z = 1-\alpha-\beta$. Obviously the choice $\alpha = \beta = 0$ gives the common point. Points on the plane lie on the first surface if

$$(1+\alpha)(1+\beta) + (1+\beta)((1-\alpha-\beta) + (1+\alpha)(1-\alpha-\beta) = 3,$$

or

$$\alpha^2 + \beta^2 + \alpha\beta = 0, \text{ or } (\alpha + \tfrac{1}{2}\beta)^2 + \tfrac{3}{4}\beta^2 = 0.$$

The only solution of this equation is $\alpha = \beta = 0$, which means the surfaces meet only at one point.

31.18. The stationary values of $f(x,y,z,\ldots)$ occur at all simultaneous solutions of

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0, \quad \cdots.$$

(a) $f(x,y,z) = x^2 + y^2 + z^2$. We require all solutions of

$$\frac{\partial f}{\partial x} = 2x = 0, \quad \frac{\partial f}{\partial y} = 2y = 0, \quad \frac{\partial f}{\partial z} = 2z = 0.$$

The only solution is $x = y = z = 0$, which is the location of the stationary point.
(b) $f(x,y,z) = x^3 - 3x + y^3 - 3yz + 2z^2$. We require all solutions of

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0, \tag{i}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3z = 0, \tag{ii}$$

$$\frac{\partial f}{\partial z} = -3y + 4z = 0, \tag{iii}$$

From (i) $x = \pm 1$. From (iii) $z = \frac{3}{4}y$ so that eliminating z in (ii),

$$y^2 - \tfrac{3}{4}y = 0, \text{ or } y(y - \tfrac{3}{4}) = 0.$$

Hence $y = 0$ or $y = \frac{3}{4}$. Finally we ensure that all combinations of x and y are included: each x can be associated with each y, but z will be determined by the choice of y. Thus the stationary points occur at

$$(-1,0,0), \quad (-1, \tfrac{3}{4}, \tfrac{9}{16}), \quad (1,0,0), \quad (1, \tfrac{3}{4}, \tfrac{9}{16}).$$

(c) $f(x,y,z) = xy + yz + zx + y - z$. We require all solutions of

$$\frac{\partial f}{\partial x} = y + z = 0, \quad \frac{\partial f}{\partial y} = x + z + 1 = 0, \frac{\partial f}{\partial z} = y + x - 1 = 0.$$

Hence $z = -y$ and

$$x - y + 1 = 0, \quad x + y - 1 = 0.$$

Solving these equations, the function has one stationary point at $(0,1,-1)$.
(d) $f(x,y,z) = x/z + y/x + z/y$: note that the function is not defined on the axes including the origin. Stationary points occur at the solutions of

$$\frac{\partial f}{\partial x} = \frac{1}{z} - \frac{y}{x^2} = 0, \quad \frac{\partial f}{\partial y} = \frac{1}{x} - \frac{z}{y^2} = 0, \quad \frac{\partial f}{\partial z} = \frac{1}{y} - \frac{x}{z^2} = 0.$$

Hence

$$x^2 = yz, \quad y^2 = zx, \quad z^2 = xy,$$

from which it follows that $x^3 = y^3 = z^3$. There are two solutions (excluding the zero solutions) $x = y = z = 1$ and $x = y = z = -1$. Therefore there are two stationary points at $(1, 1, 1)$ and $(-1, -1, -1)$.

(e) $f(x, y, z, \lambda) = (x + y + z) - \lambda(x^2 + y^2 + z^2 - 1)$. Stationary points occur at the solutions of

$$\frac{\partial f}{\partial x} = 1 - 2\lambda x = 0, \quad \frac{\partial f}{\partial y} = 1 - 2\lambda y = 0, \tag{i}$$

$$\frac{\partial f}{\partial z} = 1 - 2\lambda z = 0, \quad \frac{\partial f}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0. \tag{ii}$$

From (i), $x = y = z = 1/(2\lambda)$, and substitution in (ii) gives

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1, \text{ or } \lambda^2 = \frac{3}{4}.$$

Hence $\lambda = \pm\frac{\sqrt{3}}{2}$. Therefore there are two stationary points, at

$$(\tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{2}) \text{ and } (-\tfrac{\sqrt{3}}{3}, -\tfrac{\sqrt{3}}{3}, -\tfrac{\sqrt{3}}{3}, -\tfrac{\sqrt{3}}{2}).$$

(f) $f(x, y, z) = x^4 + y^4 + z^4 - 2(x - y + z)^2$. Stationary points occur at the solutions of

$$4x^3 - 4(x - y + z) = 0, \tag{i}$$

$$4y^3 + 4(x - y + z) = 0, \tag{ii}$$

$$4z^3 - 4(x - y + z) = 0. \tag{iii}$$

From (i) and (ii), $y^3 = -x^3$. Hence $y = -x$. Substitution back into (i) leads to $z = x^3 - 2x$. Now substitute y and z in terms of x into (iii):

$$(x^3 - 2x)^3 - x^3 = 0, \text{ or } x^3(x^2 - 2)^3 - x^3 = 0.$$

Therefore either $x = 0$ or $(x^2 - 2)^3 = 1$. The solutions of the latter are $x = \pm\sqrt{3}$. Working backwards we are now in a position to list the stationary points: they are

$$(0, 0, 0), \quad (\sqrt{3}, -\sqrt{3}, \sqrt{3}), \quad (-\sqrt{3}, \sqrt{3}, -\sqrt{3}).$$

31.19. We require the stationary points of $f(x, y, z) = x^2 + y^2 + z^2$ on the path $x = \cos t$, $y = \sin t$, $z = \sin\frac{1}{2}t$ where $0 < t < 4\pi$. Using the chain rule (see (31.22))

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = -2x\sin t + 2y\cos t + 2z\tfrac{1}{2}\cos\tfrac{1}{2}t \\ &= -2\cos t\sin t + 2\sin t\cos t + \sin\tfrac{1}{2}t\cos\tfrac{1}{2}t = \tfrac{1}{2}\sin t \end{aligned}$$

Stationary points occur where $\mathrm{d}f/\mathrm{d}t = \frac{1}{2}\sin t = 0$, that is, at $t = \pi, 2\pi, 3\pi$ in the interval. On $f(x, y, z) = x^2 + y^2 + z^2$ these values of t correspond to the points $(-1, 0, 1)$, $(1, 0, 0)$ and $(-1, 0, -1)$.

Alternatively, substitute $x = \cos t$, $y = \sin t$, $z = \sin\frac{1}{2}t$, and find the stationary points of

$$f(x(t), y(t), z(t)) = \cos^2 t + \sin^2 t + \sin^2\tfrac{1}{2}t = 1 + \sin^2\tfrac{1}{2}t$$

treated as a function of one variable.

31.20. The concentration s at (x, y, z) is given by

$$s = C\exp\{-\alpha[2(x - 1)^2 + 4y^2 + z^2]\}.$$

(a) The gradient of s is

$$\mathbf{grad}\, s = \alpha C \exp\{-\alpha[2(x-1)^2 + 4y^2 + z^2]\}(-4(x-1), -8y, -2z).$$

This gives the direction of steepest ascent so that steepest descent is minus this, namely, $(2(x-1), 4y, z)$.
(b) Suppose that the insect moves from its current point (x, y, z) to $(x+\delta x, y+\delta y, z+\delta z)$ along the steepest descent. The direction is $(\delta x, \delta y, \delta z)$ which must be the same as $(2(x-1), 4y, z)$. Hence all the components must be in the same ratio, which can be expressed as

$$\frac{\delta x}{2(x-1)} = \frac{\delta y}{4y} = \frac{\delta z}{z}.$$

(c) Dividing by δt, we can write the equations as

$$\frac{1}{2(x-1)}\frac{\delta x}{\delta t} = \frac{1}{z}\frac{\delta z}{\delta t}, \quad \frac{1}{4y}\frac{\delta y}{\delta t} = \frac{1}{z}\frac{\delta z}{\delta t}.$$

Let $\delta t \to 0$ so that

$$\frac{1}{2(x-1)}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}t}, \quad \frac{1}{4y}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

Since t does not appear explicitly in the equations, we can eliminate it from the equations and write them as

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z}{2(x-1}, \quad \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{z}{4y}.$$

(d) Both equations are of first-order separable type (see Section 22.3). Hence

$$\int \frac{\mathrm{d}z}{z} = \int \mathrm{d}x 2(x-1) + C$$

so that

$$\ln|z| = \frac{1}{2}\ln|x-1| + C, \text{ or } z^2 = M(x-1).$$

Also

$$\frac{\mathrm{d}z}{z} = \int \frac{\mathrm{d}y}{4y},$$

so that

$$\ln|z| = \frac{1}{4}\ln|y|, \text{ or } z^4 = Ny \text{ or } z = Ay^{\frac{1}{4}}.$$

Here M and N are arbitrary constants.
(e) The insect starts at $(0, 1, 1)$. From (a), the direction taken by the insect is $(-2, 4, 1)$. The path is given by the intersection of the surfaces

$$z^2 = M(x-1) \text{ and } z^4 = Ny.$$

From the initial condition, $M = -1$ and $N = 1$. Hence the path is given by $x = 1 - z^2$, $y = z^4$.

31.21. From (31.25), the stationary points of $f(x, y, z)$ subject to $g(x, y, z) = c$ are the solutions for x, y, z, λ of

$$g = c, \quad \frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0, \quad \frac{\partial f}{\partial z} - \lambda\frac{\partial g}{\partial z} = 0.$$

(a) $f(x, y, z) = x + y + z$ subject to $g(x, y, z) \equiv 1/x + 1/y + 1/z = 1$. The stationary points are solutions of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1, \tag{i}$$

$$1 + \frac{\lambda}{x^2} = 0, \quad 1 + \frac{\lambda}{y^2} = 0, \quad 1 + \frac{\lambda}{z^2} = 0. \tag{ii}$$

From (ii), $-\lambda = x^2 = y^2 = z^2$. Hence, independently, $y = \pm x$ and $z = \pm z$. Substituting y and z into (i), the four cases give:

$$y = x,\ z = x \Rightarrow x = 3,\ y = 3,\ z = 3,$$

$$y = x,\ z = -x \Rightarrow x = 1,\ y = 1,\ z = -1,$$

$$y = -x,\ z = x \Rightarrow x = 1,\ y = -1,\ z = 1,$$

$$y = -x,\ z = -x \Rightarrow x = -1,\ y = 1,\ z = 1.$$

Hence there are stationary points at $(3, 3, 3)$, $(1, 1, -1)$, $(1, -1, 1)$ and $(-1, 1, 1)$.

(b) $f(x, y, z) = xyz$ subject to $g(x, y, z) \equiv 1/x + 1/y + 1/z = 1$. The coordinates of the stationary points are solutions of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1, \tag{i}$$

$$yz + \frac{\lambda}{x^2} = 0, \quad zx + \frac{\lambda}{y^2} = 0, \quad xy + \frac{\lambda}{z^2} = 0. \tag{ii}$$

From (ii), $x^2yz = xy^2z = xyz^2 = -\lambda$. By elimination of λ, it follows that $x = y = z$. Hence from (i), $x = y = z = 3$. The function has one stationary value at $(3, 3, 3)$.

(c) $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g(x, y, z) \equiv ax + by + cz = 1$. The coordinates of the stationary points are solutions of

$$ax + by + cz = 1, \tag{i}$$

$$2x - \lambda a = 0, \quad 2y - \lambda b = 0, \quad 2z - \lambda c = 0. \tag{ii}$$

Substitute x, y and z from (ii) into (i):

$$\tfrac{1}{2}\lambda a^2 + \tfrac{1}{2}\lambda b^2 + \tfrac{1}{2}\lambda c^2 = 1.$$

Therefore

$$\lambda = \frac{2}{a^2 + b^2 + c^2},$$

and the coordinates of the only stationary value are

$$x = \frac{1}{2}\lambda a = \frac{a}{a^2 + b^2 + c^2}, \quad y = \frac{1}{2}\lambda b = \frac{b}{a^2 + b^2 + c^2}, \quad z = \frac{1}{2}\lambda a = \frac{c}{a^2 + b^2 + c^2}.$$

(d) $f(x, y, z) = xy + yz + zx$ subject to $g(x, y, z) \equiv xyz = 1$. The coordinates of the stationary points are solutions of

$$xyz = 1, \tag{i}$$

$$y + z - \lambda yz = 0, \quad z + x - \lambda zx = 0, \quad x + y - \lambda xy = 0.$$

Hence, from (ii),

$$\lambda = \frac{y + z}{yz} = \frac{z + y}{zx} = \frac{x + y}{xy}.$$

Therefore $x = y = z$ which means by (i) that $x = y = z = 1$. The stationary point is at $(1, 1, 1)$. This means that the block of given volume of smallest surface area is a cube.

(e) $f(x, y, z) = xyz$ subject to $g(x, y, z) \equiv x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. The coordinates of the stationary points are solutions of

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1, \tag{i}$$

$$yz = \frac{2\lambda x}{a^2}, \quad zx = \frac{2\lambda y}{b^2}, \quad xy = \frac{2\lambda z}{c^2}.$$

From (ii)

$$\lambda = \frac{a^2yz}{2x} = \frac{b^2zx}{2y} = \frac{c^2xy}{2z}.$$

Hence

$$\frac{x^2}{y^2} = \frac{a^2}{b^2}, \quad \frac{z^2}{x^2} = \frac{c^2}{a^2},$$

so that from (i), $3x^2 = a^2$ or $x = \pm a/\sqrt{3}$. Further $y = \pm b/\sqrt{3}$ and $z = \pm c/\sqrt{3}$. There are eight stationary points, at all sign combinations in

$$\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right).$$

(f) $f(x, y, z) = x^2 + 4y^2 + z^2$ subject to $g(x, y, z) \equiv x - y - 2z = 0$ and $h(x, y, z) \equiv z = 1$. This is a Lagrange-multiplier problem with two constraints and uses (31.26). The coordinates of the stationary points are solutions of

$$x - y - 2z = 0, \quad z = 1, \tag{i}$$

$$2x - \lambda, \quad 8y + \lambda = 0, \quad 2z + 2\lambda = 0, \quad 2z + 2\lambda - \mu = 0. \tag{ii}$$

From (ii),

$$x = \tfrac{1}{2}\lambda, \quad y = -\tfrac{1}{8}\lambda, \quad z = -\lambda + \tfrac{1}{2}\mu.$$

Substituting into (i)

$$\frac{1}{2}\lambda + \frac{1}{8}\lambda + 2\lambda - \mu = 0, \text{ or } -\frac{11}{8}\lambda - \mu = 0,$$

and

$$-\lambda + \frac{1}{2}\mu = 1.$$

The solution of these equations is $\lambda = -\frac{16}{27}$, $\mu = \frac{22}{27}$. There is, therefore, one stationary point, at

$$x = \tfrac{1}{2}\lambda = -\tfrac{8}{27}, \quad y = -\tfrac{1}{8}\lambda = \tfrac{2}{16}, \quad z = -\lambda + \tfrac{1}{2}\mu = 1.$$

(g) $f(x, y, z) = x^2 - y^2 - z^2$ subject to $(x - 1)/2 = (y - 2)/(-1) = (z - 2)/3$. The straight line defines two constraints

$$\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z-2}{3},$$

which can be expressed in the forms

$$g(x, y, z) \equiv x + 2y = 5 \text{ and } h(x, y, z) \equiv 3y + z = 8.$$

The coordinates of any stationary points are solutions of

$$x + 2y = 5, \quad 3y + z = 8, \tag{i}$$

$$2x - \lambda = 0, \quad -2y - 2\lambda - 3\mu = 0, \quad -2z - \mu = 0. \tag{ii}$$

From (ii)

$$x = \tfrac{1}{2}\lambda, \quad y = -\lambda - \tfrac{3}{2}\mu, \quad z = -\tfrac{1}{2}\mu.$$

Substsituting these into (i), we have

$$-3\lambda - 6\mu = 10, \quad -3\lambda - 5\mu = 8.$$

Hence $\lambda = \frac{2}{3}$ and $\mu = -2$. Therefore there is one stationary point, with coordinates

$$x = \tfrac{1}{3}, \quad y = \tfrac{7}{3}, \quad z = 1.$$

(h) $f(x,y,z) = xyz$ subject to $xy + yz + zx = 1$. The coordinates of stationary ponts are solutions of

$$xy + yz + zx = 1, \tag{i}$$

$$yz - \lambda(y+z) = 0, \quad zx - \lambda(z+x) = 0, \quad xy - \lambda(x+y) = 0. \tag{ii}$$

From (ii),

$$\lambda = \frac{yz}{y+z} = \frac{zx}{z+x} = \frac{xy}{x+y},$$

from which it follows that $x = y = z$. Finally from (i), $x = y = z = 1/\sqrt{3}$ which are the coordinates of the only stationary point.

(i) $f(x,y,z) = x - y - 2z$, subject to $g(x,y,z) \equiv z = 1$ and $h(x,y,z) \equiv x^2 + 4y^2 + z^2 = 6$. The coordinates of any stationary points are solutions of

$$z = 1, \quad x^2 + 4y^2 + z^2 = 6, \tag{i}$$

$$1 - 2x\mu = 0, \quad -1 - 8y\mu = 0, \quad -2 - \lambda z - 2z\mu. \tag{ii}$$

From (ii),

$$x = \frac{1}{2\mu}, \quad y = -\frac{1}{8\mu}, \quad z = -\frac{1}{2\mu}(2+\lambda).$$

The restrictions in (i) imply

$$\lambda + 2\mu = -2, \quad \frac{1}{4\mu^2} + \frac{4}{16\mu^2} + \frac{1}{4\mu^2}(\lambda+2)^2 = 6.$$

The elimination of λ between these equations leads to $\mu = \frac{1}{4}$, $\lambda = -\frac{5}{2}$ or $\mu = -\frac{1}{4}$, $\lambda = -\frac{3}{2}$. Hence the function has two stationary values at

$$(2, -\tfrac{1}{2}, 1) \text{ and } (-2, \tfrac{1}{2}, \tfrac{1}{2}).$$

31.22. (a) The following *Mathematica* program reproduces the table in Example 31.6, and can be adapted to other two- and three-dimensional steepest ascent problems.

```
<< Calculus`VectorAnalysis`
Clear[f, x, y, h]
f[x_, y_, z_] = 4 - x^2 - (1/2)y^2 - (1/2)z^2;
h = 0.05; a[0] = 1; b[0] = 1; c[0] = 1;
SetCoordinates[Cartesian[x, y, z]];
u[x_, y_, z_] = Grad[f[x, y, z]]/Sqrt[Grad[f[x, y, z]].Grad[f[x, y, z]]]
a[n_] := a[n] = a[n - 1] + h*Part[u[a[n - 1], b[n - 1], c[n - 1]], 1]
b[n_] := b[n] = b[n - 1] + h*Part[u[a[n - 1], b[n - 1], c[n - 1]], 2]
c[n_] := c[n] = c[n - 1] + h*Part[u[a[n - 1], b[n - 1], c[n - 1]], 3]
steepest = Table[i, a[i], b[i], c[i], i, 0, 5] // N
MatrixForm[%]
```

(b),(c) The following *Mathematica* program solves the problem of the steepest ascent up the hill with altitude $H = 0.5 - x^2 - 4y^2$.

```
<<Calculus`VectorAnalysis`
Clear[f, x, y, u, a, b, h]
f[x_, y_] = 0.5 - x^2 - 4*y^2; h = 0.2;
a[0] = 2; b[0] = 2;
u[x_, y_] = {D[f[x, y], x], D[f[x, y], y]}/Sqrt[(D[f[x, y], x])^2 + (D[f[x, y], y]^2];
a[n_] := a[n] = a[n - 1] + h*Part[u[a[n - 1], b[n - 1]], 1]
b[n_] := b[n] = b[n - 1] + h*Part[u[a[n - 1], b[n - 1]], 2]
```

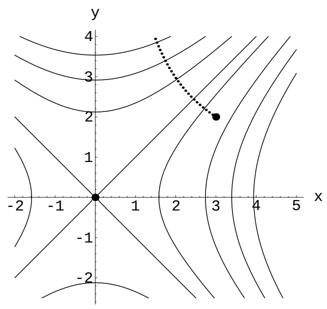

Figure 17: Problem 31.22(d)

steepest = Table[{i, a[i], b[i]}, {i, 0, 15}] // N;
MatrixForm[%]

(d) In this case the altitude is given by $H = 0.5 + x^2 - y^2$. The figure shows contours of the surface which has a saddle point at the origin. The steepest descent is shown starting at the point $(3, 2)$.
(e) Streamlines derived from the potential

$$\phi(x, y) = x\left(1 + \frac{1}{x^2 + y^2}\right)$$

are given by

$$y\left(1 - \frac{1}{x^2 + y^2}\right) = \text{constant}.$$

Some streamlines outside the circle $x^2 + y^2 = 1$ are shown in the figure.

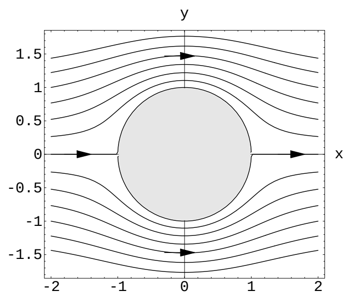

Figure 18: Problem 31.22(e)

31.23. This problem is concerned with functions of the form

$$f(x, y, z) = g(u(x, y, z)).$$

(a) Examples of functions of a function:

$$w = (xy - z)\mathrm{e}^{-xy+z}, \text{ or } w = u\mathrm{e}^{-u},\ u = xy - z;$$

$$w = (x^2 + y^2 + z^2)\cos[\sqrt{(x^2 + y^2 + z^2)}], \text{ or } w = r^2 \cos r,\ r = \sqrt{(x^2 + y^2 + z^2)}.$$

(b) In the first derivative y and z are effectively constant so that we can use the chain rule (3.3) for a function of one variable. Hence

$$\frac{\partial f}{\partial x} = g'(u)\frac{\partial u}{\partial x}, \text{ and similarly } \frac{\partial f}{\partial y} = g'(u)\frac{\partial u}{\partial y}, \quad \frac{\partial f}{\partial z} = g'(u)\frac{\partial u}{\partial z}.$$

(c) (i) $w = \mathrm{e}^{x^2-y^2+z^2}$. In this case $u(x,y,z) = x^2 - y^2 + z^2$ and $w = g(u) = \mathrm{e}^u$. Hence

$$g'(u) = \mathrm{e}^u, \text{ and } \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial u}{\partial z} = 2z.$$

By (b)

$$\frac{\partial w}{\partial x} = g'(u)\frac{\partial u}{\partial x} = \mathrm{e}^u 2x = 2x\mathrm{e}^{x^2-y^2+z^2}.$$

Similarly

$$\frac{\partial w}{\partial y} = -2y\mathrm{e}^{x^2-y^2+z^2}, \quad \frac{\partial w}{\partial z} = 2z\mathrm{e}^{x^2-y^2+z^2}.$$

(ii) $w = \sin(xy/z)$. In this case choose $u = xy/z$ so that $w = g(u) = \sin u$. Hence $g'(u) = \cos u$ and

$$\frac{\partial w}{\partial x} = \cos u \times \frac{y}{z} = \frac{y}{z}\cos(xy/z),$$

$$\frac{\partial w}{\partial y} = \frac{x}{z}\cos(xy/z), \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}\cos(xy/z).$$

(d) Assuming that $x = x(t)$, $y = y(t)$, $z = z(t)$, then

$$f(x,y,z) = g[u(x(t), y(t), z(t)],$$

and the chain rule can be expressed in the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = g'(u)\frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + g'(u)\frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + g'(u)\frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

(e) Using the chain rule given in (b)

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= g'\left(\frac{\cos t \sin t}{t}\right)\left(-\frac{y}{z}\sin t + \frac{x}{z}\cos t - \frac{xy}{z^2}\right) \\
&= g'\left(\frac{\cos t \sin t}{t}\right)\left(-\frac{\sin^2 t}{t} + \frac{\cos^2 t}{t} - \frac{\sin t \cos t}{t^2}\right) \\
&= g'\left(\frac{\sin 2t}{2t}\right)\frac{1}{2t^2}(2t\cos 2t - \sin 2t)
\end{aligned}$$

31.24. Given $f(u,v,w)$ where $u = u(x,y,z)$, $v = v(x,y,z)$ and $w = w(x,y,z)$. Then by (31.8) (this is simply a notational change)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x},$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y},$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}.$$

(a) $\phi = f(x-y, y-z, z-x)$. Here $u = x - y$, $v = y - z$, $w = z - x$. Using the chain rule above,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} - \frac{\partial \phi}{\partial w}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v}, \quad \frac{\partial \phi}{\partial z} = -\frac{\partial \phi}{\partial v} + \frac{\partial \phi}{\partial w}.$$

Adding these results

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = 0.$$

In this example $f(u, v, w) = uvw$. Hence

$$\frac{\partial \phi}{\partial x} = vw - uv, \quad \frac{\partial \phi}{\partial y} = -vw + uw, \quad \frac{\partial \phi}{\partial z} = -uw + uv,$$

and the result follows by adding these equations.
(b) $\phi = f(y/x, z/x)$. Let $u = y/x$ and $v = z/x$. Then, using the chain rule (31.8),

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\left(-\frac{y}{x^2}\right) + \frac{\partial \phi}{\partial v}\left(-\frac{z}{x^2}\right), \quad \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{1}{x}, \quad \frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial v}\frac{1}{x}.$$

Therefore

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = 0.$$

The function $\phi = x/y + y/z + z/x$ can be expressed as

$$\phi = f(u, v) = \frac{1}{u} + \frac{u}{v} + v.$$

Hence

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \left(-\frac{1}{u^2} + \frac{1}{v}\right)\left(-\frac{y}{x^2}\right) + \left(-\frac{u}{v^2} + 1\right)\left(-\frac{z}{x^2}\right) = \frac{1}{y} - \frac{z}{x^2}, \\
\frac{\partial \phi}{\partial y} &= \left(-\frac{1}{u^2} + \frac{1}{v}\right)\left(\frac{1}{x}\right) = -\frac{x}{y^2} + \frac{1}{z}, \\
\frac{\partial \phi}{\partial z} &= \left(-\frac{xy}{z^2} + 1\right)\left(\frac{1}{x}\right) = -\frac{y}{x^2} + \frac{1}{x}.
\end{aligned}$$

It can be verified that

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = 0.$$

31.25. Given $f(x, y, z, t) = \mathrm{e}^{\mathrm{i}(k_1x + k_2y + k_3z - \omega t)}$, then

$$\begin{aligned}
&\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \\
&= -k_1^2 \mathrm{e}^{\mathrm{i}(k_1x + k_2y + k_3z - \omega t)} - k_2^2 \mathrm{e}^{\mathrm{i}(k_1x + k_2y + k_3z - \omega t)} \\
&\qquad -k_3^2 \mathrm{e}^{\mathrm{i}(k_1x + k_2y + k_3z - \omega t)} + \frac{\omega^2}{c^2}\mathrm{e}^{\mathrm{i}(k_1x + k_2y + k_3z - \omega t)} \\
&= \left(-k_1^2 - k_2^2 - k_3^2 + \frac{\omega^2}{c^2}\right)\mathrm{e}^{\mathrm{i}(k_1x + k_2y + k_3z - \omega t)} \\
&= 0,
\end{aligned}$$

if $c = \omega/\sqrt{(k_1^2 + k_2^2 + k_3^2)}$. Let $f(x, y, z, t) = g(k_1x + k_2y + k_3z - \omega t)$. Then

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \left(k_1^2 + k_2^2 + k_3^2 - \frac{\omega^2}{c^2}\right) g''(k_1x + k_2y + k_3z - \omega t) = 0$$

using the the value of c defined above. Hence $g(k_1x + k_2y + k_3z - \omega t)$ satisfies the wave equation.

31.26. By (31.29) the envelope of $f(x, y, \alpha) = 0$, is obtained by eliminating the parameter α between this equation and $\partial f/\partial \alpha = 0$.
(a) Let $f(x, y, \alpha) = y - \alpha - \alpha^2 x$. Then

$$\frac{\partial f}{\partial \alpha} = -1 - 2\alpha x.$$

Eliminate α between

$$y - \alpha - \alpha^2 x = 0, \tag{i}$$

and

$$-1 - 2\alpha x = 0. \tag{ii}$$

From (ii), $\alpha = -1/(2x)$, so that (i) becomes

$$y + \frac{1}{2x} - \frac{x}{4x^2} = 0, \text{ or } 4xy = -1.$$

The envelope is given by $4xy = -1$, which is a rectangular hyperbola.
(b) Let $f(x, y, \alpha) = y + \alpha^2 x - \alpha$. Then

$$\frac{\partial f}{\partial \alpha} = 2\alpha x - 1.$$

Eliminate α between

$$y + \alpha^2 x - \alpha = 0, \tag{i}$$

and

$$2\alpha x - 1 = 0.$$

From (ii) $\alpha = 1/(2x)$, so that (i(becomes

$$y + \frac{x}{4x^2} - \frac{1}{2x} = 0.$$

The envelope is given by $4xy = 1$, which is a rectangular hyperbola.
(c) Express the function f in the form $f(x, y, \alpha) = (1 - \alpha)x + \alpha y - \alpha(1 - \alpha)$. Then

$$\frac{\partial f}{\partial \alpha} = -x + y - 1 + 2\alpha.$$

Eliminate α between

$$(1 - \alpha)x + \alpha y - \alpha(1 - \alpha) = 0, \tag{i}$$

and

$$-x + y - 1 + 2\alpha = 0. \tag{ii}$$

From (ii), $\alpha = \frac{1}{2}(x - y + 1)$ so that (i) becomes

$$x(-x + y + 1) + y(x - y + 1) - \frac{1}{2}(x - y + 1)(-x + y + 1) = 0,$$

which, after expansion, is the envelope

$$x^2 + y^2 - 2xy - 2x - 2y + 1 = 0.$$

(d) Let $f(x, y, \theta) = x \cos\theta + y \sin\theta - 1$. Then

$$\frac{\partial f}{\partial \theta} = -x \sin\theta + y \cos\theta.$$

Eliminate θ between

$$x \cos\theta + y \sin\theta - 1 = 0, \tag{i}$$

and

$$-x \sin\theta + y \cos\theta = 0. \tag{ii}$$

From (ii), $\tan\theta = y/x$. Hence, $\cos\theta = x/\sqrt{(x^2 + y^2)}$ and $\sin\theta = y/\sqrt{(x^2 + y^2)}$, so that (i) becomes

$$\frac{x^2}{\sqrt{(x^2 + y^2)}} + \frac{y^2}{\sqrt{(x^2 + y^2)}} = 1,$$

which is the equation of the circle $x^2 + y^2 = 1$.

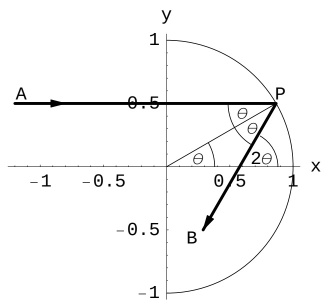

Figure 19: Problem 31.27

31.27. (a) The semicircular mirror is shown in Figure 19 with a ray AP falling on the mirror at P, and reflected along PB. Since P has coordinates $(\cos\theta, \sin\theta)$, the radius to P makes an angle θ with the x axis. The ray is reflected at the same angle θ to the radius at P, and therefore makes an angle 2θ with the x axis. Hence the slope of PB is $\tan\theta$, and its equation is

$$y - \sin\theta = \tan 2\theta(x - \cos\theta)$$

or

$$y\cos 2\theta - \sin\theta\cos 2\theta = x\sin 2\theta - \cos\theta\sin 2\theta.$$

Since $\sin 2\theta\cos\theta - \cos 2\theta\sin\theta = \sin\theta$, the equation of the reflected ray is

$$x\sin 2\theta - y\cos 2\theta = \sin\theta.$$

(b) Let

$$f(x, y, \theta) = x\sin 2\theta - y\cos 2\theta - \sin\theta.$$

By (31.29), the caustic is given by eliminating θ between $f(x, y, \theta) = 0$ and $\partial f(x, y, \theta)/\partial\theta$. Hence θ has to be eliminated between

$$x\sin 2\theta - y\cos 2\theta = \sin\theta, \qquad \text{(i)}$$

and

$$2x\cos 2\theta - 2y\sin 2\theta = \cos\theta.\text{(ii)}$$

Divide (ii) through by 2, and then square and add (i) and (ii):

$$(x\sin 2\theta - y\cos 2\theta)^2 + (x\cos 2\theta - y\sin 2\theta)^2 = \sin^2\theta + \frac{1}{4}\cos^2\theta,$$

or

$$x^2 + y^2 = \tfrac{1}{4} + \tfrac{3}{4}\sin^2\theta.(iii)$$

Eliminate x between (i) and (ii), so that

$$\begin{aligned} -y &= \sin\theta\cos 2\theta - \frac{1}{2}\cos\theta\sin 2\theta = \sin\theta - 2\sin^3\theta - \cos^2\theta\sin\theta \\ &= \sin\theta - 2\sin^3\theta - \sin\theta + \sin^3\theta \\ &= -\sin^3\theta \end{aligned}$$

Hence $\sin\theta = y^{\frac{1}{3}}$. Finally eliminating $\sin\theta$ in (iii) we obtain the equation of the caustic.

$$x^2 + y^2 = \frac{1}{4}(3y^{\frac{2}{3}} + 1).$$

The grey curve in Figure 20 shows the caustic.

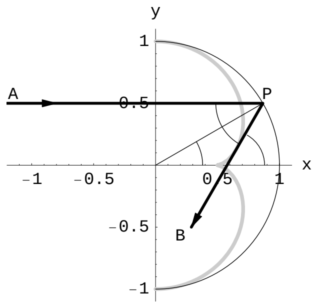

Figure 20: Problem 31.27(b)

31.28. The length cut of by the line between the axes is L. Let the line make angle θ with the x axis: in other words its slope is $\tan\theta$. The line cuts the x axis at $(-L\cos\theta, 0)$ and the y axis at $(0, L\sin\theta)$ (note that, if the intercept is in the first quadrant, then $\frac{1}{2}\pi < \theta < \pi$). the equation of the straight line is

$$y - L\sin\theta = x\tan\theta. \tag{i}$$

Differentiate this equation with respect to θ so that

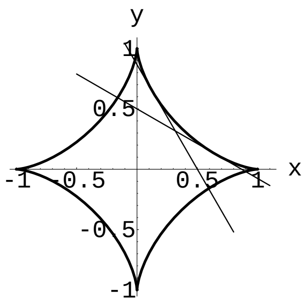

Figure 21: Problem 31.28

$$-L\cos\theta = x\sec^2\theta. \tag{ii}$$

The envelope is given by the elimination of θ between (i) and (ii). From (ii)

$$\cos^3\theta = -\frac{x}{L}, \text{ or } \cos\theta = \left(-\frac{x}{L}\right)^{\frac{1}{3}}. \tag{iii}$$

From (i)

$$y = L\sin\theta + x\tan\theta = L\sin\theta - L\cos^3\theta\tan\theta = L\sin^3\theta.$$

Therefore

$$\sin\theta = \left(\frac{y}{L}\right)^{\frac{1}{3}}. \tag{iv}$$

Squaring and adding (iii) and (iv),

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = L^{\frac{2}{3}},$$

which is the envelope of the straight lines. The envelope is shown in Figure 21.

Chapter 32: Double integration

32.1. The integration is in two stages: with respect to the inner variable first and then with respect to the outer variable. However, with constant limits of integration the order of integration can be changed with out affecting the limits. Thus, by (32.1)

$$I = \int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \int_c^d \left(\int_a^b f(x,y)\mathrm{d}x \right) \mathrm{d}y,$$

and, changing the order of integration,

$$I = \int_a^b \int_c^d f(x,y)\mathrm{d}y\mathrm{d}x = \int_a^b \left(\int_c^d f(x,y)\mathrm{d}y \right) \mathrm{d}x$$

also.

(a)

$$\int_0^1 \int_1^2 xy^2\mathrm{d}x\mathrm{d}y = \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{x=1}^2 \mathrm{d}y = \int_0^1 \left[\frac{3}{2}y^2\right] \mathrm{d}y = \left[\frac{1}{2}y^3\right]_0^1 = \frac{1}{2}.$$

(b)

$$\int_0^1 \int_0^1 y\mathrm{e}^{xy}\mathrm{d}x\mathrm{d}y = \int_0^1 [\mathrm{e}^{xy}]_{x=0}^1 \mathrm{d}y = \int_0^1 (\mathrm{e}^y - 1)\mathrm{d}y = [\mathrm{e}^y - y]_0^1 = \mathrm{e} - 2.$$

(c)

$$\int_c^d \int_a^b \mathrm{d}x\mathrm{d}y = \int_c^d [x]_{x=a}^b \mathrm{d}y = \int_c^d (b-a)\ dy = (b-a)[y]_c^d = (b-a)(d-c).$$

(d)

$$\int_a^b \int_c^d \mathrm{d}x\mathrm{d}y = \int_a^b [x]_{x=c}^d \mathrm{d}y = \int_a^b (d-c)\mathrm{d}y = (d-c)[y]_a^b = (b-a)(d-c):$$

the answer is the same as that for (c).

(e) Note that in this problem the integration is with respect to y first. Hence

$$\int_c^d \int_a^b \mathrm{d}y\mathrm{d}x = \int_c^d [y]_{y=a}^b \mathrm{d}x = \int_c^d (b-a)\ dx = (b-a)[x]_c^d = (b-a)(d-c),$$

which is the same answer as in (c) and (d).

(f)

$$\begin{aligned}\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} y\sin(xy)\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{1}{2}} [-\cos(xy)]_{x=0}^{\frac{1}{2}\pi}\,\mathrm{d}y = \int_0^{\frac{1}{2}} \left\{-\cos\left(\tfrac{1}{2}\pi y\right) + 1\right\}\mathrm{d}y \\ &= \left[-\frac{2}{\pi}\sin\left(\frac{1}{2}\pi y\right) + y\right]_0^{\frac{1}{2}} \\ &= -\frac{2}{\pi}\sin\frac{1}{4}\pi + \frac{1}{2} = -\frac{\sqrt{2}}{\pi} + \frac{1}{2}.\end{aligned}$$

(g)

$$\int_{-1}^1 \int_{-1}^1 x^2\mathrm{d}x\mathrm{d}y = \int_{-1}^1 \left[\frac{1}{3}x^3\right]_{x=-1}^1 \mathrm{d}y = \frac{2}{3}\int_{-1}^1 \mathrm{d}y = \frac{4}{3}.$$

(h) Integrate with respect ot y first:

$$\int_1^2 \int_0^1 x^2\mathrm{d}y\mathrm{d}x = \int_1^2 x^2[y]_0^1\mathrm{d}x = \int_1^2 x^2\mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}(8-1) = \frac{7}{3}.$$

(i)

$$\begin{aligned}
\int_0^1 \int_{-1}^1 (xy^2 - x^2y)\mathrm{d}x\mathrm{d}y &= \int_0^1 \left[\frac{1}{2}x^2y^2 - \frac{1}{3}x^3y\right]_{x=-1}^1 \mathrm{d}y \\
&= \int_0^1 \left(\frac{1}{2}y^2 - \frac{1}{3}y - \frac{1}{2}y^2 - \frac{1}{3}y\right)\mathrm{d}y = \int_0^1 \left(-\frac{2}{3}y\right)\mathrm{d}y \\
&= \left[-\frac{2}{3}\frac{1}{2}y^2\right]_0^1 = -\frac{1}{3}
\end{aligned}$$

(j) Integrate with respect to y first:

$$\begin{aligned}
\int_{-1}^1 \int_0^1 (xy^2 - x^2y)\mathrm{d}y\mathrm{d}x &= \int_{-1}^1 \left[\frac{1}{3}xy^3 - \frac{1}{2}x^2y\right]_{y=0}^1 \mathrm{d}x = \int_{-1}^1 \left(\frac{1}{3}x - \frac{1}{2}x^2\right)\mathrm{d}x \\
&= \left[\frac{1}{6}x^2 - \frac{1}{6}x^3\right]_{-1}^1 = -\frac{1}{3}
\end{aligned}$$

Alternatively, change the order of integration and the limits, and use the answer to the previous problem:

$$\int_{-1}^1 \int_0^1 (xy^2 - x^2y)\mathrm{d}y\mathrm{d}x = \int_0^1 \int_{-1}^1 (xy^2 - x^2y)\mathrm{d}x\mathrm{d}y = -\frac{1}{3}.$$

(k)

$$\begin{aligned}
\int_0^1 \int_0^1 (x + y^2 + 1)^2\mathrm{d}x\mathrm{d}y &= \int_0^1 \left[\frac{1}{3}(x + y^2 + 1)^3\right]_{x=0}^1 \mathrm{d}y \\
&= \int_0^1 \frac{1}{3}[(2 + y^2)^3 - (1 + y^2)^3]\ dy \\
&= \int_0^1 \frac{1}{3}[3y^4 + 9y^2 + 7]\mathrm{d}y = \frac{1}{3}\left[\frac{3}{5}y^5 + 3y^3 + 7y\right]_0^1 \\
&= \frac{53}{15}
\end{aligned}$$

(l) Integrate with respect to y first:

$$\begin{aligned}
\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \cos(x + y)\mathrm{d}y\mathrm{d}x &= \int_0^{\frac{1}{2}\pi} [\sin(x + y)]_{y=0}^{\frac{1}{2}\pi} \mathrm{d}x \\
&= \int_0^{\frac{1}{2}\pi} \left[\sin\left(x + \frac{1}{2}\pi\right) - \sin x\right]\mathrm{d}x \\
&= \left[-\cos\left(x + \frac{1}{2}\pi\right) + \cos x\right]_0^{\frac{1}{2}\pi} \\
&= 0
\end{aligned}$$

(m)

$$\int_1^2 \int_0^1 \frac{x}{y}\mathrm{d}x\mathrm{d}y = \int_1^2 \left[\frac{x^2}{2y}\right]_{x=0}^1 \mathrm{d}y = \int_1^2 \frac{1}{2y}\mathrm{d}y = \frac{1}{2}[\ln y]_1^2 = \frac{1}{2}\ln 2.$$

32.2. The signed volume $\mathcal{V}$ between the surface $z = f(x, y)$ and the plane $z = 0$ over the rectangle $a \leq x \leq b$, $c \leq y \leq d$ is given by

$$\mathcal{V} = \int_c^d \int_a^b f(x, y)\mathrm{d}x\mathrm{d}y, \text{ or } \int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x.$$

(a) $z = xy$, $0 \le x \le 1$, $0 \le y \le 1$. The signed volume is

$$\mathcal{V} = \int_0^1 \int_0^1 xy\mathrm{d}x\mathrm{d}y = \int_0^1 \left[\frac{1}{2}x^2\right]_0^1 y\mathrm{d}y = \frac{1}{2}\int_0^1 y\mathrm{d}y = \frac{1}{4}.$$

(b) $z = xy$, $-1 \le x \le 1$, $0 \le y \le 1$. Then

$$\mathcal{V} = \int_0^1 \int_{-1}^1 xy\mathrm{d}x\mathrm{d}y = \int_0^1 \left[\frac{1}{2}x^2\right]_{-1}^1 y\mathrm{d}y = 0.$$

If (x_0, y_0, z_0) is any point on the surface, then the point $(-x_0, y_0, -z_0)$ also lies on the surface. Elements of volume about these points cancel out so that the total signed volume is zero.
(c) $z = x + y$, $-1 \le x \le 2$, $-2 \le y \le 1$. The signed volume is

$$\begin{aligned}\mathcal{V} &= \int_{-2}^1 \int_{-1}^2 (x+y)\mathrm{d}x\mathrm{d}y = \int_{-2}^1 \left[\frac{1}{2}x^2 + xy\right]_{-1}^2 \mathrm{d}y \\ &= \int_{-2}^1 \left[\frac{3}{2} + y\right] \mathrm{d}y = \left[\frac{3}{2}y + 3\frac{1}{2}y^2\right]_{-2}^1 \\ &= \frac{3}{2} + \frac{3}{2} + 3 - 6 = 0.\end{aligned}$$

(d) $z = -1$, $a \le x \le b$, $c \le y \le d$. The signed volume is

$$\mathcal{V} = \int_c^d \int_a^b (-1)\mathrm{d}x\mathrm{d}y = -\int_c^d [x]_a^b \mathrm{d}y = -\int_c^d (b-a)\mathrm{d}y = -(b-a)(d-c).$$

(e) $z = 2x - y + 3$, $0 \le x \le 1$, $0 \le y \le 1$. The signed volume is

$$\mathcal{V} = \int_0^1 \int_0^1 (2x - y + 3)\mathrm{d}x\mathrm{d}y = \int_0^1 (4-y)\mathrm{d}y = \left[4y - \frac{1}{2}y^2\right]_0^1 = \frac{7}{2}.$$

(f) $z = 1/(x+y)$, $1 \le x \le 2$, $0 \le y \le 1$. The signed volume is

$$\begin{aligned}\int_0^1 \int_1^2 \frac{1}{x+y}\mathrm{d}x\mathrm{d}y &= \int_0^1 [\ln(x+y)]_{x=1}^2 \mathrm{d}y \\ &= \int_0^1 [\ln(2+y) - \ln(1+y)]\mathrm{d}y. \\ &= \int_2^3 \ln s\mathrm{d}s - \int_1^2 \ln t\mathrm{d}t \quad (s = y+2, t = y+1) \\ &= [s\ln s - s]_2^3 - [t\ln t - t]_1^2 \text{ (see Appendix E)} \\ &= 3\ln 3 - 4\ln 2\end{aligned}$$

(g) $z = (x + 2y - 1)^2$, $-2 \le x \le 1$, $-1 \le y \le 1$. The signed volume is given by

$$\begin{aligned}\mathcal{V} &= \int_{-1}^1 \int_{-2}^1 (x+2y-1)^2\mathrm{d}x\mathrm{d}y = \int_{-1}^1 \frac{1}{3}(x+2y-1)^3]_{-2}^1\mathrm{d}y \\ &= \int_{-1}^1 \frac{1}{3}[8y^3 - (2y-3)^3]\ dy \\ &= \frac{1}{3}\left[2y^4 - \frac{1}{8}(2y-3)^4\right]_{-1}^1 = 26\end{aligned}$$

32.3. The region of integration is shown in each case. The values of the repeated integrals are also stated.

(a) $$\int_0^1 \int_0^y \mathrm{d}x\mathrm{d}y = \int_0^1 [x]_{x=0}^y \mathrm{d}y = \int_0^1 y\mathrm{d}y = \frac{1}{2}.$$

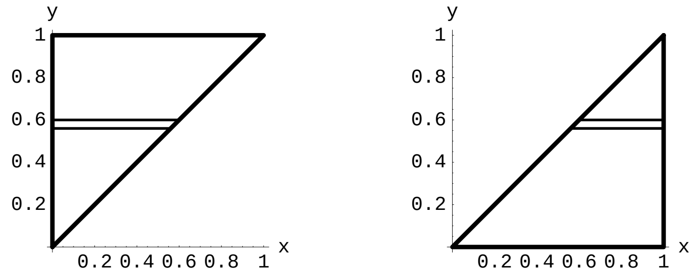

Figure 22: Problem 32.3(a), (b)

(b)
$$\int_0^1 \int_y^1 \mathrm{d}x\mathrm{d}y = \frac{1}{2}.$$

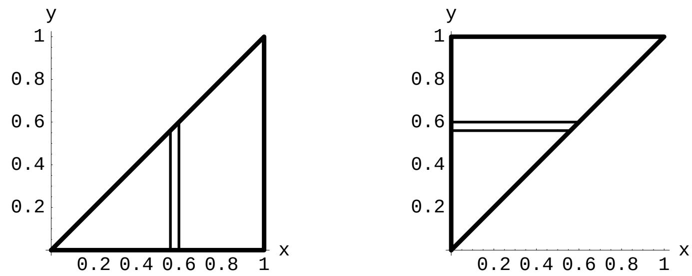

Figure 23: Problem 32.3(c), (d)

(c)
$$\int_0^1 \int_0^y x^2 y \mathrm{d}y\mathrm{d}x = \int_0^1 x^2 \left[\frac{1}{2}y^2\right]_{y=0}^x \mathrm{d}x = \frac{1}{2}\int_0^1 x^4 \mathrm{d}x = \frac{1}{10}.$$

(d)
$$\int_0^1 \int_0^x (x+y^2)^2 \mathrm{d}x\mathrm{d}y = \frac{9}{20}.$$

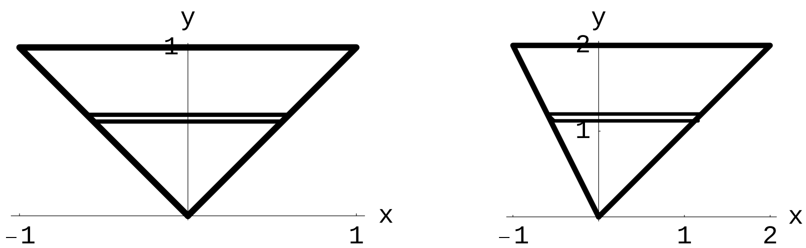

Figure 24: Problem 32.3(e), (f)

(e)
$$\int_0^1 \int_{-y}^y y \mathrm{d}x\mathrm{d}y = \frac{2}{3}.$$

(f)
$$\int_0^2 \int_{-\frac{1}{2}y}^{y} y^2 \sin(xy)\mathrm{d}x\mathrm{d}y = 4\cos 1 \sin^3 1.$$

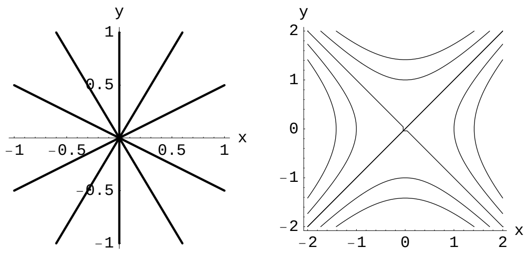

Figure 25: Problem 32.3(g), (h)

(g)
$$\int_0^2 \int_0^{1-\frac{1}{2}x} x^2 \mathrm{d}y\mathrm{d}x = \int_0^2 x^2 [y]_{y=0}^{1-\frac{1}{2}x} \mathrm{d}x = \int_0^2 x^2 (1 - \frac{1}{2}x)\mathrm{d}x = \frac{2}{3}.$$

(h)
$$\int_0^1 \int_0^{\sqrt{(1-y^2)}} x\mathrm{d}x\mathrm{d}y = \frac{1}{3}.$$

(i) The region of integration is the same as that shown for (h) but with the strip parallel to the x axis. The value of the integral is

$$\int_0^1 \int_0^{\sqrt{(1-x^2)}} x\mathrm{d}y\mathrm{d}x = \frac{1}{3},$$

which, as expected, is the same as that in (h).

32.4. The base of the wedge is the semi-circle $x^2 + y^2 = 1$, $y \geq 0$, and its height at any point (x, y) on the base is $z = 2y$. Take a strip in the (x, y) plane of width δy which is parallel to the x axis. Its length will be $2\sqrt{(1 - y^2)}$, and the semi-circle will be covered if $0 \leq y \leq 1$. Hence the volume $\mathcal{V}$ is given by

$$\begin{aligned} \mathcal{V} &= \int_0^1 \int_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} z\mathrm{d}x\mathrm{d}y = \int_0^1 \int_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} 2y\mathrm{d}x\mathrm{d}y \\ &= 2\int_0^1 y[x]_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} \mathrm{d}y = 4\int_0^1 y\sqrt{(1-y^2)}\mathrm{d}y \\ &= -\frac{4}{3}\left[(1-y^2)^{\frac{3}{2}}\right]_0^1 = \frac{4}{3} \end{aligned}$$

32.5. The region of integration should be sketched as shown in each case together with the new strip.

(a)
$$\int_0^1 \int_0^y f(x, y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_x^1 f(x, y)\mathrm{d}y\mathrm{d}x.$$

(b)
$$\int_0^1 \int_y^1 f(x, y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^x f(x, y)\mathrm{d}y\mathrm{d}x.$$

(c) It can be seen from the shape of the region that the integration with respect to y will be the sum of two integrals over regions separated by the line $x = 2$. Thus

$$\int_1^2 \int_0^{y+1} f(x, y)\mathrm{d}x\mathrm{d}y = \int_1^2 \int_0^2 f(x, y)\mathrm{d}y\mathrm{d}x + \int_2^3 \int_{x-1}^2 f(x, y)\mathrm{d}y\mathrm{d}x.$$

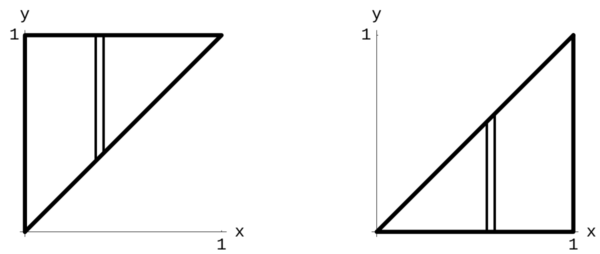

Figure 26: Problem 32.5(a), (b)

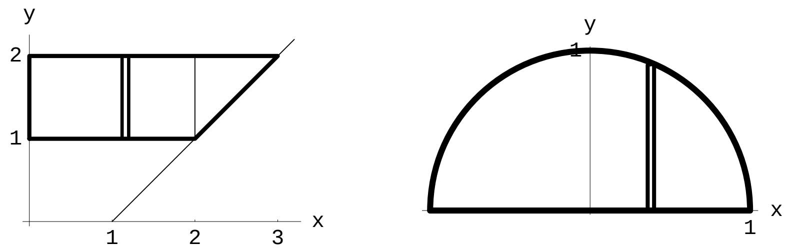

Figure 27: Problem 32.5(c), (d)

(d)
$$\int_0^1 \int_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{-1}^1 \int_0^{\sqrt{(1-x^2)}} f(x,y)\mathrm{d}y\mathrm{d}x.$$

(e) Two integrals are required for the change of the order of integration. Thus

$$\int_2^4 \int_0^{\frac{1}{2}y} f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \int_2^4 f(x,y)\mathrm{d}y\mathrm{d}x + \int_1^2 \int_{2x}^4 f(x,y)\mathrm{d}y\mathrm{d}x.$$

(f) The curves $y = x^2$ and $y = x^3$ intersect at the points $(0,0)$ and $(1,1)$, so the limits of integration

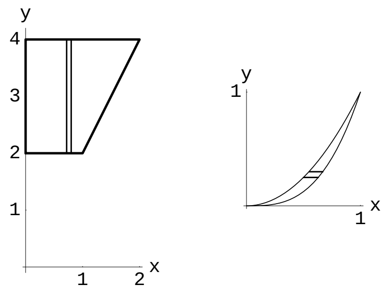

Figure 28: Problem 32.5(e), (f)

for x are 0 and 1. Thus

$$\int_0^1 \int_{x^3}^{x^2} f(x,y)\mathrm{d}y\mathrm{d}x = \int_0^1 \int_{y^{\frac{1}{2}}}^{y^{\frac{1}{3}}} f(x,y)\mathrm{d}x\mathrm{d}y.$$

(g) In the reversed order, the integral becomes the sum of two integrals separated at $y = 0$. Thus

$$\int_0^1 \int_{-1+x}^{1-x} f(x,y)\mathrm{d}y\mathrm{d}x = \int_{-1}^0 \int_0^{1+y} f(x,y)\mathrm{d}x\mathrm{d}y + \int_0^1 \int_0^{1-y} f(x,y)\mathrm{d}x\mathrm{d}y.$$

(h) The region of integration is the circle shown which has centre at $(0,1)$ and radius 1. In

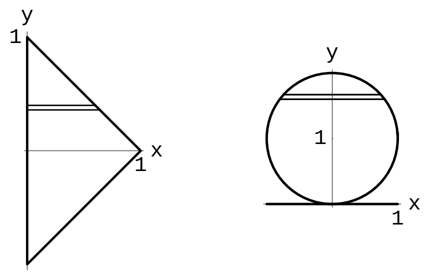

Figure 29: Problem 32.5(g), (h)

the reversed order of integration the strip parallel to the x axis lies between $-\sqrt{(2y-y^2)}$ and $\sqrt{(2y-y^2)}$. Thus

$$\int_{-1}^{1}\int_{1-\sqrt{(1-x^2)}}^{1+\sqrt{(1-x^2)}} f(x,y)\mathrm{d}y\mathrm{d}x = \int_0^2\int_{-\sqrt{(2y-y^2)}}^{\sqrt{(2y-y^2)}} f(x,y)\mathrm{d}x\mathrm{d}y.$$

32.6. (a) The limits of integration are all constants, which means that the region of integration is a rectangle with two sides along the axes. To reverse the order of integration simply transpose the integrals with their limits. Hence

$$\begin{aligned}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi} x\sin(xy)\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{1}{2}\pi}\left(\int_0^{\frac{1}{2}} x\sin(xy)\mathrm{d}y\right)\mathrm{d}x\\ &= \int_0^{\frac{1}{2}\pi}[-\cos(xy)]_{y=0}^{\frac{1}{2}}\,\mathrm{d}x\\ &= \int_0^{\frac{1}{2}\pi}\left(-\cos\tfrac{1}{2}x+1\right)\mathrm{d}x\\ &= \left[-2\sin\tfrac{1}{2}x+x\right]_0^{\frac{1}{2}\pi} = -\sqrt{2}+\tfrac{1}{2}\pi.\end{aligned}$$

(b) The straight boundaries of the region are $x=0$, $y=2$ and $x=2(y-1)$. Hence when the

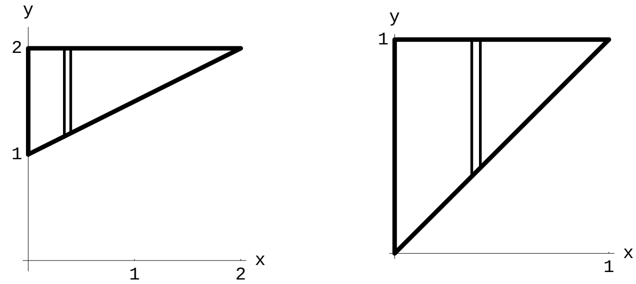

Figure 30: Problem 32.6(b), (c)

order of integration is reversed the limits on y are $y=\frac{1}{2}x+1$ and $y=2$ as shown. Therefore

$$\begin{aligned}\int_1^2\int_0^{2(y-1)} x^2\mathrm{d}x\mathrm{d}y &= \int_0^2\int_{\frac{1}{2}x+1}^{2} x^2\mathrm{d}y\mathrm{d}x = \int_0^2 \left[x^2y\right]_{y=\frac{1}{2}x+1}^{2}\\ &= \int_0^2[x^2-\frac{1}{2}x^3]\mathrm{d}x = \left[\frac{1}{3}x^3-\frac{1}{8}x^4\right]_0^2 = \frac{2}{3}\end{aligned}$$

(c) From Figure 30

$$\begin{aligned}\int_0^1\int_0^y x^2\mathrm{e}^{xy}\mathrm{d}x\mathrm{d}y &= \int_0^1\int_x^1 x^2\mathrm{e}^{xy}\mathrm{d}y\mathrm{d}x\\ &= \int_0^1 [x\mathrm{e}^{xy}]_x^1 = \int_0^1 [x\mathrm{e}^x - x\mathrm{e}^{x^2}]\mathrm{d}x\end{aligned}$$

The first integral can be evaluated by integration by parts:

$$\int_0^1 x\mathrm{e}^x\mathrm{d}x = [x\mathrm{e}^x]_0^1 - \int_0^1 \mathrm{e}^x\mathrm{d}x = \mathrm{e} - \mathrm{e} + 1 = 1,$$

and, using the substitution $u = x^2$, the second becomes

$$\int_0^1 x\mathrm{e}^{x^2}\mathrm{d}x = [\frac{1}{2}\mathrm{e}^{x^2}]_0^1 = \frac{1}{2}\mathrm{e} - \frac{1}{2}.$$

Hence

$$\int_0^1\int_0^y x^2\mathrm{e}^{xy}\mathrm{d}x\mathrm{d}y = 1 - \tfrac{1}{2}\mathrm{e} + \tfrac{1}{2} = \tfrac{3}{2} - \tfrac{1}{2}\mathrm{e}.$$

(d) This is an infinite integral in y. The region of integration lies between the two curves shown

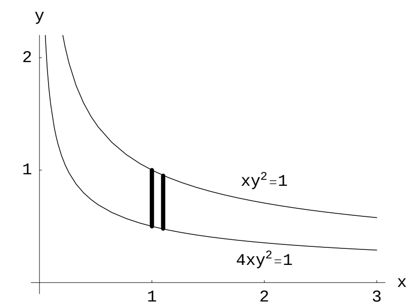

Figure 31: Problem 32.6(d)

in Figure 31. Hence, reversing the order of integration

$$\begin{aligned}\int_0^\infty\int_{\frac{1}{4}y^{-2}}^{y^{-2}} x^2y\mathrm{e}^{-x^2y^2}\mathrm{d}x\mathrm{d}y &= \int_0^\infty\int_{y=\frac{1}{2}x^{-\frac{1}{2}}}^{x^{-\frac{1}{2}}} x^2y\mathrm{e}^{-x^2y^2}\mathrm{d}y\mathrm{d}x\\ &= \frac{1}{2}\int_0^\infty \left[-\mathrm{e}^{-x^2y^2}\right]_{\frac{1}{2}x^{-\frac{1}{2}}}^{x^{-\frac{1}{2}}}\mathrm{d}x\\ &= \frac{1}{2}\int_0^\infty [-\mathrm{e}^{-x} + \mathrm{e}^{-\frac{1}{4}x}]\mathrm{d}x\\ &= \frac{1}{2}\left[\mathrm{e}^{-x} - 4\mathrm{e}^{-\frac{1}{4}x}\right]_0^\infty = \frac{3}{2}\end{aligned}$$

(f) The limits are all constants so that we need only transpose the integrals. Hence

$$\begin{aligned}\int_1^2\int_0^1 \frac{y}{x^2+y^2}\mathrm{d}x\mathrm{d}y &= \int_0^1\int_1^2 \frac{y}{x^2+y^2}\mathrm{d}y\mathrm{d}x\\ &= \frac{1}{2}\int_0^1\int_1^4 \frac{1}{x^2+u}\mathrm{d}u\mathrm{d}x \text{ (substituting } u = y^2)\\ &= \frac{1}{2}\int_0^1 \left[\ln(x^2+u)\right]_1^4\mathrm{d}x = \frac{1}{2}\int_0^1 [\ln(x^2+4) - \ln(x^2+1)\mathrm{d}x\end{aligned}$$

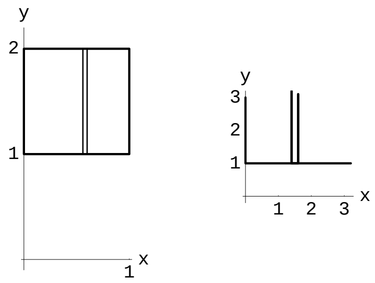

Figure 32: Problem 32.6(f), (g)

A typical integral on the right is

$$\begin{aligned}\int \ln(x^2+a^2)\mathrm{d}x &= x\ln(x^2+a^2) - \int \frac{2x^2\mathrm{d}x}{x^2+a^2}\mathrm{d}x \\ &= x\ln(x^2+a^2) - 2\int\left(1-\frac{a^2}{x^2+a^2}\right)\mathrm{d}x \\ &= x\ln(x^2+a^2) - 2x + 2a\arctan(x/a).\end{aligned}$$

after integrating by parts. Now put a successively equal to 2 and 1. Then

$$\begin{aligned}&\int_1^2\int_0^1 \frac{y}{x^2+y^2}\mathrm{d}x\mathrm{d}y \\ &= \tfrac{1}{2}[x\{\ln(x^2+4)-\ln(x^2+1)\} + 4\arctan(x/2) - 2\arctan x]_0^1 \\ &= \tfrac{1}{2}\ln\tfrac{5}{2} - \tfrac{1}{4}\pi + 4\arctan\tfrac{1}{2}\end{aligned}$$

(g) (*Note*: this repeated integral does not converge, that is, its value is not finite.) Replace by

$$\int_1^\infty\int_0^\infty \frac{1}{(x+y)^3}\mathrm{d}x\mathrm{d}y.)$$

Then

$$\begin{aligned}\int_1^\infty\int_0^\infty \frac{1}{(x+y)^3}\mathrm{d}x\mathrm{d}y &= \int_0^\infty\int_1^\infty \frac{1}{(x+y)^3}\mathrm{d}y\mathrm{d}x = -\frac{1}{2}\int_0^\infty\left[\frac{1}{(x+y)^2}\right]_{y=1}^\infty \mathrm{d}x \\ &= \frac{1}{2}\int_0^\infty \frac{1}{(x+1)^2}\mathrm{d}x = -\frac{1}{2}\left[\frac{1}{x+1}\right]_0^\infty \\ &= \frac{1}{2}\end{aligned}$$

(h)

$$\begin{aligned}\int_0^1\int_y^1 y(x^2-y^2)^{\frac{1}{2}}\mathrm{d}x\mathrm{d}y &= \int_0^1\int_0^x y(x^2-y^2)^{\frac{1}{2}}\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{2}\int_0^1\int_0^{x^2}(x^2-u)^{\frac{1}{2}}\mathrm{d}u\mathrm{d}x \text{ (using the substitution } u=y^2) \\ &= -\frac{1}{3}\int_0^1\left[(x^2-u)^{\frac{3}{2}}\right]_{u=0}^{x^2}\mathrm{d}x = \frac{1}{3}\int_0^1 x^3\mathrm{d}x = \frac{1}{12}\end{aligned}$$

(i) The integral must be split into two parts. Thus

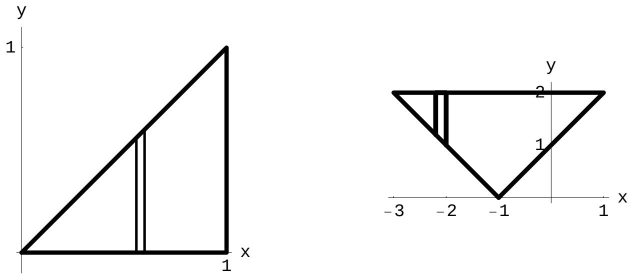

Figure 33: Problem 32.6(h), (i)

$$
\begin{aligned}
\int_0^2\int_{-y-1}^{y-1} x^2\mathrm{d}x\mathrm{d}y &= \int_0^2\int_{-y-1}^{-1} x^2\mathrm{d}x\mathrm{d}y + \int_0^2\int_{-1}^{y-1} x^2\mathrm{d}x\mathrm{d}y \\
&= \int_{-3}^{-1}\int_{-x-1}^{2} x^2\mathrm{d}y\mathrm{d}x + \int_{-1}^{1}\int_{x+1}^{2} x^2\mathrm{d}y\mathrm{d}x \\
&= \int_{-3}^{-1} x^2[y]_{-x-1}^{2}\mathrm{d}x + \int_{-1}^{1} x^2[y]_{x+1}^{2}\mathrm{d}x \\
&= \int_{-3}^{-1} (3x^2+x^3)\mathrm{d}x + \int_{-1}^{1} (x^2-x^3)\mathrm{d}x \\
&= \left[x^3+\frac{x^4}{4}\right]_{-3}^{-1} + \left[\frac{x^3}{3}-\frac{x^4}{4}\right]_{-1}^{1} \\
&= \left(-1+\frac{1}{4}+27-\frac{81}{4}\right) + \left(\frac{1}{3}-\frac{1}{4}+\frac{1}{3}+\frac{1}{4}\right) = \frac{20}{3}
\end{aligned}
$$

(j) The region of integration is the same as that shown for Problem 32.6(h). Thus

$$
\begin{aligned}
\int_0^1\int_y^1 \frac{y\mathrm{d}x\mathrm{d}y}{(x^2-y^2)^{\frac{1}{2}}} &= \int_0^1\int_0^x \frac{y\mathrm{d}y\mathrm{d}x}{(x^2-y^2)^{\frac{1}{2}}} = \frac{1}{2}\int_0^1\int_0^{x^2} \frac{\mathrm{d}u\mathrm{d}x}{(x^2-u)^{\frac{1}{2}}} \\
&= -\int_0^1 [(x^2-u)^{\frac{1}{2}}]_0^{x^2}\mathrm{d}x = \int_0^1 x\mathrm{d}x = \frac{1}{2}
\end{aligned}
$$

32.7. (a) Integrate with respect to x first. Then, using the figure,

$$
\begin{aligned}
\iint_{\mathcal{R}} f(P)\mathrm{d}A &= \int_1^4\int_1^2 (x^2+y^2)\mathrm{d}x\mathrm{d}y = \int_1^4 \left[\frac{1}{3}x^3+xy^2\right]_{x=1}^{2}\mathrm{d}y \\
&= \int_1^4\left(\frac{7}{3}+y^2\right)\mathrm{d}y = \left[\frac{7}{3}y+\frac{1}{3}y^3\right]_1^4 = 28
\end{aligned}
$$

(b) To avoid having to split the integral integrate with respect to y first. The upper and lower limits for y lie on the lines $y=-(x-\sqrt{3})/\sqrt{3}$ and $y=(x-\sqrt{3})/\sqrt{3}$ respectively. Therefore

$$
\begin{aligned}
\iint_{\mathcal{R}} f(P)\mathrm{d}A &= \int_0^{\sqrt{3}}\int_{(x-\sqrt{3})/\sqrt{3}}^{-(x-\sqrt{3})/\sqrt{3}} x\mathrm{d}y\mathrm{d}x \\
&= \int_0^{\sqrt{3}} [y]_{(x-\sqrt{3})/\sqrt{3}}^{-(x-\sqrt{3})/\sqrt{3}}\mathrm{d}x \\
&= -\frac{2}{\sqrt{3}}\int_0^{\sqrt{3}}(x^2-x\sqrt{3})\mathrm{d}x
\end{aligned}
$$

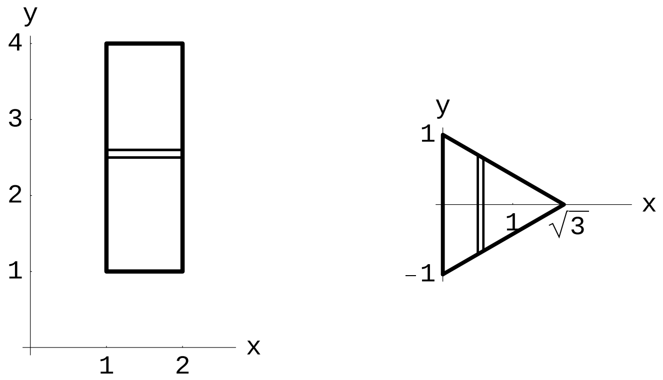

Figure 34: Problem 32.7(a), (b)

$$= -\frac{2}{\sqrt{3}}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\sqrt{3}\right]_0^{\sqrt{3}}$$

$$= -\frac{2}{\sqrt{3}}\left[\frac{(\sqrt{3})^3}{3} - \frac{(\sqrt{3})^3}{2}\right] = 1.$$

(c) Integrating with respect to x first,

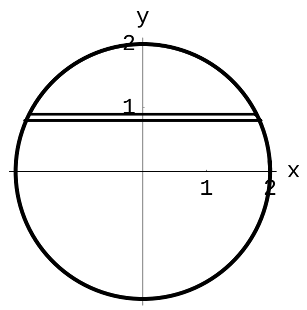

Figure 35: Problem 32.7(c)

$$\begin{aligned}\iint_{\mathcal{R}} f(P)\mathrm{d}A &= \int_{-2}^{2}\int_{-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} y^2\mathrm{d}x\mathrm{d}y = 2\int_{-2}^{2} y^2\sqrt{(4-y^2)}\mathrm{d}y \\ &= 32\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^2 t\cos^2 t\mathrm{d}t \text{ (substituting } y = 2\sin t) \\ &= 8\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^2 2t\mathrm{d}t = 4\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(1-\cos 4t)\mathrm{d}t = 4\pi\end{aligned}$$

32.8. If the sector $\mathcal{R}$ is the region $a \le r \le b$, $\alpha \le \theta \le \beta$, and P is a representative point in $\mathcal{R}$, then, in polar coordinates, the double integral of $f(P)$ over $\mathcal{R}$ is

$$\iint_{\mathcal{R}} f(P)\mathrm{d}A = \int_{\alpha}^{\beta}\int_{a}^{b} f(r,\theta)r\mathrm{d}r\mathrm{d}\theta,$$

(see (32.4) and Figure 32.9).

(a) $\mathcal{R}$ is $x^2+y^2 \le 1$, and $f(P) = x^2+y^2 = r^2$. In this problem $a=0$, $b=1$, $\alpha=0$ and $\beta = 2\pi$. Hence

$$\iint_{\mathcal{R}} f(P)\mathrm{d}A = \int_0^{2\pi}\int_0^1 r^3 \mathrm{d}r\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi}[r^4]_0^1 \mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi}\mathrm{d}\theta = \frac{1}{2}\pi.$$

(b) $\mathcal{R}$ is $x^2+y^2 \le 1$, and $f(P) = y^2 = r^2\sin^2\theta$. In this problem $a=0$, $b=1$, $\alpha=0$ and $\beta=2\pi$. Hence

$$\begin{aligned}\iint_{\mathcal{R}} f(P)\mathrm{d}A &= \int_0^{2\pi}\int_0^1 r^3\sin^2\theta \mathrm{d}r\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi}\left[r^4\right]_0^1 sin^2\theta \mathrm{d}\theta \\ &= \frac{1}{4}\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\mathrm{d}\theta = \frac{1}{4}\pi.\end{aligned}$$

(c) $\mathcal{R}$ is the region $r \le 2$, $x \ge 0$, $y \ge 0$, and $f(P) = xy = r^2\sin\theta\cos\theta$. In the this problem $a=0$, $b=2$, $\theta=0$ and $\theta = \frac{1}{2}\pi$. Therefore

$$\begin{aligned}\iint_{\mathcal{R}} f(P)\mathrm{d}A &= \int_0^{\frac{1}{2}\pi}\int_0^2 r^3\sin\theta\cos\theta \mathrm{d}r\mathrm{d}\theta = \frac{1}{4}\int_0^{\frac{1}{2}\pi}\left[r^4\right]_0^2 \sin\theta\cos\theta \mathrm{d}\theta \\ &= 4\int_0^{\frac{1}{2}\pi}\frac{1}{2}\sin 2\theta \mathrm{d}\theta \\ &= [-\cos 2\theta]_0^{\frac{1}{2}\pi} = 2\end{aligned}$$

(d) $\mathcal{R}$ is the sector $1 \le r \le 2$, $0 \le \theta \le \frac{1}{2}\pi$, and $f(P) = xy = r^2\sin\theta\cos\theta$. This is like (c) but with a change of lower limit for r. Thus

$$\begin{aligned}\iint_{\mathcal{R}} f(P)\mathrm{d}A &= \int_0^{\frac{1}{2}\pi}\int_1^2 r^3\sin\theta\cos\theta \mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{4}\int_0^{\frac{1}{2}\pi}\left[r^4\right]_1^2 \sin\theta\cos\theta \mathrm{d}\theta \\ &= \frac{15}{4}\int_0^{\frac{1}{2}\pi}\frac{1}{2}\sin 2\theta \mathrm{d}\theta \\ &= \frac{15}{16}[-\cos 2\theta]_0^{\frac{1}{2}\pi} = \frac{15}{8}\end{aligned}$$

(e) $\mathcal{R}$ is the disc $x^2+y^2 \le 4$, and $f(x,y) = \arctan(y/x) = \theta$. Therefore

$$\begin{aligned}\iint_{\mathcal{R}} f(P)\mathrm{d}A &= \int_0^{2\pi}\int_0^2 r\theta \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\theta\left[\frac{1}{2}r^2\right]_0^2 \mathrm{d}\theta \\ &= 2\int_0^{2\pi}\theta \mathrm{d}\theta = 4\pi^2\end{aligned}$$

(f) $\mathcal{R}$ is the first quadrant $r \ge 0$, $0 \le \theta\frac{1}{2} \le \pi$, and $f(x,y) = \mathrm{e}^{-4(x^2+y^2)} = \mathrm{e}^{-4r^2}$. Then

$$\begin{aligned}\iint_{\mathcal{R}} f(x,y)\mathrm{d}A &= \int_0^{\frac{1}{2}\pi}\int_0^{\infty} r\mathrm{e}^{-4r^2}\mathrm{d}r\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{\frac{1}{2}\pi}\int_0^{\infty}\mathrm{e}^{-4u}\mathrm{d}u\mathrm{d}\theta \text{ (substituting } u=r^2) \\ &= \frac{1}{2}\int_0^{\frac{1}{2}\pi}\left[-\frac{1}{4}\mathrm{e}^{-4u}\right]_0^{\infty}\mathrm{d}\theta = \frac{1}{8}\int_0^{\frac{1}{2}\pi}\mathrm{d}\theta \\ &= \frac{1}{16}\pi\end{aligned}$$

(g) Assuming that the equation of the sphere is $x^2+y^2+z^2=a^2$, consider the section $z=0$ (that is, the (x,y) plane) of the sphere. The volume $\mathcal{V}$ of the sphere is twice the volume of the hemisphere $z=(a^2-x^2-y^2)^{\frac{1}{2}}$ which can be expressed as the integral of $f(z)$ over the circle $x^2+y^2=a^2$. Therefore

$$\begin{aligned}\mathcal{V} &= 2\int_0^{2\pi}\int_0^a r(a^2-r^2)^{\frac{1}{2}}\mathrm{d}r\mathrm{d}\theta = 2\int_0^{2\pi}\left[\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^a \mathrm{d}\theta \\ &= \frac{2}{3}a^3\int_0^{2\pi}\mathrm{d}\theta = \frac{4}{3}\pi a^3\end{aligned}$$

(h) $\mathcal{R}$ is the half-plane $y\geq 0$ and $f(P)=y\mathrm{e}^{-(x^2+y^2)}=r\sin\theta\mathrm{e}^{-r^2}$. Hence, since the integral is separable,

$$\begin{aligned}\iint_{\mathcal{R}} f(x,y)\mathrm{d}A &= \int_0^{\pi}\int_0^{\infty} r^2\sin\theta\mathrm{e}^{-r^2}\mathrm{d}r\mathrm{d}\theta = \int_0^{\pi}\sin\theta\mathrm{d}\theta\int_0^{\infty} r^2\mathrm{e}^{-r^2}\mathrm{d}r \\ &= [-\cos\theta]_0^{\pi}\int_0^{\infty} r\left(-\frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathrm{e}^{-r^2}\right)\mathrm{d}r \\ &= -\left([r\mathrm{e}^{-r^2}]_0^{\infty} - \int_0^{\pi}\mathrm{e}^{-r^2}\mathrm{d}r\right) \quad \text{(integrating by parts)} \\ &= \int_0^{\infty}\mathrm{e}^{-r^2}\mathrm{d}r = \frac{1}{2}\sqrt{\pi},\end{aligned}$$

using the special formula given in Example 32.11.

32.9. A cylindrical hole of equation $(x-\frac{1}{2}a)^2+y^2=\frac{1}{4}a^2$ is drilled through a sphere of equation $x^2+y^2+z^2=a^2$ as shown in Figure 32.20 in the book. Consider the polar coordinates (r,θ) in the (x,y) plane. The polar equation of the cylindrical hole is $r=a\cos\theta$ (since the angle subtended by a diameter at a point on the circle is a right angle). At the location (r,θ) the length drilled from the sphere within the hole is $2\sqrt{(a^2-r^2)}$ perpendicular to the (x,y) plane. Hence the element of volume removed at (r,θ) is $2r\sqrt{(a^2-r^2)}r\delta r\delta\theta$. The total volume removed from the sphere is (note that $\sin\theta$ is missing from one of the upper limits in the problem)

$$\begin{aligned}V_c &= 2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^{a\cos\theta} r\sqrt{(a^2-r^2)}\mathrm{d}r\mathrm{d}\theta = \frac{2}{3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left[-(a^2-r^2)^{\frac{3}{2}}\right]_0^{a\cos\theta}\mathrm{d}\theta \\ &= \frac{2a^3}{3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(1-\cos^3\theta)\mathrm{d}\theta \\ &= \frac{2a^3}{3}\left([\theta]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi}(1-\sin^2\theta)\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\mathrm{d}\theta\right) \\ &= \frac{2a^3}{3}\left(\pi - \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\right) = a^3\left(\frac{2\pi}{3}-\frac{8}{9}\right)\end{aligned}$$

Hence the volume of the remaining part of the sphere is

$$V=\tfrac{4}{3}\pi a^3 - V_c = \tfrac{4}{3}\pi a^3 = -\tfrac{2}{3}\pi a^3 + \tfrac{8}{9}a^3 = \tfrac{2}{3}\pi a^3 + \tfrac{8}{9}a^3$$

32.10. The Jacobian determinant is given by

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}, \text{ or, alternatively, } \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}.$$

(a) $x=u^2-v^2$, $y=uv$. Then

$$J(u,v)=\begin{vmatrix} 2u & -2v \\ v & u\end{vmatrix} = 2(u^2+v^2)\geq 0.$$

(b) $x = u - v$, $y = 2v$. Then

$$J(u,v) = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2.$$

(c) $u = 2x - y$, $v = x + 2y$, or $x = \frac{2}{5}u + \frac{1}{5}v$, $y = -\frac{1}{5}u + \frac{2}{5}v$. Then

$$J(u,v) = \begin{vmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{vmatrix} = \frac{4}{25} + \frac{1}{25} = \frac{1}{5}.$$

(d) $x = u - \mathrm{e}^{-v}$, $y = u - \mathrm{e}^{v}$. Then

$$J(u,v) = \begin{vmatrix} 1 & \mathrm{e}^{-v} \\ 1 & -\mathrm{e}^{v} \end{vmatrix} = -\mathrm{e}^{v} - \mathrm{e}^{-v} = -2\cosh v.$$

32.11. Given $x = u/v$, $y = uv$, then (see introduction to the previous problem)

$$J(u,v) = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = 2\frac{u}{v}.$$

The region of integration in the (x,y) plane bounded by the lines $y = 2x$, $y = x$ and the rectangular

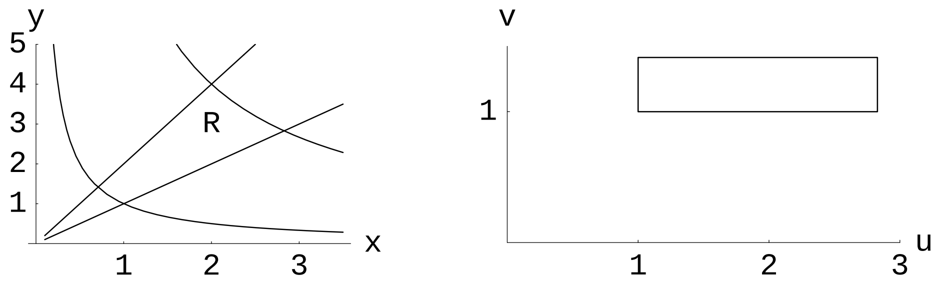

Figure 36: Problem 32.11

hyperbolas $xy = 1$ and $xy = 8$ is shown in the figure. In the (u,v) plane this region becomes a rectangle bounded by the straight lines $u = 1$, $u = 2\sqrt{2}$, $v = 1$ and $v = \sqrt{2}$. Using (32.12),

$$\begin{aligned} \iint_{\mathcal{R}} xy^2 \mathrm{d}x\mathrm{d}y &= \int_1^{\sqrt{2}} \int_1^{2\sqrt{2}} \frac{u}{v}(uv)^2 |J(u,v)| \mathrm{d}u\mathrm{d}v \\ &= \int_1^{\sqrt{2}} \int_1^{2\sqrt{2}} u^3 v \frac{2u}{v} \mathrm{d}u\mathrm{d}v \\ &= 2\int_1^{\sqrt{2}} \int_1^{2\sqrt{2}} u^4 \mathrm{d}u\mathrm{d}v = 2\left[\frac{u^5}{5}\right]_1^{2\sqrt{2}} [v]_1^{\sqrt{2}} \\ &= \frac{2}{5}[257 - 129\sqrt{2}] \end{aligned}$$

32.12. The intersection of the parabolas $y = x^2$, $y = 2x^2$, $x = y^2$, $x = 2y^2$ occurs in the first quadrant as shown in the first figure. The transformation $u = y/x^2$, $v = x/y^2$ maps the region in the (x,y) plane into the square with edges $u = 1$, $u = 2$, $v = 1$, $v = 2$. If $\mathcal{R}$ denotes the region between the parabolas in the (x,y) plane, then the area $\mathcal{A}$ of $\mathcal{R}$ is given by

$$\mathcal{A} = \iint_{\mathcal{R}} \mathrm{d}x\mathrm{d}y = \int_1^2 \int_1^2 |J(u,v)| \mathrm{d}u\mathrm{d}v,$$

where the Jacobian

$$J(u,v) = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}, \text{ or } \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

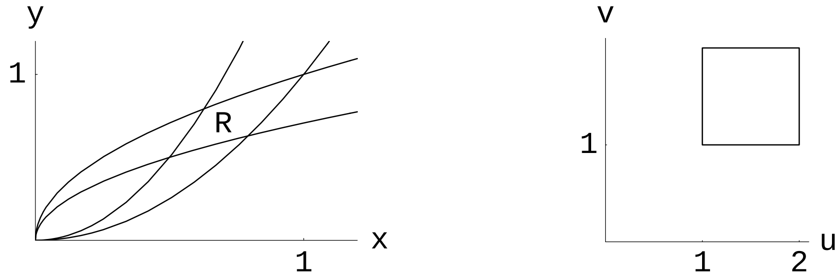

Figure 37: Problem 32.12

In terms of u and v, $x = u^{-\frac{1}{3}}v^{-\frac{2}{3}}$, $y = u^{-\frac{2}{3}}v^{-\frac{1}{3}}$. Hence

$$J(u,v) = \begin{vmatrix} -\frac{2}{3}u^{-\frac{5}{3}}v^{-\frac{1}{3}} & -\frac{1}{3}u^{-\frac{2}{3}}v^{-\frac{4}{3}} \\ -\frac{1}{3}u^{-\frac{4}{3}}v^{-\frac{2}{3}} & -\frac{2}{3}u^{-\frac{1}{3}}v^{-\frac{5}{3}} \end{vmatrix} = \frac{1}{3}u^{-2}v^{-2} \geq 0.$$

Therefore

$$\mathcal{A} = \frac{1}{3}\int_1^2\int_1^2 u^{-2}v^{-2}\mathrm{d}u\mathrm{d}v = \frac{1}{3}\int_1^2 u^{-2}\mathrm{d}u\int_1^2 v^{-2}\mathrm{d}v = \frac{1}{12}.$$

32.13. The region $\mathcal{R}$ is shown in the figure. If integration with respect to x is taken first, then the integral has to be split into two integrals, for $y > 0$ and $y < 0$. The edge AB is $x + y = 1$, edge AD is $-x + y = 1$, edge BC is $x - y = 1$, and edge CD is $-x - y = 1$. Hence

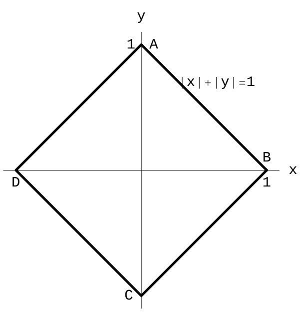

Figure 38: Problem 32.13

$$\iint_{\mathcal{R}} x\mathrm{e}^{x+y}\mathrm{d}A = \int_0^1\int_{y-1}^{-y+1} x\mathrm{e}^{x+y}\mathrm{d}x\mathrm{d}y + \int_{-1}^0\int_{-y-1}^{y+1} x\mathrm{e}^{x+y}\mathrm{d}x\mathrm{d}y.$$

Using integration by parts

$$\int x\mathrm{e}^x\mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x\mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x.$$

Therefore

$$\begin{aligned} \iint_{\mathcal{R}} x\mathrm{e}^{x+y}\mathrm{d}A &= \int_0^1 \mathrm{e}^y\left[(x-1)\mathrm{e}^x\right]_{y-1}^{-y+1}\mathrm{d}y + \int_{-1}^0 \mathrm{e}^y\left[(x-1)\mathrm{e}^x\right]_{-y-1}^{y+1}\mathrm{d}y \\ &= \int_0^1 [-y\mathrm{e} - (y-2)\mathrm{e}^{2y-1}\mathrm{d}y + \int_{-1}^0 [y\mathrm{e}^{2y+1} + (y+2)\mathrm{e}^{-1}]\mathrm{d}y \\ &= \left[-\frac{5}{4e} + \frac{e}{4}\right] + \left[\frac{9}{4e} - \frac{e}{4}\right] = \frac{1}{e} \end{aligned}$$

32.14. The equations of the edges of the rhombus $\mathcal{R}$ are for AB, $y = 1 - \frac{1}{2}x$, for AD, $y = 1 + \frac{1}{2}x$, for BC, $y = -1 + \frac{1}{2}x$ and for CD, $y = -1 - \frac{1}{2}x$. As in the previous problem, the volume is the sum of two repeated integrals. In this case integrate with respect to y first and then x so that separate integrals are required for $x > 0$ and $x < 0$. However, the integrand is even in x and independent of y so we need only double the value of the integral over $x > 0$. Hence the volume is given by

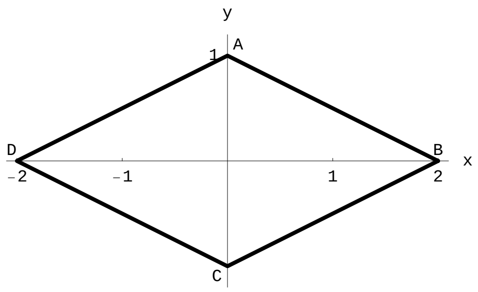

Figure 39: Problem 32.14

$$
\begin{aligned}
V = 2\iint_{\mathcal{R}} (x^2+2)\mathrm{d}A &= 4\int_0^2 \int_{-1+\frac{1}{2}x}^{1-\frac{1}{2}x} (x^2+2)\mathrm{d}y\mathrm{d}x \\
&= 4\int_0^2 (x^2+2)[y]_{-1+\frac{1}{2}x}^{1-\frac{1}{2}x}\mathrm{d}x = \int_0^2 (x^2+2)(2-x)\mathrm{d}x \\
&= 4\int_0^2 (-x^3+2x^2-2x+4)\mathrm{d}x \\
&= -4\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + x^2 - 4x\right]_0^2 \\
&= \frac{64}{3}
\end{aligned}
$$

32.15. For $x = r\cos\theta$, $y = r\sin\theta$, the Jacobian of the transformation is

$$
\begin{aligned}
J(r,\theta) &= \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r(\cos^2\theta + \sin^2\theta) = r.
\end{aligned}
$$

Elimination of θ and r gives the inverse of the transformation:

$$r = \sqrt{(x^2+y^2)}, \quad \tan\theta = \frac{y}{x}.$$

Hence

$$J(x,y) = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} x/\sqrt{(x^2+y^2)} & y/\sqrt{(x^2+y^2)} \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{vmatrix} = \frac{1}{r}.$$

Therefore

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r} = 1\Big/\frac{\partial(x,y)}{\partial(r,\theta)}.$$

If $u = y/x^2$ and $v = x/y^2$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -2y/x^3 & 1/x^2 \\ 1/y^2 & -2x/y^3 \end{vmatrix} = \frac{3}{x^2y^2} = 3u^2v^2.$$

Hence, using inverse rule above,

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3u^2v^2}$$

32.16. The Jacobian of the transformation $u = x^2 - y^2$, $v = 2xy$ is, using the general inverse result in Example 30.16,

$$\frac{\partial(x,y)}{\partial(u,v)} = 1\Bigg/\frac{\partial(u,v)}{\partial(x,y)} = 1\Bigg/\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = \frac{1}{4(x^2+y^2)}.$$

The first figure shows the intersection of the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $xy = 2$, $xy = 4$.

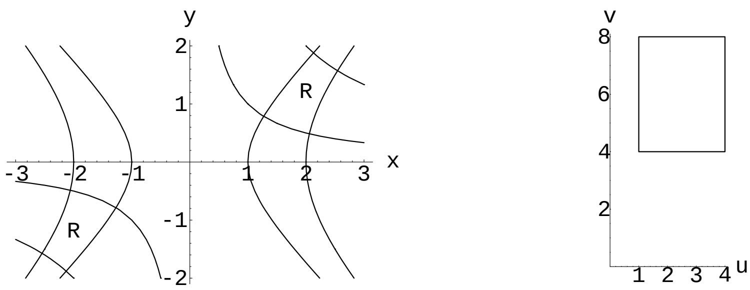

Figure 40: Problem 32.16

The region $\mathcal{R}$ is in two pieces, in the first and third quadrants, which are symmetric about the origin. With $u = x^2 - y^2$, $v = 2xy$, both regions map on to the same rectangle bounded by the straight lines $u = 1$, $u = 4$, $v = 4$, $v = 8$. Since the integrand is $x^2 + y^2$, by symmetry, the values of the double integral over the two regions of $\mathcal{R}$ are the same. Hence

$$\begin{aligned}\int\!\!\int_{\mathcal{R}} (x^2+y^2)\mathrm{d}x\mathrm{d}y &= \int_4^8\int_1^4 (x^2+y^2)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v \\ &= \int_4^8\int_1^4 (x^2+y^2)\frac{1}{(x^2+y^2)}\mathrm{d}u\mathrm{d}v \\ &= \int_4^8\int_1^4 \mathrm{d}u\mathrm{d}v = [u]_1^4[v]_4^8 = 3\times 4 = 12\end{aligned}$$

32.17. Move the origin to the corner P as shown in the figure. The coordinates of Q becomes $(x_Q - x_P, y_Q - y_P)$ and of S becomes $(x_S - x_P, y_S - y_P)$. We define a transformation which maps the parallelogram on to a rectangle. The equations of PQ and PS can be expressed respectively as

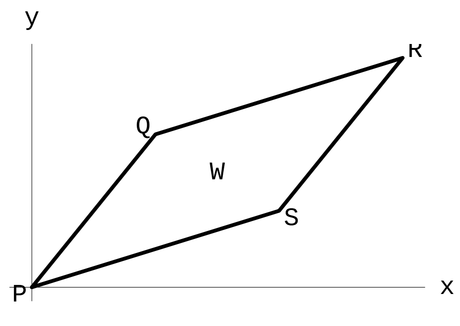

Figure 41: Problem 32.17

$$(x_Q - x_P)y = (y_Q - y_P)x, \quad (x_S - x_P)y = (y_S - y_P)x.$$

Let

$$u = (x_Q - x_P)y - (y_Q - y_P)x, \quad v = -(x_S - x_P)y + (y_S - y_P)x.$$

Hence on PQ, $u = 0$ and on PS, $v = 0$.

Since SR is parallel to PQ, its equation can be written as

$$(x_Q - x_P)(y - y_S + y_P) = (y_Q - y_P)(x - x_S + x_P),$$

from which it follows that, on SR,

$$\begin{aligned} u &= (x_Q - x_P)(y_S - y_P) - (y_Q - y_P)(x_S - x_P) \\ &= \begin{vmatrix} x_Q - x_P & x_S - x_P \\ y_Q - y_P & y_S - y_P \end{vmatrix} = \Delta, \end{aligned}$$

say. Similarly, the equation of QR can be expressed as

$$(x_S - x_P)(y - y_Q + y_P) = (y_S - y_P)(x - x_Q + y_P).$$

Hence on QR, $v = \Delta$ also.

The Jacobian of the transformation is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -(y_Q - y_P) & x_Q - x_P \\ y_S - y_P & -(x_S - x_P) \end{vmatrix} = -\Delta.$$

Using (32.12) and Example 30.16, the area $\mathcal{W}$ of the parallelogram is given by

$$\mathcal{W} = \iint_{\mathcal{R}} \mathrm{d}x\mathrm{d}y = \int_0^{\Delta} \int_0^{\Delta} \frac{1}{|\Delta|}\mathrm{d}u\ dv = \frac{\Delta^2}{|\Delta|} = |\Delta|,$$

as required.

32.18. (a) Integrate both sides of the given identity with respect to x:

$$\begin{aligned} \int_0^{\infty} \frac{\mathrm{e}^{-ax} - \mathrm{e}^{-bx}}{x}\mathrm{d}x &= \int_0^{\infty}\int_a^b \mathrm{e}^{-xy}\mathrm{d}y\mathrm{d}x \\ &= \int_a^b \int_0^{\infty} \mathrm{e}^{-xy}\mathrm{d}x\mathrm{d}y \text{ (changing the order of integration)} \\ &= \int_a^b \left[-\frac{-\mathrm{e}^{xy}}{y}\right]_0^{\infty} \mathrm{d}y \\ &= \int_a^b \frac{\mathrm{d}y}{y} = [\ln y]_a^b \\ &= \ln(b/a) \end{aligned}$$

(b) Consider the following integral: for any a and b,

$$\int_a^b \frac{\sin xy}{x}\mathrm{d}y = \left[-\frac{\cos(xy)}{x^2}\right]_a^b = \frac{\cos ax - \cos bx}{x^2}.$$

Since $\cos ax = \cos(-ax)$ and $\cos bx = \cos(-b)$, all signs of a and b are covered by

$$\frac{\cos ax - \cos bx}{x^2} = \int_{|a|}^{|b|} \frac{\sin xy}{x}\mathrm{d}y.$$

y is always positive in this integral, which is required in the substitution below. Now integrate both sides with respect to x and interchange the order of integration in the repeated integral so that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2}\mathrm{d}x &= \int_{-\infty}^{\infty}\int_{|a|}^{|b|} \frac{\sin(xy)}{x}\mathrm{d}y\mathrm{d}x = \int_{|a|}^{|b|}\int_{-\infty}^{\infty} \frac{\sin(xy)}{x}\mathrm{d}x\mathrm{d}y \\ &= \int_{|a|}^{|b|}\int_{-\infty}^{\infty} \frac{\sin u}{u}\mathrm{d}u\mathrm{d}y \text{ (putting } x = u/y\text{: } y \text{ is positive)} \\ &= \int_{|a|}^{|b|} \sqrt{\pi}\mathrm{d}y = \sqrt{\pi}(|b| - |a|) \end{aligned}$$

Chapter 33: Line integrals

33.1. The path AOB consists of two straight lines AO along which $\delta x < 0$, and OB along which $\delta x > 0$. In each case note where δx or δy changes sign and split the integral into separate segments on which δx and δy have constant sign.
(a) δxchanges sign at O, so we split the integral there:

$$\begin{aligned}\int_{(AOB)} x\mathrm{d}x &= \int_{(AO)} x\mathrm{d}x + \int_{(OB)} x\mathrm{d}x = \int_1^0 x\mathrm{d}x + \int_0^1 x\mathrm{d}x \\ &= -\int_0^1 x\mathrm{d}x + \int_0^1 x\mathrm{d}x = 0.\end{aligned}$$

(Notice that the integral is independent of the path connecting A and B.)
(b) On AO, $y = -x$ and on OB, $y = x$ and δx changes sign at O. Therefore

$$\begin{aligned}\int_{(AOB)} y\mathrm{d}x &= \int_{(AO)} y\mathrm{d}x + \int_{(OB)} y\mathrm{d}x = \int_1^0 (-x)\mathrm{d}x + \int_0^1 y\mathrm{d}x \\ &= -\frac{1}{2}[x^2]_1^0 + \frac{1}{2}[x^2]_0^1 \\ &= \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

(c) As in (a),

$$\begin{aligned}\int_{(AOB)} x^2\mathrm{d}x &= \int_{(AO)} x^2\mathrm{d}x + \int_{(OB)} x^2\mathrm{d}x = \int_1^0 x^2\mathrm{d}x + \int_0^1 x^2\mathrm{d}x \\ &= -\int_0^1 x^2\mathrm{d}x + \int_0^1 x^2\mathrm{d}x = 0\end{aligned}$$

33.2. On the parabola $y^2 = x$, $\delta x > 0$ for $y > 0$, $\delta x < 0$ for $y < 0$, and δy has constant sign.
(a)

$$\int_{\mathcal{P}} x\mathrm{d}x = \int_{(AO)} x\mathrm{d}x + \int_{(OB)} x\mathrm{d}x = \int_1^0 x\mathrm{d}x + \int_0^1 x\mathrm{d}x = 0.$$

(b) Since $y = x^2$ on the parabola

$$\int_{\mathcal{P}} y\mathrm{d}x = \int_1^0 (-x^{\frac{1}{2}})\mathrm{d}x + \int_0^1 x^{\frac{1}{2}}\mathrm{d}x = 2\int_0^1 x^{\frac{1}{2}}\mathrm{d}x = \frac{4}{3}.$$

(c)

$$\int_{\mathcal{P}} x^2\mathrm{d}x = \int_1^0 x^2\mathrm{d}x + \int_0^1 x^2\mathrm{d}x = 0.$$

(d) The element $\delta y > 0$ on AOB. Hence

$$\int_{\mathcal{P}} (x+y)\mathrm{d}y = \int_{-1}^1 (y^2+y)\mathrm{d}y = \left[\frac{y^3}{3} + \frac{y^2}{2}\right]_{-1}^1 = \frac{2}{3}.$$

(e) As in (d)

$$\int_{\mathcal{P}} xy^2\mathrm{d}y = \int_{-1}^1 y^4\mathrm{d}y = \left[\frac{y^5}{5}\right]_{-1}^1 = \frac{2}{5}.$$

(f) δx changes sign at o and $\delta y > 0$ throughout, so

$$\begin{aligned}\int_{\mathcal{P}} (x\mathrm{d}x + y\mathrm{d}y) &= \int_1^0 x\mathrm{d}x + \int_0^1 \mathrm{d}x + \int_{-1}^1 y\mathrm{d}y \\ &= 0 + \left[\frac{y^2}{2}\right]_{-1}^1 = 0\end{aligned}$$

(g) As in (f)

$$\int_{\mathcal{P}}(\tfrac{1}{2}\mathrm{d}x - y\mathrm{d}y) = \tfrac{1}{2}\int_1^0 \mathrm{d}x + \tfrac{1}{2}\int_0^1 \mathrm{d}x - \left[\tfrac{1}{2}y^2\right]_{-1}^1 = 0 - 0 = 0.$$

(h) On AO, $y = -x^{\frac{1}{2}}$ and on OB, $y = x^{\frac{1}{2}}$. Therefore

$$\begin{aligned}\int_{\mathcal{P}}(y\mathrm{d}x - x\mathrm{d}y) &= \int_1^0 (-x^{\frac{1}{2}})\mathrm{d}x + \int_0^1 x^{\frac{1}{2}}\mathrm{d}x - \int_{-1}^1 y^2\mathrm{d}y \\ &= 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 - \left[\frac{y^3}{3}\right]_{-1}^1 \\ &= \frac{4}{3} - \frac{2}{3} = \frac{2}{3}\end{aligned}$$

33.3. (a) $\mathcal{P}$ is given parametrically by $x = t^2$, $y = t$, $0 \le t \le 1$. Then

$$\int_{\mathcal{P}} xy^2\mathrm{d}x = \int_0^1 xy^2\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \int_0^1 t^4.2t\mathrm{d}t = 2\int_0^1 t^5\mathrm{d}t = \frac{1}{3}.$$

(b) $\mathcal{P}$ is given parametrically by $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$. Then

$$\begin{aligned}\int_{\mathcal{P}}(x\mathrm{d}y - y\mathrm{d}x) &= \int_0^\pi \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_0^\pi [\cos t.\cos t - \sin t(-\sin t)]\mathrm{d}t = \int_0^\pi \mathrm{d}t = \pi\end{aligned}$$

(c) $\mathcal{P}$ is given parametrically by $x = t+1$, $y = t$, $z = 2t$, $0 \le t \le 1$: this is a path in three dimensions. Then

$$\begin{aligned}\int_{\mathcal{P}}(z\mathrm{d}x - x\mathrm{d}y + y\mathrm{d}z) &= \int_0^1 \left(z\frac{\mathrm{d}x}{\mathrm{d}t} - x\frac{\mathrm{d}y}{\mathrm{d}t} + y\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_0^1 [2t - (t+1) + 2t]\mathrm{d}t = \int_0^1 (3t-1)\mathrm{d}t \\ &= \left[\frac{3}{2}t^2 - t\right]_0^1 = \frac{1}{2}.\end{aligned}$$

(d) $\mathcal{P}$ is given parametrically by $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_{\mathcal{P}}(x^2\mathrm{d}x + y^2\mathrm{d}y + z^2\mathrm{d}z) &= \int_0^{2\pi}\left(x^2\frac{\mathrm{d}x}{\mathrm{d}t} + y^2\frac{\mathrm{d}y}{\mathrm{d}t} + z^2\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_0^{2\pi}[\cos^2 t(-\sin t) + \sin^2 t\cos t + t^2]\mathrm{d}t \\ &= \left[\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t + \frac{1}{3}t^3\right]_0^{2\pi} \\ &= 0 + 0 + \frac{1}{3}(2\pi)^3 = \frac{8\pi^3}{3}\end{aligned}$$

(e) $\mathcal{P}$ is given parametrically by $x = t^2+1$, $y = 2t - t^2$, $z = 2t^2$, $0 \le t \le 1$. Then

$$\begin{aligned}\int_{\mathcal{P}}(z\mathrm{d}x - x\mathrm{d}y + y\mathrm{d}z) &= \int_0^1\left(z\frac{\mathrm{d}x}{\mathrm{d}t} - x\frac{\mathrm{d}y}{\mathrm{d}t} + y\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_0^1 [2t^2(2t) - (t^2+1)(2-2t) + (2t-t^2)(4t)]\mathrm{d}t \\ &= \int_0^1 [2t^3 + 6t^2 + 2t - 2]\mathrm{d}t = \left[\frac{1}{2}t^4 + 2t^3 + t^2 - 2\right]_0^1 \\ &= \frac{3}{2}\end{aligned}$$

33.4. On AO, $y = -x^{\frac{1}{2}}$, and on OB, $y = x^{\frac{1}{2}}$. Hence, with $f(x, y) = x + y$,

$$\begin{aligned}
&\int_{(AB)} f(x, k(x))\frac{\mathrm{d}k}{\mathrm{d}x}\mathrm{d}y \\
&= \int_{(AO)} (x+y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x + \int_{(OB)} (x+y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x \\
&= \int_1^0 (x - x^{\frac{1}{2}})\left(-\frac{1}{2}x^{-\frac{1}{2}}\right)\mathrm{d}x + \int_0^1 (x + x^{\frac{1}{2}})\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\mathrm{d}x \\
&= -\int_0^1 (x - x^{\frac{1}{2}})\left(-\frac{1}{2}x^{-\frac{1}{2}}\right)\mathrm{d}x + \int_0^1 (x + x^{\frac{1}{2}})\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\mathrm{d}x \\
&= \int_0^1 x^{\frac{1}{2}}\mathrm{d}x = \frac{2}{3}[x^{\frac{3}{2}}]_0^1 \\
&= \frac{2}{3},
\end{aligned}$$

which agrees with the answer to Problem 33.2(d).

33.5. On AB, $x = 1$ and $\delta x = 0$, on BC, $y = 2$ and $\delta y = 0$, on AO, $y = 0$ and $\delta y = 0$, and on OC, $x = 0$ and $\delta x = 0$.

(a) $$\int_{(ABC)} \mathrm{d}x = \int_{(AB)} \mathrm{d}x + \int_{(BC)} \mathrm{d}x = 0 + \int_1^0 \mathrm{d}x = [x]_1^0 = -1.$$

(b) $$\int_{(AOC)} \mathrm{d}y = \int_{(AO)} \mathrm{d}y + \int_{(OC)} \mathrm{d}y = \int_0^2 \mathrm{d}y + 0 = [y]_0^2 = 2.$$

(c)

$$\begin{aligned}
\int_{(ABC)} (x\mathrm{d}y - y\mathrm{d}x) &= \int_{(AB)} (x\mathrm{d}y - y\mathrm{d}x) + \int_{(BC)} (x\mathrm{d}y_y\mathrm{d}x) \\
&= \int_0^2 \mathrm{d}y + \int_1^0 (-2)\mathrm{d}x = [y]_0^2 - 2[x]_1^0 = 2 + 2 = 4.
\end{aligned}$$

(d) $$\int_{(AOC)} (x\mathrm{d}y - y\mathrm{d}x) = \int_{(AO)} (x\mathrm{d}y - y\mathrm{d}x) + \int_{(OC)} (x\mathrm{d}y - y\mathrm{d}x) = 0 + 0 = 0.$$

(e) $$\int_{(ABC)} y\mathrm{d}y = \int_{(AB)} y\mathrm{d}y + \int_{(BC)} y\mathrm{d}y = \int_0^2 y\mathrm{d}y + 0 = \frac{1}{2}[y^2]_0^2 = 2.$$

(f) $$\int_{(AOC)} y\mathrm{d}y = \int_{(AO)} y\mathrm{d}y + \int_{(OC)} y\mathrm{d}y = 0 + \int_0^2 y\mathrm{d}y = \frac{1}{2}[y^2]_0^2 = 2.$$

(g) $$\int_{(ABC)} (y\mathrm{d}x + x\mathrm{d}y) = \int_{(AB)} x\mathrm{d}y + \int_{(BC)} y\mathrm{d}x = \int_0^2 \mathrm{d}y + \int_1^0 2\mathrm{d}x = 2 - 2 = 0.$$

(h) $$\int_{(AOC)} (y\mathrm{d}x + x\mathrm{d}y) = \int_{(AO)} y\mathrm{d}x + \int_{(OC)} x\mathrm{d}y = 0 + 0 = 0,$$

since $y = 0$ on AO, and $x = 0$ on OC.

33.6. The integrand $f(x,y,z)\mathrm{d}x + g(x,y,z)\mathrm{d}y + h(x,y,z)\mathrm{d}z$ a perfect differential if there exists a function $S(x,y,z)$ such that

$$f(x,y,z) = \frac{\partial S}{\partial x}, \quad g(x,y,z) = \frac{\partial S}{\partial y}, \quad h(x,y,z) = \frac{\partial S}{\partial z}.$$

If S is single-valued in a region $\mathcal{R}$ and $\mathcal{P}$ in $\mathcal{R}$ is any path joining the points A and B, then

$$\int_{\mathcal{P}} [f(x,y,z)\mathrm{d}x + ig(x,y,z)\mathrm{d}y + h(x,y,z)]\mathrm{d}z = S_B - S_A.$$

(a) $\mathcal{P}$ is a path joining $(-1,1,-1)$ to $(1,-1,1)$. In this example $S(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$. Hence

$$\int_{\mathcal{P}} (x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z) = \frac{1}{2}\int_{\mathcal{P}} \mathrm{d}(x^2+y^2+z^2) = \frac{1}{2}[x^2+y^2+z^2]_{(-1,1,-1)}^{(1,-1,1)} = 3-3=0.$$

(b) $\mathcal{P}$ is a path joining $(0,0,0)$ to $(1,1,1)$. In this case $S(x,y,z) = xyz$. Hence

$$\int_{\mathcal{P}} (yz\mathrm{d}x + zx\mathrm{d}y + xy\mathrm{d}z) = \int_{\mathcal{P}} \mathrm{d}(xyz) = [xyz]_{(0,0,0)}^{(1,1,1)} = 1.$$

(c) $\mathcal{P}$ is a path joining $(0,0,0)$ to $(1,1,1)$. In this case $S(x,y,z) = \frac{1}{2}\mathrm{e}^{x^2+y^2+z^2}$. Hence

$$\begin{aligned}\int_{\mathcal{P}} \mathrm{e}^{x^2+y^2+z^2}(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z) &= \frac{1}{2}\int_{\mathcal{P}} \mathrm{d}(\mathrm{e}^{x^2+y^2+z^2}) \\ &= [\mathrm{e}^{x^2+y^2+z^2}]_{(0,0,0)}^{(1,1,1)} = \frac{1}{2}(\mathrm{e}^3-1).\end{aligned}$$

(d) $\mathcal{P}$ is a path joining $(1,1,1)$ to $(0,1,0)$. In this case, $S(x,y,z)$ is given by

$$\frac{\partial S}{\partial x} = y+z, \quad \frac{\partial S}{\partial y} = z+x, \quad \frac{\partial S}{\partial z} = x+y.$$

Integrating these partial derivatives in turn, we obtain

$$S(x,y,z) = xy + zx + f(y,z) \quad S(x,y,z) = yz + xy + g(x,z),$$
$$S(x,y,z) = zx + yz + h(x,y).$$

These equations are consistent if we choose

$$f(y,z) = yz, \quad g(x,z) = zx, \quad h(x,y) = xy.$$

Hence $S(x,y,z) = yz + zx + xy$, and

$$\begin{aligned}\int_{\mathcal{P}} [(y+z)\mathrm{d}x + (z+x)\mathrm{d}y + (x+y)\mathrm{d}z] &= \int_{\mathcal{P}} d(yz+zx+xy) \\ &= [yz+zx+xy]_{(1,1,1)}^{(0,1,0)} = 0-3=-3\end{aligned}$$

(e) $\mathcal{P}$ is a path joining $(1,0,\pi)$ to $(0,\pi,1)$. In this case $S(x,y,z)$ is given by

$$\frac{\partial S}{\partial x} = (y+z)\cos(xy+yz+zx) = \frac{\partial}{\partial x}(\sin(xy+yz+zx)),$$

$$\frac{\partial S}{\partial y} = (z+x)\cos(xy+yz+zx) = \frac{\partial}{\partial y}(\sin(xy+yz+zx)),$$

$$\frac{\partial S}{\partial z} = (x+y)\cos(xy+yz+zx) = \frac{\partial}{\partial z}(\sin(xy+yz+zx)).$$

Hence $S(x,y,z) = \sin(xy+yz+zx)$, and

$$\begin{aligned}\int [\cos(xy+yz+zx)][(y+z)\mathrm{d}x+(z+x)\mathrm{d}y+(x+y)\mathrm{d}z] & \\ &= \int_{\mathcal{P}} \mathrm{d}[\sin(xy+yz+zx)] \\ &= [\sin(xy+yz+zx)]_{(0,\pi,1)}^{(1,0,\pi)} = 0\end{aligned}$$

(f) $\mathcal{P}$ is a path joining $(1,1)$ to $(2,2)$. This is a path in the (x,y) plane. In this case $S(x,y) = \frac{1}{2}x^2y^2$. Hence

$$\int_{\mathcal{P}}(xy^2\mathrm{d}x + x^2y\mathrm{d}y) = \tfrac{1}{2}\int_{\mathcal{P}} \mathrm{d}(x^2y^2) = \tfrac{1}{2}[x^2y^2]_{(1,1)}^{(2,2)} = 8 - \tfrac{1}{2} = \tfrac{15}{2}.$$

33.7. (a) The circle $\mathcal{C} : x^2+y^2 = 4$ can be parametrized in an anticlockwise sense by $x = 2\cos t$, $y = 2\sin t$, $(0 \le t < 2\pi)$. Then

$$\begin{aligned}\int_{\mathcal{C}}(x^2\mathrm{d}y - y^2\mathrm{d}x) &= \int_0^{2\pi}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}t} - y^2\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_0^{2\pi}[4\cos^2 t.2\cos t - 4\sin^2 t(-2\sin t)]\mathrm{d}t \\ &= 8\int_0^{2\pi}(\cos^3 t + \sin^3 t)\mathrm{d}t = 0,\end{aligned}$$

since both $\cos^3 t$ and $\sin^3 t$ have mean value of zero over their period 2π.
(b) Given the parametrization $x = 2\cos\theta$, $y = 3\sin\theta$, $(0 \le \theta < 2\pi)$,

$$\begin{aligned}\int_{\mathcal{C}}\left(\frac{x}{y}\mathrm{d}x + \frac{y}{x}\mathrm{d}y\right) &= \int_0^{2\pi}\left(\frac{2\cos\theta}{3\sin\theta}(-2\sin\theta) + \frac{3\sin\theta}{2\cos\theta}(3\sin\theta)\right)\mathrm{d}\theta \\ &= \int_0^{2\pi}\left(-\frac{4}{3}\cos\theta + \frac{9}{4}\sin\theta\right)\mathrm{d}\theta = 0\end{aligned}$$

33.8. (a) The path $\mathcal{C}$ is given by $x = \sin t$, $y = \cos t$, $z = \sin t$. Then

$$\begin{aligned}\int_{\mathcal{C}}(y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z) &= \int_0^{2\pi}\left(y\frac{\mathrm{d}x}{\mathrm{d}t} + z\frac{\mathrm{d}y}{\mathrm{d}t} + x\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_0^{2\pi}[\cos t(\cos t) + \sin t(-\sin t) + \sin t(\cos t)]\mathrm{d}t \\ &= \int_0^{2\pi}[\cos^2 t - \sin^2 t + \sin t\cos t]\mathrm{d}t \\ &= \int_0^{2\pi}[\cos 2t + \tfrac{1}{2}\sin 2t]\mathrm{d}t = 0\end{aligned}$$

(b) ABC is the triangle $A : (1,0,0)$, $B : (0,1,0)$, $C : (0,0,1)$. On AB, $x+y = 1$ and $z = 0$, on BC, $y+z = 1$ and $x = 0$, on CA, $z+x =$ and $y = 0$. Hence

$$\begin{aligned}\int_{(ABC)}(y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z) &= \int_{(AB)}(y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z) \\ &\quad + int_{(BC)}(y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z) + \int_{(CA)}(y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z) \\ &= \int_1^0(1-x)\mathrm{d}x + \int_1^0(1-y)\mathrm{d}y + \int_1^0(1-z)\mathrm{d}z \\ &\quad \text{(using } x,\ y,\ z \text{ as parameters on } AB,\ BC,\ CA)\end{aligned}$$

$$= \left[x - \frac{1}{2}x^2\right]_1^0 + \left[y - \frac{1}{2}y^2\right]_1^0 + \left[z - \frac{1}{2}z^2\right]_1^0$$
$$= -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{3}{2}$$

(c) In differentials $\mathrm{d}S(x, y, z) = \mathrm{d}(xyz) = yz\mathrm{d}x + zx\mathrm{d}y + xy\mathrm{d}z$. Hence for any path between any two points, say, A and B,

$$\int_A^B (yz\mathrm{d}x + zx\mathrm{d}y + xy\mathrm{d}z) = \int_A^B \mathrm{d}(xyz) = S_B - S_A.$$

In other words the integral is independent of the path joining A and B. If A and B coincide than the path is a closed path $\mathcal{C}$ and $S_B = S_A$. Hence

$$\int_{\mathcal{C}} (yz\mathrm{d}x + zx\mathrm{d}y + xy\mathrm{d}z) = 0.$$

33.9. In terms of differentials $\mathrm{d}S(x, y, z) = \mathrm{d}(\frac{1}{3}x^3y) = yx^2\mathrm{d}x + \frac{1}{3}x^3\mathrm{d}y$. Hence

$$\int_{(AB)} (yx^2\mathrm{d}x + \frac{1}{3}x^3\mathrm{d}y) = \int_A^B \mathrm{d}(\tfrac{1}{3}x^3y) = S_B - S_A,$$

which depends only on the values of S at A and B.

Given the polar equation $r = \mathrm{e}^{\theta/(2\pi)}$ for $0 \le \theta \le \pi$, the point A occurs where $\theta = 0$, $r = 1$, and B is located at $\theta = 2\pi$, $r = \sqrt{\mathrm{e}}$. Hence A has coordinates $(1, 0)$, B has coordinates $(-\sqrt{\mathrm{e}}, 0)$. Therefore, since the integral is path-independent,

$$\textstyle\int_{(AB)} (yx^2\mathrm{d}x + \frac{1}{3}x^3\mathrm{d}y) = \int_{(AB)} \mathrm{d}(\frac{1}{3}x^3y) = [\frac{1}{3}x^3y]_{(1,0)}^{(-\sqrt{\mathrm{e}},0)} = 0.$$

33.10. Let $\mathcal{C}$ be any closed curve, and A and B any two points on $\mathcal{C}$. Consider the paths $\mathcal{C}_1 = (APB)$ and $\rfloor_2 = (BQA)$. Then

$$\int_{\mathcal{C}} (f\mathrm{d}x + g\mathrm{d}y) = \int_{\mathcal{C}_1} (f\mathrm{d}x + g\mathrm{d}y) + \int_{\mathcal{C}_2} (f\mathrm{d}x + g\mathrm{d}y).$$

But

$$\int_{\mathcal{C}_2} (f\mathrm{d}x + g\mathrm{d}y) = -\int_{AQB} (f\mathrm{d}x + g\mathrm{d}y) = -\int_{\mathcal{C}_1} (f\mathrm{d}x + g\mathrm{d}y)$$

(by hypothesis). Therefore

$$\int_{\mathcal{C}} (f\mathrm{d}x + g\mathrm{d}y) = \int_{\mathcal{C}_1} (f\mathrm{d}x + g\mathrm{d}y) - \int_{\mathcal{C}_1} (f\mathrm{d}x + g\mathrm{d}y) = 0.$$

33.11. We shall consider the two-dimensional case (in higher dimensions the procedure is the same). A differential form $f\mathrm{d}x + g\mathrm{d}y$ is 'perfect' if (and only it) there exists a function $S(x, y)$ such that $f = \partial S/\partial x$ and $g = \partial S/\partial y$. Given a perfect differential in the form

$$\frac{\partial S}{\partial x}\mathrm{d}x + \frac{\partial S}{\partial y}\mathrm{d}y,$$

change the variables from x, y to u, v, where

$$x = p(u, v), \quad y = q(u, v), \tag{i}$$

and put

$$S(x, y) = S(p(u, v), q(u, v)) = E(u, v). \tag{ii}$$

From the incremental formula and eqn (i),

$$\begin{aligned}
\frac{\partial S}{\partial x}\mathrm{d}x + \frac{\partial S}{\partial y}\mathrm{d}y &= \frac{\partial S}{\partial x}\left(\frac{\partial p}{\partial u}\mathrm{d}u + \frac{\partial p}{\partial v}\mathrm{d}v\right) + \frac{\partial S}{\partial y}\left(\frac{\partial q}{\partial u}\mathrm{d}u + \frac{\partial q}{\partial v}\mathrm{d}v\right) \\
&= \left(\frac{\partial S}{\partial x}\frac{\partial p}{\partial u} + \frac{\partial S}{\partial y}\frac{\partial q}{\partial u}\right)\mathrm{d}u + \left(\frac{\partial S}{\partial x}\frac{\partial p}{\partial v} + \frac{\partial S}{\partial y}\frac{\partial q}{\partial v}\right)\mathrm{d}v \\
&= \frac{\partial E}{\partial u}\mathrm{d}u + \frac{\partial E}{\partial v}\mathrm{d}v, \qquad \text{(iii)}
\end{aligned}$$

by the chain rule (30..6), using the notations in (i) and (ii). This is a perfect differential form in u and v.

In polar coordinates put $u = r$, $v = \theta$, with the change of variable

$$x = p(r,\theta) = r\cos\theta, \quad y = q(r,\theta) = r\sin\theta. \qquad \text{(iv)}$$

We have to verify independently that for the given case $\mathrm{d}(xy) = y\mathrm{d}x + x\mathrm{d}y$ takes the form (iii) in polar coordinates. In the above notation $S(x,y) = xy$, so from (ii),

$$E(r,\theta) = r^2\cos\theta\sin\theta, \quad \frac{\partial E}{\partial r} = 2r\cos\theta\sin\theta, \quad \frac{\partial E}{\partial \theta} = r^2(\cos^2\theta - \sin^2\theta) \qquad \text{(v)}$$

From (iv) and the incremental formula

$$\begin{aligned}
y\mathrm{d}x + x\mathrm{d}y &= r\sin\theta(\cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta) + r\cos\theta(\sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta) \\
&= 2r\cos\theta\sin\theta + r^2(\cos^2\theta - \sin^2\theta)\mathrm{d}\theta \\
&= \frac{\partial E}{\partial r}\mathrm{d}r + \frac{\partial E}{\partial \theta}, \qquad \text{(vi)}
\end{aligned}$$

by comparison with (v), confirming that under the change of coordinates to polars, the differential form remains 'perfect'.

33.12. Green's theorem in a plane states that (see 33.12)), if P and Q are smooth on $\mathcal{C}$ and its interior $\mathcal{A}$, then

$$\int_{\mathcal{C}} (P\mathrm{d}x + Q\mathrm{d}y) = \iint_{\mathcal{A}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}\mathcal{A},$$

where $\mathcal{A}$ is the region enclosed by a simple closed curve $\mathcal{C}$.
(a) As an example, let $\mathcal{C}$ be the circle $x^2+y^2 = 1$, and let $P(x,y) = x$, $Q(x,y) = x$. The curve $\mathcal{C}$ can be represented parametrically in an counterclockwise sense by $x = \cos t$, $y = \sin t$, $(0 \le t < 2\pi)$. Then

$$\begin{aligned}
\int_{\mathcal{C}} (P\mathrm{d}x + Q\mathrm{d}y) &= \int_{\mathcal{C}} (x\mathrm{d}x + x\mathrm{d}y = \int_0^{2\pi} [\cos t(-\sin t) + \cos t\cos t]\mathrm{d}t \\
&= \int_0^{2\pi} \left[-\frac{1}{2}\sin 2t + \frac{1}{2}(1+\cos 2t)\right]\mathrm{d}t \\
&= \left[\frac{1}{4}\cos 2t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} \\
&= \pi
\end{aligned}$$

The double integral becomes

$$\begin{aligned}
\iint_{\mathcal{A}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}\mathcal{A} &= \iint_{\mathcal{A}} \mathrm{d}\mathcal{A} \\
&= \text{(area of a circle of unit radius)} \\
&= \pi.
\end{aligned}$$

Hence Green's theorem is verified.

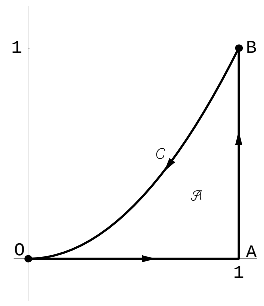

Figure 42: Problem 33.12(b)

(b) Let $\mathcal{C}$ be the closed curve bounded by $y = x^2$, $y = 0$ and $x = 1$, and let $P(x, y) = xy - y^2$, $Q(x, y) = x^2$. On $y = x^2$. The boundary $\mathcal{C}$ must be traversed in the counterclockwise sense. Hence

$$\begin{aligned}
\int_{\mathcal{C}} (P\mathrm{d}x + Q\mathrm{d}y) &= \int_{(OA)} [(xy - y^2)\mathrm{d}x + x^2\mathrm{d}y] + \int_{(AB)} [(xy - y^2)\mathrm{d}x + x^2\mathrm{d}y] \\
&\quad + \int_{(BO)} [(xy - y^2)\mathrm{d}x + x^2\mathrm{d}y] \\
&= 0 + \int_0^1 \mathrm{d}y + \int_1^0 [(x^3 - x^4)\mathrm{d}x + x^2(2x)\mathrm{d}x] \\
&= [y]_0^1 + \int_1^0 (3x^3 - x^4)\mathrm{d}x \\
&= 1 - \frac{3}{4} + \frac{1}{5} = \frac{9}{20}
\end{aligned}$$

The double integral gives

$$\begin{aligned}
\iint_{\mathcal{A}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}\mathcal{A} &= \iint_{\mathcal{A}} (x + 2y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^1 \int_0^{x^2} (x + 2y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left[xy + y^2\right]_{y=0}^{x^2} \mathrm{d}x \\
&= \int_0^1 [x^3 + x^4]\mathrm{d}x \\
&= \frac{1}{4} + \frac{1}{5} = \frac{9}{20},
\end{aligned}$$

which agrees with the line integral above.

33.13. By Green's theorem, the area $\mathcal{A}$ enclosed by the curve $\mathcal{C}$ taken counterclockwise is given by

$$\mathcal{A} = \tfrac{1}{2} \int_{\mathcal{C}} (x\mathrm{d}y - y\mathrm{d}x).$$

(a) The circle $x^2 + y^2 = 4$ can be described parametrically by $x = 2\cos t$, $y = \sin t$ for $0 \le t < 2\pi$. Therefore

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2} \int_0^{2\pi} \left[x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} \right] \mathrm{d}t = \frac{1}{2} \int_0^{2\pi} [2\cos t(2\cos t) - (2\sin t)(-2\sin t)]\mathrm{d}t \\
&= 2\int_0^{2\pi} \mathrm{d}t = 4\pi
\end{aligned}$$

which is the area of a circle of radius 2.

(b) The ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ can be described parametrically by $x = 2\cos t,\ y = 3\sin t$ for $0 \le t < 2\pi$. Therefore

$$\mathcal{A} = \frac{1}{2}\int_0^{2\pi} [2\cos t(3\cos t) - 3\sin t(-2\sin t)]\mathrm{d}t = 3\int_0^{2\pi} \mathrm{d}t = 6\pi,$$

which is the area of an ellispe with semi-axes 2 and 3.

(c) The path $\mathcal{C}$ is the triangle with vertices $A : (-1, 0)$, $B : (2, 0)$, $C : (0, 4)$. By the usual formula the area of the triangle is

$$\tfrac{1}{2}(\text{base}) \times \text{ (height)} = \tfrac{1}{2}(3 \times 4) = 6.$$

On AB, $y = 0$and $\delta y = 0$, on BC, $y = 4 - 2x$, and on CA, $y = 4 + 4x$. Therefore

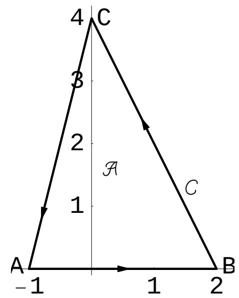

Figure 43: Problem 33.13(c)

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\int_{(AB)} (x\mathrm{d}y - y\mathrm{d}x) + \frac{1}{2}\int_{(BC)} (x\mathrm{d}y - y\mathrm{d}x) + \frac{1}{2}\int_{(CA)} (x\mathrm{d}y - y\mathrm{d}x) \\
&= 0 + \frac{1}{2}\int_2^0 [x(-2) - (4 - 2x)]\mathrm{d}x + \frac{1}{2}\int_0^{-1} [x(-4) - (4 - 4x)]\mathrm{d}x \\
&= \frac{1}{2}\int_2^0 (-4)\mathrm{d}x + \frac{1}{2}\int_0^{-1} (-4)\mathrm{d}x \\
&= \frac{1}{2}[-4x]_2^0 + \frac{1}{2}[-4x]_0^{-1} = 4 + 2 = 6,
\end{aligned}$$

which agrees with the result above.

33.14. The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ (shown in the figure) can be parametrized by putting $x = \cos^3 t$, $y = \sin^3 t$ for $0 \le t < 2\pi$. As in Example 33.9, the area $\mathcal{A}$ enclosed by the curve $\mathcal{C}$ is given by

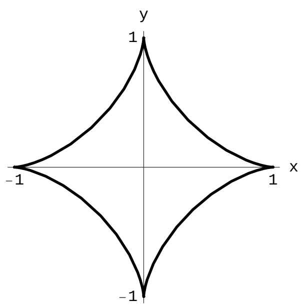

Figure 44: Problem 33.14: graph of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

$$\mathcal{A} = \frac{1}{2}\int_{\mathcal{C}} (x\mathrm{d}y - y\mathrm{d}x) = \frac{1}{2}\int_0^{2\pi} \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{2\pi}[\cos^3 t(3\sin^2 t\cos t) - \sin^3 t(-3\cos^2 t\sin t)]\mathrm{d}t \\
&= \frac{3}{2}\int_0^{2\pi}\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)\mathrm{d}t \\
&= \frac{3}{8}\int_0^{2\pi}\sin^2 2t\mathrm{d}t = \frac{3}{16}\int_0^{2\pi}(1-\cos 4t)\mathrm{d}t \\
&= \frac{3}{8}\pi.
\end{aligned}
$$

33.15. From (33.14), the work done is

$$
W = \int_{\mathcal{C}} \mathbf{F}.\mathrm{d}\mathbf{r},
$$

where $\mathcal{C}$ is a path from infinity to A, a point with position vector $\mathbf{R}$, and $\mathbf{F} = -\gamma Mm\mathbf{r}/r^3$. Therefore

$$
\begin{aligned}
W &= -\gamma Mm\int_{\mathcal{C}}\frac{\mathbf{r}}{r^3}.\mathrm{d}\mathbf{r} = -\gamma Mm\int_{\mathcal{C}}\frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{r^3} \\
&= -\gamma Mm\int_{\infty}^{R}\frac{r\mathrm{d}r}{r^3} = \gamma Mm\int_{\infty}^{R}\mathrm{d}(r^{-1}) \\
&= \frac{\gamma Mm}{R}.
\end{aligned}
$$

It is simply a notational change to replace $\mathbf{R}$ by $\mathbf{r}$ to obtain the work done to the point $\mathbf{r}$.

33.16. (a) Let $\mathbf{f} = (x^2 - y^2, 2xy)$. Then, for any closed curve $\mathcal{C}$ enclosing the region $\mathcal{A}$,

$$
\begin{aligned}
\int_{\mathcal{C}}\mathbf{f}\cdot\mathrm{d}\mathbf{r} &= \int_{\mathcal{C}}[(x^2-y^2)\mathrm{d}x + 2xy\mathrm{d}y] \\
&= \iint_{\mathcal{A}}\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right]\mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathcal{A}}[2y+2y]\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{A}}4y\mathrm{d}x\mathrm{d}y \neq 0
\end{aligned}
$$

in general. We conclude that $\mathbf{f}$ is not conservative.
(b) Let $\mathbf{f} = (\frac{1}{2}\ln(x^2+y^2), \arctan(y/x))$ for $x > 0$. Then, using (33.12), for any closed curve $\mathcal{C}$,

$$
\begin{aligned}
\iint_{\mathcal{C}}\mathbf{f}\cdot\mathrm{d}\mathbf{r} &= \int_{\mathcal{C}}\left[\frac{1}{2}\ln(x^2+y^2)\mathrm{d}x + \arctan\left(\frac{y}{x}\right)\mathrm{d}y\right] \\
&= \iint_{\mathcal{A}}\left[\frac{\partial}{\partial x}\left\{\arctan\left(\frac{y}{x}\right)\right\} - \frac{\partial}{\partial y}\left\{\frac{1}{2}\ln(x^2+y^2)\right\}\right]\mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathcal{A}}\left[-\frac{y}{x^2+y^2} - \frac{y}{x^2+y^2}\right]\mathrm{d}x\mathrm{d}y \\
&= -\iint_{\mathcal{A}}\frac{2y\mathrm{d}x\mathrm{d}y}{x^2+y^2}
\end{aligned}
$$

which is not zero in general. Hence $\mathbf{f}$ is not conservative.

33.17. We first decide whether $\mathbf{f} = y\hat{\mathbf{i}} + \hat{\mathbf{j}} + x\hat{\mathbf{k}}$ has a potential. If $\partial V/\partial x = -y$, then $V = -xy + g(y,z)$. Hence $\partial V/\partial y = -x + \partial g(y,z)/\partial y$. This can never be consistent with the $\hat{\mathbf{j}}$ component of $\mathbf{f}$ for any choice of $g(y,z)$. Hence $\mathbf{f}$ is not conservative. As a consequence work done will be path-dependent.

The path between $(0,0,0)$ and $(1,1,1)$ can be given parametrically by $x=t$, $y=t$, $z=t$ for $0 \le t \le 1$. Hence the work done on the path against $\mathbf{f}$ given by

$$W = -\int_{t=0}^{t=1} \mathbf{f}\cdot \mathrm{d}\mathbf{r} = -\int_0^1 (t,1,t)\cdot(1,1,1)\mathrm{d}t = -\int_0^1 (2t+1)\mathrm{d}t = -[t^2+t]_0^1 = -2.$$

33.18. Given $\mathbf{f}(x,y,z) = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$, it is obvious that $V(x,y,z) = -xyz$ is the potential of $\mathbf{f}$. The potential is single-valued, so that $\mathbf{f}$ is conservative.

The path $\mathcal{C}$, $x=\cos t$, $y=\sin t$, $z=\sin t\cos t$ for $-\frac{1}{2}\pi \le t \le \frac{1}{2}\pi$ joins the points $A:(0,-1,0)$ and $B:(0,1,0)$. Since $\mathbf{f}$ is conservative, the work done is independent of the path. Hence the work done against $\mathbf{f}$ is given by

$$W = -\int_{\mathcal{C}} \mathbf{f}\cdot\mathrm{d}\mathbf{r} = \int_{(AB)} \mathbf{grad}V\cdot\mathrm{d}\mathbf{r} = V_B - V_A = 0.$$

33.19. The vector field $\mathbf{f}$ can be written as

$$\mathbf{f} = r^{\alpha}\hat{\mathbf{r}} = r^{\alpha-1}\mathbf{r} = (x^2+y^2+z^2)^{(\alpha-1)/2}(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}).$$

Consider the potential $V(x,y,z) = k(x^2+y^2+z^2)^{\beta}$ where k and β are constants to be determined. The first components match if

$$-\frac{\partial V}{\partial x} = 2\beta kx(x^2+y^2+z^2)^{\beta-1} = x(x^2+y^2+z^2)^{(\alpha-1)/2}.$$

They are the same if $2\beta k = 1$ and $\beta - 1 = \frac{1}{2}(\alpha-1)$. Hence $\beta = \frac{1}{2}(\alpha+1)$ and $k = 1/(\alpha+1)$. Hence

$$V(x,y,z) = \frac{1}{\alpha+1}(x^2+y^2+z^2)^{(\alpha+1)/2}.$$

It can be seen by symmetry that $\partial V/\partial y$ and $\partial V/\partial z$ give the other components of $\mathbf{f}$. Hence $\mathbf{f}$ has a potential, and is single-valued.

33.20. With $\mathbf{f} = \hat{\mathbf{r}}f(r)$, $r = (x^2+y^2+z^2)^{\frac{1}{2}}$,

$$\mathbf{f} = \frac{f(r)}{r}\mathbf{r} = \frac{f(r)}{r}(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}).$$

Suppose that $V(x,y,z) = -g(r)$. Then the first component of $\mathbf{f}$ is given by

$$\frac{\partial V}{\partial x} = -g'(r)x(x^2+y^2+z^2)^{-\frac{1}{2}} = -x\frac{g'(r)}{r} = x\frac{f(r)}{r}$$

(and similarly for the other components of $\mathbf{f}$). Therefore $g'(r) = -f(r)$ so that

$$g(r) = -\int f(r)\mathrm{d}r :$$

any indefinite integral will do.

33.21. We transform the annular region into a simple closed path by bridging the points $(1,0)$ and $(2,0)$ by two paths in opposite directions AB and BA, whose contributions to the integral cancel. The closed path $\mathcal{C}$ consists of the circle $x^2+y^2=1$ taken clockwise, the line AB, the circle $x^2+y^2=4$ taken counterclockwise and the line BA. In the integral

$$\int_{\mathcal{C}} [(2x-y^3)\mathrm{d}x - xy\mathrm{d}y],$$

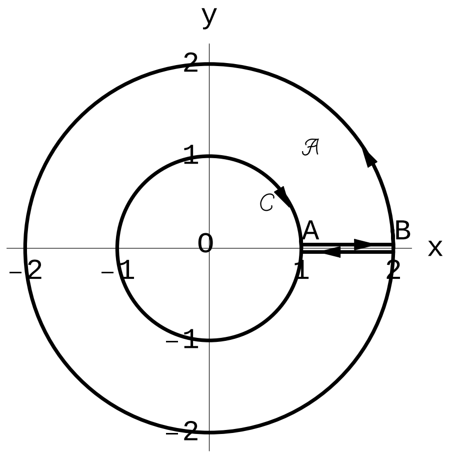

Figure 45: Problem 33.21

$P(x,y) = 2x - y^3$ and $Q(x,y) = -xy$. The circle $\mathcal{C}_1 : x^2 + y^2 = 4$ can be represented parametrically by $x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$, and the circle $\mathcal{C}_2 : x^2 + y^2 = 1$ can be represented by $x = \cos t$, $y = -\sin t$ for $0 \le t \le 2\pi$. Hence

$$\begin{aligned}
\int_{\mathcal{C}} [(2x - y^3)\mathrm{d}x - xy\mathrm{d}y] &= \int_{\mathcal{C}_2} + \int_{(BA)} + \int_{\mathcal{C}_1} + \int_{(AB)} [(2x - y^3)\mathrm{d}x - xy\mathrm{d}y] \\
&= \int_0^{2\pi} [-8\cos t \sin t + 16\sin^4 t - 4\sin t\cos^2 t]\mathrm{d}t \\
&\qquad + \int_0^{2\pi} [-2\sin t\cos t - \sin^4 t - \sin t\cos^2 t]\mathrm{d}t \\
&= 12\pi - \tfrac{3}{4}\pi = \tfrac{45}{4}\pi,
\end{aligned}$$

since the integrals along AB and BA cancel.

The double integral becomes

$$\begin{aligned}
\int\int_{\mathcal{A}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}\mathcal{A} &= \int\int_{\mathcal{A}} (-y + 3y^2)\mathrm{d}x\mathrm{d}y \\
&= \int_0^{2\pi}\int_1^2 (-r\sin\theta + 3r^2\sin^2\theta) r\mathrm{d}r\mathrm{d}\theta \\
&\qquad \text{(in polar coordinates)} \\
&= \tfrac{45}{4}\pi,
\end{aligned}$$

which agrees with the line integral.

33.22. In the integral

$$\int_{\mathcal{C}} (5x^4 y\mathrm{d}x + x^5\mathrm{d}y),$$

$P(x,y) = 5x^4 y$, and $Q(x,y) = x^5$. In Green's theorem the integrand of the double integral is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 5x^4 - 5x^4 = 0.$$

Hence the line integral is zero.

33.23. The closed curve generated by $(\cos t - \frac{1}{2}\sin 2t, \sin t)$ is shown in the figure. The area is

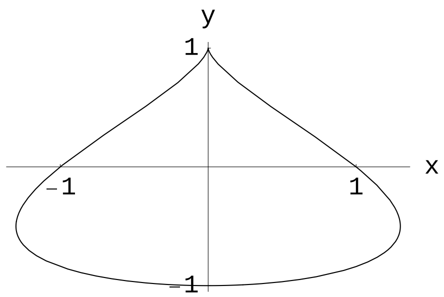

Figure 46: Problem 33.23

given by the formula in Example 33.9. Then

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\int_{\mathcal{C}}(x\mathrm{d}y - y\mathrm{d}x) = \int_0^{2\pi}[(\cos t - \tfrac{1}{2}\sin 2t)\cos t - \sin t(-\sin t - \cos 2t)]\mathrm{d}t \\
&= \int_0^{2\pi}(1 - \tfrac{1}{2}\sin 2t\cos t + \sin t\cos 2t)\mathrm{d}t \\
&= \int_0^{2\pi}[1 - \sin t\cos^2 t + \sin t(\cos^2 t - \sin^2 t)]\mathrm{d}t \\
&= \int_0^{2\pi}(1 - \sin^3 t)\mathrm{d}t \\
&= 2\pi,
\end{aligned}$$

Chapter 34: Vector fields: divergence and curl

34.1. Let $\mathcal{R}$ be the projection of the surface

$$\mathcal{S} : z = \sqrt{(a^2 - x^2 - y^2)} - a + h, \quad (0 < h \leq a).$$

on to the (x, y) plane. From Section 34.3, the surface area S is given by

$$S = \iint_{\mathcal{R}} \frac{\mathrm{d}A}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|},$$

where $\hat{\mathbf{n}}$ is a unit normal to S. By (28.7),

$$\hat{\mathbf{n}} = \frac{(-x, -y, -1)}{\sqrt{(a^2 - x^2 - y^2)}}.$$

Therefore

$$|\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}| = \frac{1}{\sqrt{(a^2 - x^2 - y^2)}}.$$

Also, $\mathcal{R}$ is a circle of radius $\sqrt{[a^2 - (a-h)^2]}$. Therefore

$$S = \iint_{\mathcal{R}} \sqrt{(a^2 - x^2 - y^2)}\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_0^{\sqrt{[a^2-(a-h)^2]}} \sqrt{(a^2 - r^2)}r\mathrm{d}r\mathrm{d}\theta$$

in polar coordinates. Hence, since the integrand is separable,

$$\begin{aligned}
S &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{[a^2-(a-h)^2]}} \sqrt{(a^2 - r^2)}r\mathrm{d}r \\
&= 2\pi\frac{1}{3}[-(a^2 - r^2)^{\frac{3}{2}}]_0^{\sqrt{[a^2-(a-h)^2]}} \\
&= \frac{2}{3}\pi[a^3 - (a-h)^3].
\end{aligned}$$

34.2. (a)

$$\begin{aligned}\int_0^1\int_0^z\int_y^{2y} x\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\int_0^z\left[\frac{1}{2}x^2\right]_y^{2y}\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\int_0^z\left(2y^2-\frac{1}{2}y^2\right)\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\left[\frac{2}{3}y^3-\frac{1}{6}y^3\right]_0^z\mathrm{d}z\\ &= \int_0^1\left(\frac{2}{3}z^3-\frac{1}{6}z^3\right)\mathrm{d}z\\ &= \left[\frac{1}{6}z^4-\frac{1}{24}z^4\right]_0^1=\frac{1}{6}-\frac{1}{24}=\frac{1}{8}\end{aligned}$$

(b)

$$\begin{aligned}\int_0^1\int_0^z\int_0^{\sqrt{(1-y^2)}} x\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\int_0^z\left[\frac{1}{2}x^2\right]_0^{\sqrt{(1-y^2)}}\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\int_0^z\left(\frac{1}{2}-\frac{1}{2}y^2\right)\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\left[\frac{1}{2}y-\frac{1}{6}y^3\right]_0^z\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\left(\frac{1}{2}z-\frac{1}{6}z^3\right)\mathrm{d}z\\ &= \left[\frac{1}{4}z^2-\frac{1}{24}z^4\right]_0^1=\frac{1}{4}-\frac{1}{24}=\frac{5}{24}\end{aligned}$$

(c)

$$\begin{aligned}\int_0^1\int_0^z\int_{-\frac{1}{2}\sqrt{(1-y^2-z^2)}}^{\sqrt{(1-y^2-z^2)}} x^3\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1\int_0^z\left[\frac{1}{4}x^4\right]_{-\frac{1}{2}\sqrt{(1-y^2-z^2)}}^{\sqrt{(1-y^2-z^2)}}\mathrm{d}y\mathrm{d}z\\ &= \int_0^1\int_0^z\left(\frac{1}{4}(1-y^2-z^2)^2-\frac{1}{64}(1-y^2-z^2)^2\right)\mathrm{d}y\mathrm{d}z\\ &= \frac{15}{64}\int_0^1\int_0^z(1+y^4+z^4+2y^2z^2-2y^2-2z^2)\mathrm{d}y\mathrm{d}z\\ &= \frac{15}{64}\int_0^1\left[y+\frac{1}{5}y^5+yz^4+\frac{2}{3}y^3z^2-\frac{2}{3}y^3-2yz^2\right]_0^z\mathrm{d}z\\ &= \frac{15}{64}\int_0^1\left(z-\frac{8}{3}z^3+\frac{28}{15}z^5\right)\mathrm{d}z\\ &= \frac{15}{64}\left[\frac{1}{2}-\frac{2}{3}+\frac{14}{45}\right]=\frac{13}{384}\end{aligned}$$

34.3. The figure shows that part of the sphere which lies in the first octant, and a box element of side-lengths δx, δy and δz. Imagine that the box slides parallel to the x axis between the sphere and the (y,z) plane. For a general point the limits on x are $x=0$ and $x=\sqrt{(a^2-y^2-z^2)}$. The resulting column is now moved in a plane parallel to the (x,y) between $y=0$ and $y=\sqrt{(a^2-z^2)}$. Finally this slab is moved in the z direction between $z=0$ and $z=a$ so that the whole octant has now been covered. The complete repeated integral reads

$$\int_0^a\int_0^{\sqrt{(a^2-z^2)}}\int_0^{\sqrt{(a^2-y^2-z^2)}} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

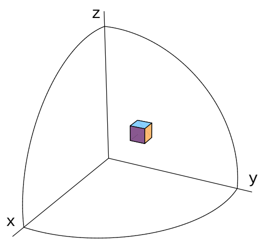

Figure 47: Problem 34.3

34.4. The tetrahedron is shown in the figure together with a box element with side-lengths δx, δy and δz. Slide the element in the x, then y and finally z directions to cover the interior of the

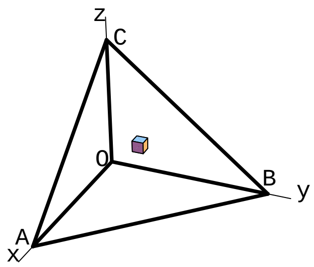

Figure 48: Problem 34.4

tetrahedron. Thus the volume V is given by

$$\begin{aligned}
V &= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^c \int_0^{b(1-\frac{z}{c})} a\left(1-\frac{y}{b}-\frac{z}{c}\right) \mathrm{d}y\mathrm{d}z \\
&= a\int_0^c \left[y-\frac{y^2}{2b}-\frac{yz}{c}\right]_0^{b(1-\frac{z}{c})} \mathrm{d}z \\
&= a\int_0^c \left(b\left(1-\frac{z}{c}\right)-\frac{b}{2}\left(1-\frac{z}{c}\right)^2-\frac{bz}{c}\left(1-\frac{z}{c}\right)\right)\mathrm{d}z \\
&= ab\int_0^c \left(\frac{1}{2}-\frac{z}{c}+\frac{z^2}{2c^2}\right)\mathrm{d}z \\
&= abc\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right] = \frac{1}{6}abc.
\end{aligned}$$

34.5. A unit normal to the surface $z = x^2 + y$ is (see (28.7))

$$\hat{\mathbf{n}} = \frac{(2x, 1, -1)}{\sqrt{(4x^2+1+1)}} = \frac{(2x, 1, -1)}{\sqrt{(4x^2+2)}}.$$

The surface area S is given by

$$S = \iint_{\mathcal{R}} \mathrm{d}\mathcal{S} = \iint_{\mathcal{R}} \frac{\mathrm{d}A}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|},$$

where $\mathcal{R}$ is the projection of $\mathcal{S}$ on to the (x, y) plane. In this case $\mathcal{R}$ is the square $|x| \le 1$, $|y| \le 1$ and

$$|\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}| = \frac{1}{\sqrt{(4x^2+2)}}.$$

Therefore

$$\begin{aligned} S &= \int_{-1}^{1}\int_{-1}^{1} \sqrt{(4x^2+2)}\mathrm{d}x\mathrm{d}y = \int_{-1}^{1} \sqrt{(4x^2+2)}\mathrm{d}x \int_{-1}^{1} \mathrm{d}y \\ &= 4\textstyle\int_{-1}^{1} \sqrt{(x^2+\frac{1}{2})}\mathrm{d}x. \end{aligned}$$

Using the substitution $x = (\sinh t)/\sqrt{2}$,

$$\begin{aligned} S &= 2\int_{-\sinh^{-1}(\sqrt{2})}^{\sinh^{-1}(\sqrt{2})} \sqrt{(1+\sinh^2 t)}\cosh t\mathrm{d}t = 2\int_{-\sinh^{-1}(\sqrt{2})}^{\sinh^{-1}(\sqrt{2})} \cosh^2 t\mathrm{d}t \\ &= \int_{-\sinh^{-1}(\sqrt{2})}^{\sinh^{-1}(\sqrt{2})} (1+\cosh 2t)\mathrm{d}t \\ &= \left[t + \frac{1}{2}\sinh 2t\right]_{-\sinh^{-1}(\sqrt{2})}^{\sinh^{-1}(\sqrt{2})} \\ &= [t + \sinh t\cosh t]_{-\sinh^{-1}(\sqrt{2})}^{\sinh^{-1}(\sqrt{2})} \\ &= 2\sinh^{-1}(\sqrt{2}) + 2\sqrt{2}\sqrt{3} = 2\sinh^{-1}(\sqrt{2}) + 2\sqrt{6} \end{aligned}$$

34.6. The vector field $\mathbf{F}$ is irrotational if $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$. For the given vector field $\mathbf{F}$, using (34.8),

$$\begin{aligned} \mathbf{curl}\,F &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz\mathrm{e}^{xyz} - y\sin xy + z & xz\mathrm{e}^{xyz} - x\sin xy & xy\mathrm{e}^{xyz} + x \end{vmatrix} \\ &= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}(xy\mathrm{e}^{xyz} + x) - \frac{\partial}{\partial z}(xz\mathrm{e}^{xyz} - x\sin xy)\right] \\ &\qquad +\hat{\mathbf{j}}\left[\frac{\partial}{\partial z}(yz\mathrm{e}^{xyz} - y\sin xy + z) - \frac{\partial}{\partial x}(xy\mathrm{e}^{xyz} + x)\right] \\ &\qquad\quad +\hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(xz\mathrm{e}^{xyz} - x\sin xy) - \frac{\partial}{\partial y}(yz\mathrm{e}^{xyz} - y\sin xy + z)\right] \\ &= \mathbf{0}. \end{aligned}$$

Since $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$, there exists a function ϕ such that $\mathbf{F} = \mathbf{grad}\,\phi$. Therefore

$$\frac{\partial \phi}{\partial x} = yz\mathrm{e}^{xyz} - y\sin xy + z,$$

$$\frac{\partial \phi}{\partial y} = xz\mathrm{e}^{xyz} - x\sin xy,$$

$$\frac{\partial \phi}{\partial z} = xy\mathrm{e}^{xyz} + x.$$

Integrate the partial derivatives with respect to x, y and z respectively:

$$\phi = \int (yz\mathrm{e}^{xyz} - y\sin xy + z)\mathrm{d}x + f(y,z) = \mathrm{e}^{xyz} + xz + \cos xy + f(y,z),$$

$$\phi = \int (xz\mathrm{e}^{xyz} - x\sin xy)\mathrm{d}y + g(z,x) = \mathrm{e}^{xyz} + \cos xy + g(z,x),$$

$$\phi = \int (xye^{xyz} + x)\mathrm{d}z + h(x,y) = e^{xyz} + xz + h(x,y).$$

These expressions for ϕ are consistent if

$$f(y,z) = C, \quad g(z,x) = xz + C, \quad h(x,y) = \cos xy + C,$$

in which case

$$\phi = e^{xyz} + xz + \cos xy + C.$$

34.7. If $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and

$$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \frac{1}{2}(u^2 - v^2),$$

the scale factors of this paraboloidal transformation are (see (34.18))

$$\begin{aligned} h_1 &= \left|\frac{\partial \mathbf{r}}{\partial u}\right| = |v\cos\phi\hat{\mathbf{i}} + v\sin\phi\hat{\mathbf{j}} + u\hat{\mathbf{k}}|, \\ &= \sqrt{[v^2\cos^2\phi + v^2\sin^2\phi + u^2]} = \sqrt{(u^2+v^2)}, \\ h_2 &= \left|\frac{\partial \mathbf{r}}{\partial v}\right| = |u\cos\phi\hat{\mathbf{i}} + u\sin\phi\hat{\mathbf{j}} - v\hat{\mathbf{k}}|, \\ &= \sqrt{[u^2\cos^2\phi + u^2\sin^2\phi + v^2]} = \sqrt{(u^2+v^2)} \\ h_3 &= \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = |-uv\sin\phi\hat{\mathbf{i}} + uv\cos\phi\hat{\mathbf{j}}|, \\ &= \sqrt{(u^2v^2\sin^2\phi + u^2v^2\cos^2\phi)} = uv, \end{aligned}$$

since $u \geq 0$, $v \geq 0$.

By (34.20), for any vector field $\mathbf{F} = F_u\hat{\mathbf{e}}_u + F_v\hat{\mathbf{e}}_v + F_\phi\hat{\mathbf{e}}_\phi$,

$$\begin{aligned} \operatorname{div}\mathbf{F} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3F_u) + \frac{\partial}{\partial v}(h_3h_1F_v) + \frac{\partial}{\partial \phi}(h_2h_3F_\phi)\right] \\ &= \frac{1}{uv(u^2+v^2)}\left[\frac{\partial}{\partial u}[uv\sqrt{(u^2+v^2)}F_u] + \frac{\partial}{\partial v}[uv\sqrt{(u^2+v^2)}F_v] \right. \\ &\qquad \left. + \frac{\partial}{\partial \phi}[(u^2+v^2)F_\phi]\right] \end{aligned}$$

34.8. In the identities assume that $\mathbf{F} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}$ and $\mathbf{G} = G_1\hat{\mathbf{i}} + G_2\hat{\mathbf{j}} + G_3\hat{\mathbf{k}}$.

(a)

$$\begin{aligned} \mathbf{grad}\,(UV) &= \frac{\partial(UV)}{\partial x}\hat{\mathbf{i}} + \frac{\partial(UV)}{\partial y}\hat{\mathbf{j}} + \frac{\partial(UV)}{\partial z}\hat{\mathbf{k}} \\ &= \left(U\frac{\partial V}{\partial x} + V\frac{\partial U}{\partial x}\right)\hat{\mathbf{i}} + \left(U\frac{\partial V}{\partial y} + V\frac{\partial U}{\partial y}\right)\hat{\mathbf{j}} + \left(U\frac{\partial V}{\partial z} + V\frac{\partial U}{\partial z}\right)\hat{\mathbf{k}} \\ &= U\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) + V\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}}\right) \\ &= U\mathbf{grad}\,V + V\mathbf{grad}\,U. \end{aligned}$$

(b)

$$\begin{aligned} \operatorname{div}(U\mathbf{F}) &= \frac{\partial}{\partial x}(UF_1) + \frac{\partial}{\partial y}(UF_2) + \frac{\partial}{\partial z}(UF_3) \\ &= \frac{\partial U}{\partial x}F_1 + U\frac{\partial F_1}{\partial x} + \frac{\partial U}{\partial y}F_2 + U\frac{\partial F_2}{\partial y} + \frac{\partial U}{\partial z}F_3 + U\frac{\partial F_3}{\partial z} \\ &= \left(\frac{\partial U}{\partial x}F_1 + \frac{\partial U}{\partial y}F_2 + \frac{\partial U}{\partial z}F_3\right) + U\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \\ &= (\mathbf{grad}\,U)\cdot\mathbf{F} + U\operatorname{div}\mathbf{F} \end{aligned}$$

(c) The vector product is given by

$$\mathbf{F}\times\mathbf{G} = (F_2G_3 - F_3G_2)\hat{\mathbf{i}} + (F_3G_1 - F_1G_3)\hat{\mathbf{j}} + (F_1G_2 - F_2G_1)\hat{\mathbf{k}}.$$

Hence

$$\begin{aligned}
\operatorname{div}(\mathbf{F}\times\mathbf{G}) &= \frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \frac{\partial}{\partial y}(F_3G_1 - F_1G_3) + \frac{\partial}{\partial z}(F_1G_2 - F_2G_1)\\
&= G_1\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + G_2\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + G_3\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\\
&\quad - \left[F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) + F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) + F_3\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\right]\\
&= (\mathbf{curl}\,\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\mathbf{curl}\,\mathbf{G}).
\end{aligned}$$

(d) Using (34.8),

$$\begin{aligned}
\mathbf{curl}\,\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}\\
&= \left(\frac{\partial^2 F_2}{\partial y\partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial x\partial z}\right)\hat{\mathbf{i}}\\
&\quad + \left(\frac{\partial^2 F_3}{\partial z\partial y} - \frac{\partial^2 F_2}{\partial z^2} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y}\right)\hat{\mathbf{j}}\\
&\quad + \left(\frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} - \frac{\partial^2 F_2}{\partial y\partial z}\right)\hat{\mathbf{k}}
\end{aligned}$$

Consider the first component of $\mathbf{curl}\,\mathbf{curl}\,\mathbf{F}$. Then, by adding and subtracting a term $\partial^2 F_1/\partial x^2$ to the $\hat{\mathbf{i}}$ component, we obtain,

$$\begin{aligned}
&\left(\frac{\partial^2 F_2}{\partial y\partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial x\partial z}\right)\hat{\mathbf{i}}\\
&= \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y\partial x} + \frac{\partial^2 F_3}{\partial x\partial z} - \frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_3}{\partial z^2}\right)\hat{\mathbf{i}}\\
&= \left[\left(\frac{\partial}{\partial x}\operatorname{div}\mathbf{F}\right) + \operatorname{div}\mathbf{grad}\,F_1\right]\hat{\mathbf{i}}.
\end{aligned}$$

Similar expansions occur can be found for the $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ components. Therefore addition of these terms leads to

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{F} = \mathbf{grad}\,(\operatorname{div}\mathbf{F}) - \operatorname{div}\mathbf{grad}\,\mathbf{F}.$$

(e) Consider the $\hat{\mathbf{i}}$ component of each term of

$$(\mathbf{F}\times\mathbf{curl}\,\mathbf{G}) + (\mathbf{G}\times\mathbf{curl}\,\mathbf{F}) + (\mathbf{F}\cdot\mathbf{grad}\,)\mathbf{G} + \mathbf{G}\cdot\mathbf{grad}\,)\mathbf{F}.$$

The expansions of the individual terms are as follows:

$$\begin{aligned}
\mathbf{F}\times\mathbf{curl}\,\mathbf{G} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ F_1 & F_2 & F_3 \\ \frac{\partial G_2}{\partial y} - \frac{\partial G_2}{\partial z} & \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} & \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \end{vmatrix}\\
&= \left[F_2\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) - F_3\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right)\right]\hat{\mathbf{i}} + \cdots,
\end{aligned}$$

$$\begin{aligned}
\mathbf{G}\times\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ G_1 & G_2 & G_3 \\ \frac{\partial F_2}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}\\
&= \left[G_2\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - G_3\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]\hat{\mathbf{i}} + \cdots,
\end{aligned}$$

$$\begin{aligned}(\mathbf{F}\cdot\mathbf{grad}\,)\mathbf{G} &= \left(F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}\right)(G_1\hat{\mathbf{i}}+G_2\hat{\mathbf{j}}+G_3\hat{\mathbf{k}})\\ &= \left(F_1\frac{\partial G_1}{\partial x}+F_2\frac{\partial G_1}{\partial y}+F_3\frac{\partial G_1}{\partial z}\right)\hat{\mathbf{i}}+\cdots,\end{aligned}$$

$$\begin{aligned}(\mathbf{G}\cdot\mathbf{grad}\,)\mathbf{F} &= \left(G_1\frac{\partial}{\partial x}+G_2\frac{\partial}{\partial y}+G_3\frac{\partial}{\partial z}\right)(F_1\hat{\mathbf{i}}+F_2\hat{\mathbf{j}}+F_3\hat{\mathbf{k}})\\ &= \left(G_1\frac{\partial F_1}{\partial x}+G_2\frac{\partial F_1}{\partial y}+G_3\frac{\partial F_1}{\partial z}\right)\hat{\mathbf{i}}+\cdots,\end{aligned}$$

Adding these $\hat{\mathbf{i}}$ components on the right-hand side:

$$\begin{aligned}&\hat{\mathbf{i}}\text{ component of }\mathbf{grad}\,(\mathbf{F}\cdot\mathbf{G}) =\\ &= \left(F_1\frac{\partial G_1}{\partial x}+G_1\frac{\partial F_1}{\partial x}+F_2\frac{\partial G_2}{\partial x}+G_2\frac{\partial F_2}{\partial x}+F_3\frac{\partial G_3}{\partial x}+G_3\frac{\partial F_3}{\partial x}\right)\hat{\mathbf{i}}\\ &= \left(\frac{\partial(F_1G_1)}{\partial x}+\frac{\partial(F_2G_2)}{\partial x}+\frac{\partial(F_3G_3)}{\partial x}\right)\hat{\mathbf{i}}\\ &= \frac{\partial}{\partial x}(\mathbf{F}\cdot\mathbf{G})\hat{\mathbf{i}},\end{aligned}$$

The other components can be verified in a similar manner.

34.9. Since

$$\mathbf{grad}\,\phi=\frac{\partial\phi}{\partial x}\hat{\mathbf{i}}+\frac{\partial\phi}{\partial y}\hat{\mathbf{j}}+\frac{\partial\phi}{\partial z}\hat{\mathbf{k}},$$

then, from the definition (34.4),

$$\begin{aligned}\operatorname{div}\mathbf{grad}\,\phi &= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)\\ &= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}.\end{aligned}$$

If $\phi=1/\sqrt{(x^2+y^2+z^2)}$, then

$$\frac{\partial\phi}{\partial x}=-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} &= -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\\ &= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.\end{aligned}$$

Similarly

$$\frac{\partial^2\phi}{\partial y^2}=\frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}},\qquad \frac{\partial^2\phi}{\partial z^2}=\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

The sum of these second derivatives is zero, confirming that $\phi=1/\sqrt{(x^2+y^2+z^2)}$ satisfies Laplace's equation.

34.10. (a) Let $\mathbf{F}=F_1\hat{\mathbf{i}}+F_2\hat{\mathbf{j}}+F_3\hat{\mathbf{k}}$ and $\mathbf{G}=G_1\hat{\mathbf{i}}+G_2\hat{\mathbf{j}}+G_3\hat{\mathbf{k}}$. Then, from the definition (34.4),

$$\begin{aligned}\operatorname{div}(\mathbf{F}+\mathbf{G}) &= \frac{\partial}{\partial x}(F_1+G_1)+\frac{\partial}{\partial y}(F_2+G_2)+\frac{\partial}{\partial z}(F_3+G_3)\\ &= \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)+\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)\\ &= \operatorname{div}\mathbf{F}+\operatorname{div}\mathbf{G}\end{aligned}$$

(b) From (34.8)

$$\begin{aligned}
\mathbf{curl}\,(\mathbf{F}+\mathbf{G}) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1+G_1 & F_2+G_2 & F_3+G_3 \end{vmatrix} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} \\
&= \mathbf{curl}\,\mathbf{F} + \mathbf{curl}\,\mathbf{G}.
\end{aligned}$$

The last step uses a property of determinants illustrated in Problem 8.10.

34.11. (a) Using definition (34.4),

$$\begin{aligned}
\mathrm{div}\,(\mathrm{e}^{xyz}\hat{\mathbf{i}} + \mathrm{e}^{y^2z}\hat{\mathbf{j}} + \mathrm{e}^{xz}\hat{\mathbf{k}}) &= \frac{\partial}{\partial x}(\mathrm{e}^{xyz}) + \frac{\partial}{\partial y}(\mathrm{e}^{y^2z}) + \frac{\partial}{\partial z}\mathrm{e}^{xz}) \\
&= yz\mathrm{e}^{xyz} + 2yz\mathrm{e}^{y^2z} + x\mathrm{e}^{xz}
\end{aligned}$$

(b) Using definition (3.4)

$$\begin{aligned}
\mathrm{div}\,((xz-y)\hat{\mathrm{i}} + yz\hat{\mathrm{j}} + 2xy\hat{\mathrm{j}} &= \frac{\partial}{\partial x}(xz-y) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(2xy) \\
&= z = z = 2z.
\end{aligned}$$

(c) Using definition (34.4),

$$\begin{aligned}
\mathrm{div}\,[(xz-y^2)\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + 2x^2y\hat{\mathbf{k}}] &= \frac{\partial}{\partial x}(xz-y^2) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(2x^2y) \\
&= z + z = 2z.
\end{aligned}$$

Hence

$$\mathrm{div}\,\mathbf{F} = 2z.$$

None of the vector fields is solenoidal.

34.12. Use the definition (34.8).

(a) $\mathbf{F} = \mathrm{e}^{xyz}\hat{\mathbf{i}} + \mathrm{e}^{y^2z}\hat{\mathbf{j}} + \mathrm{e}^{xz}\hat{\mathbf{k}}$. Then

$$\begin{aligned}
\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{e}^{xyz} & \mathrm{e}^{y^2z} & \mathrm{e}^{xz} \end{vmatrix} \\
&= -y^2\mathrm{e}^{y^2z}\hat{\mathbf{i}} + (xy\mathrm{e}^{xyz} - z\mathrm{e}^{xz})\hat{\mathbf{j}} - xz\mathrm{e}^{xyz}\hat{\mathbf{k}}.
\end{aligned}$$

(b) $\mathbf{F} = (xz-y)\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + 2xy\hat{\mathbf{k}}$. Then

$$\begin{aligned}
\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz-y & yz & 2xy \end{vmatrix} \\
&= (2x-y)\hat{\mathbf{i}} + (x-2y)\hat{\mathbf{j}} + \hat{\mathbf{k}}
\end{aligned}$$

(c) $\mathbf{F} = (2xy+yz)\hat{\mathbf{i}} + (x^2+xz)\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$. Then

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+yz & x^2+xz & xy \end{vmatrix} = \mathbf{0}.$$

Hence the $\mathbf{F}$ is irrotational.

34.13. Since the vector field is irrotational, $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$, which implies that there exists a scalar potential such that $\mathbf{v} = \mathbf{grad}\,\Phi$. If the vector field $\mathbf{v}$ is also solenoidal, then $\operatorname{div}\mathbf{v} = 0$. Therefore

$$\operatorname{div}\mathbf{grad}\,\Phi = 0, \text{ or } \nabla^2\Phi = 0.$$

34.14. We are given $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = (x^2+y^2+z^2)^{\frac{1}{2}}$.

(a)

$$\begin{aligned}
\operatorname{div}(r^2\mathbf{r}) &= \frac{\partial}{\partial x}(xr^2) + \frac{\partial}{\partial y}(yr^2) + \frac{\partial}{\partial z}(zr^2) \\
&= (r^2+2x^2) + (r^2+2y^2) + (r^2+2z^2) = 5r^2.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{curl}\,(r^3\mathbf{r}) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^3x & r^3y & r^3z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(zr^3) - \frac{\partial}{\partial z}(yr^3)\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}(xr^3) - \frac{\partial}{\partial x}(zr^3)\right)\hat{\mathbf{j}} \\
&\qquad + \left(\frac{\partial}{\partial x}(yr^3) - \frac{\partial}{\partial y}(xr^3)\right)\hat{\mathbf{k}} \\
&= \mathbf{0},
\end{aligned}$$

since

$$\frac{\partial}{\partial y}(zr^3) - \frac{\partial}{\partial z}(yr^3) = z\frac{\partial(r^3)}{\partial y} - y\frac{\partial(r^3)}{\partial z} = 3zyr - 3zyr = 0,$$

and similarly for the other two components. The function $r^3\mathbf{r}$ is therefore irrotational.

(c)

$$\mathbf{grad}\,(r^3) = \mathbf{grad}\,(x^2+y^2+z^2)^{\frac{3}{2}} = 3r\mathbf{r}.$$

(d)

$$\begin{aligned}
\operatorname{div}(\mathbf{r}/r^3) &= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) \\
&= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \\
&= 0.
\end{aligned}$$

The function $\mathbf{r}/r^3$ is therefore solenoidal.

(e)

$$\begin{aligned}
\mathbf{curl}\,(\mathbf{r}/r^2) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x/r^2 & y/r^2 & z/r^2 \end{vmatrix} \\
&= \left(-\frac{2zy}{r^3} + \frac{2yz}{r^3}\right)\hat{\mathbf{i}} + \left(-\frac{2zx}{r^3} + \frac{2xz}{r^3}\right)\hat{\mathbf{j}} + \left(-\frac{2yx}{r^3} + \frac{2xy}{r^3}\right)\hat{\mathbf{k}} \\
&= \mathbf{0}
\end{aligned}$$

Therefore the function $\mathbf{r}/r^2$ is irrotational.

(f)

$$\begin{aligned}\mathbf{grad}\,(r^3) &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{3}{2}}\hat{\mathbf{i}} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{3}{2}}\hat{\mathbf{j}} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{3}{2}}\hat{\mathbf{k}}\\ &= 3xr\hat{\mathbf{i}} + 3yr\hat{\mathbf{j}} + 3zr\hat{\mathbf{k}} = 3r\mathbf{r}.\end{aligned}$$

Secondly

$$\begin{aligned}\mathrm{div}\,\mathbf{grad}\,r^3 &= \mathrm{div}\,(3r\mathbf{r}) = 3\left[\frac{\partial}{\partial x}(rx) + \frac{\partial}{\partial y}(ry) + \frac{\partial}{\partial z}(rz)\right]\\ &= 3\left[r + \frac{x^2}{r} + r + \frac{y^2}{r} + r + \frac{z^2}{r}\right] = 12r\end{aligned}$$

34.15. Compare the components on both sides of

$$(\mathbf{v}\cdot\mathbf{grad}\,)\mathbf{v} = \tfrac{1}{2}\mathbf{grad}\,v^2 - (\mathbf{v}\times\mathbf{curl}\,\mathbf{v}).$$

Let $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$. The $\hat{\mathbf{i}}$ component of $(\mathbf{v}\cdot\mathbf{grad}\,)\mathbf{v}$ is

$$v_1\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_1}{\partial y} + v_3\frac{\partial v_1}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x}(v_1^2) + v_2\frac{\partial v_1}{\partial y} + v_3\frac{\partial v_1}{\partial z}.$$

On the right, the $\hat{\mathbf{i}}$ component of $\frac{1}{2}\mathbf{grad}\,v^2 - \mathbf{v}\times\mathbf{curl}\,\mathbf{v}$ is

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial x}(v_1^2+v_2^2+v_3^2) - v_2\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) + v_3\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\\ &= \frac{1}{2}\frac{\partial}{\partial x}(v_1^2+v_2^2+v_3^2) - \frac{1}{2}\frac{\partial}{\partial x}v_2^2 - \frac{1}{2}\frac{\partial}{\partial x}v_3^2 + v_2\frac{\partial v_1}{\partial y} + v_3\frac{\partial v_1}{\partial z}\\ &= \frac{1}{2}\frac{\partial}{\partial x}(v_1^2) + v_2\frac{\partial v_1}{\partial y} + v_3\frac{\partial v_1}{\partial z},\end{aligned}$$

which agrees with the $\hat{\mathbf{i}}$ component of $(\mathbf{v}\cdot\mathbf{grad}\,)\mathbf{v}$. A similar argument applies to the other components.

34.16. Refer to Section 34.6. Laplace's equation is given by $\mathrm{div}\,\mathbf{grad}\,U = 0$. In cylindrical polar coordinates (ρ,ϕ,z) (see p. 689),

$$\mathbf{grad}\,U = \frac{\partial U}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial U}{\partial z}\hat{\mathbf{e}}_z.$$

Let $\mathbf{F} = F_\rho\hat{\mathbf{e}}_\rho + F_\phi\hat{\mathbf{e}}_\phi + F_z\hat{\mathbf{e}}_z = \mathbf{grad}\,U$. Then

$$\begin{aligned}\mathrm{div}\,\mathbf{F} = \mathrm{div}\,\mathbf{grad}\,U &= \frac{1}{\rho}\left[\frac{1}{\rho}(\rho F_\rho) + \frac{\partial}{\partial\phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right]\\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2} + \frac{\partial^2 U}{\partial z^2}\end{aligned}$$

from which Laplace's equation follows.

If $U = f(\rho)$, then the partial derivatives $\partial U/\partial\phi = 0$ and $\partial U/\partial z = 0$. Hence $f(\rho)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}U}{\mathrm{d}\rho}\right) = 0,\ \text{ or }\ \rho f''(\rho) + f'(\rho) = 0.$$

This is a separable equation which for $f'(\rho)$:

$$\int\frac{\mathrm{d}f'}{f'} = -\int\frac{\mathrm{d}\rho}{\rho} + A,$$

with solution (since $\rho > 0$)

$$\ln f' = -\ln \rho + C, \text{ or } f' = \frac{B}{\rho}.$$

This is a further separable equation with solution given by

$$\int \mathrm{d}f = \int \frac{B\mathrm{d}\rho}{\rho} + A, \text{ or } F = A + B\ln\rho.$$

34.17. Refer to Section 34.6. Laplace's equation is given by $\operatorname{div}\mathbf{grad}\, U = 0$. In spherical polar coordinates (r, θ, ϕ) (see Example 34.7),

$$\mathbf{grad}\, U = \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi.$$

Let $\mathbf{F} = F_r\hat{\mathbf{e}}_r + F_\theta\hat{\mathbf{e}}_\theta + F_\phi\hat{\mathbf{e}}_\phi = \mathbf{grad}\, U$. Then

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= \operatorname{div}\mathbf{grad}\, U \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(F_\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}.
\end{aligned}$$

from which Laplace's equation follows.

If $U = f(r)$, then Laplace's equation reduces to

$$(r^2 f'(r))' = 0.$$

By integration $r^2 f'(r) = \text{constant} = -B$, say. Hence

$$f'(r) = -\frac{B}{r}.$$

Integrating

$$f(r) = -\int \frac{B\mathrm{d}r}{r} + A = \frac{B}{r} + A,$$

where A is a constant.

34.18. The divergence theorem (34.7) states that

$$\iiint_{\mathcal{V}} \operatorname{div}\mathbf{F}\mathrm{d}\mathcal{V} = \iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}\mathcal{S},$$

where $\hat{\mathbf{n}}$is the unit outward normal to the surface $\mathcal{S}$ of the region $\mathcal{V}$.

In this problem $\mathcal{V}$ is the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. For

$$\mathbf{F} = xy^2\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xyz\hat{\mathbf{j}},$$

the divergence of $\mathbf{F}$ is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(xz)\frac{\partial}{\partial z}(xyz) = y^2 + xy.$$

Applying the divergence theorem,

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{n}}\mathrm{d}\mathcal{S} &= \iiint_{\mathcal{V}} \operatorname{div}\mathbf{F}\mathrm{d}\mathcal{V} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(y^2 + xy)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_{-1}^{1}\int_{-1}^{1}\left[xy^2 + \frac{1}{2}x^2 y\right]_{x=-1}^{1}\mathrm{d}y\mathrm{d}z = \int_{-1}^{1}\int_{-1}^{1} y\mathrm{d}y\mathrm{d}z \\
&= \int_{-1}^{1}\left[\frac{1}{2}y^2\right]_{-1}^{1}\mathrm{d}z = 0
\end{aligned}$$

34.19. The divergence theorem is quoted in Problem 34.18. Put the vector field in the theorem equal to $\mathbf{curl}\,\mathbf{F}$. Then

$$\iint_{\mathcal{S}} \hat{\mathbf{n}} \cdot \mathbf{curl}\,\mathbf{F} \mathrm{d}\mathcal{S} = \iiint_{\mathcal{V}} \mathrm{div}\,\mathbf{curl}\,\mathbf{F} \mathrm{d}\mathcal{V} = 0,$$

since $\mathrm{div}\,\mathbf{curl}\,\mathbf{F} \equiv 0$ (see 34.10). Hence

$$\iint_{\mathcal{S}} \hat{\mathbf{n}} \cdot \mathbf{curl}\,\mathbf{F} \mathrm{d}\mathcal{S} = 0.$$

34.20. The divergence theorem is quoted in Problem 34.18. The vector field $\mathbf{F} = \hat{\mathbf{n}}$ on the surface $\mathcal{S}$. Hence

$$\iint_{\mathcal{S}} \hat{\mathbf{n}} \cdot \mathbf{F} \mathrm{d}\mathcal{S} = \iint_{\mathcal{S}} \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} \mathrm{d}\mathcal{S} = \iint_{\mathcal{S}} \mathrm{d}\mathcal{S} = A,$$

which is the surface area of $\mathcal{S}$. By the divergence theorem

$$A = \iiint_{\mathcal{V}} \mathrm{div}\,\mathbf{F} \mathrm{d}\mathcal{V}.$$

34.21. In the divergence theorem let $\mathbf{F} = \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Then, since $\mathrm{div}\,\mathbf{r} = 3$,

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{r} \cdot \hat{\mathbf{n}} \mathrm{d}\mathcal{S} &= \iiint \mathrm{div}\,\mathbf{r} \mathrm{d}\mathcal{V} \\ &= 3 \iiint \mathrm{d}\mathcal{V} = 3V, \end{aligned}$$

where V is the volume enclosed by $\mathcal{S}$. Hence

$$V = \frac{1}{3} \iint_{\mathcal{S}} \mathbf{r} \cdot \hat{\mathbf{n}} \mathrm{d}\mathcal{S}.$$

(a) For the sphere, $\hat{\mathbf{n}} = \mathbf{r}/r = \mathbf{r}/a$, where $\mathbf{r}$ is the position vector of a point on $\mathcal{S}$. By the result in Problem 34.20, the volume of the sphere is given by

$$\begin{aligned} V &= \frac{1}{3} \iint_{\mathcal{S}} \mathbf{r} \cdot \frac{\mathbf{r}}{a} \mathrm{d}\mathcal{S} \\ &= \frac{a}{3} \iint_{\mathcal{S}} \mathrm{d}\mathcal{S} = \frac{a}{3} \times \text{(surface area of the sphere)} \\ &= \frac{a}{3}(4\pi a^2) = \frac{4}{3}\pi a^3 \end{aligned}$$

(b) On the curved surface the position vector is perpendicular to the normal to the surface. On the base of the cone $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. Hence the volume is given by

$$V = \frac{1}{3} \iint_{\mathcal{S}} \mathbf{r} \cdot \hat{\mathbf{n}} \mathrm{d}\mathcal{S} = \frac{1}{3} \iint_{\mathcal{S}_1} h \mathrm{d}\mathcal{S},$$

where $\mathcal{S}_1$ is the base of the cone and $\mathbf{r} \cdot \hat{\mathbf{n}} = h$. The integral

$$\frac{1}{3} \iint_{\mathcal{S}_1} \mathrm{d}\mathcal{S}$$

is the surface area A of the base. Therefore the volume of the cone is $\frac{1}{3}Ah$.

34.22. Apply the divergence theorem with $\mathrm{div}\,\mathbf{F} = 1$. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}\mathcal{S} = \iiint_{\mathcal{V}} \mathrm{div}\,\mathbf{F} \mathrm{d}\mathcal{V} = \iiint_{\mathcal{V}} \mathrm{d}\mathcal{V} = V,$$

the volume of the region enclosed by $\mathcal{S}$.