PART III: Integration and differential equations

Chapter 14: Antidifferentiation and area

14.1. In each case we require the general form of y such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

where $f(x)$ is given.
(a) $\frac{1}{6}x^6 + C$; $\frac{3}{5}x^5 + C$; $\frac{1}{2}x^4 + C$; $\frac{1}{9}x^3 + C$; $3x^2 + C$; $3x + C$; C.
(b) $\frac{1}{4}x^{-2} + C$; $-2x^{-1} + C$; $3\ln x + C$.
(c) $\frac{2}{5}x^{\frac{5}{2}} + C$; $\frac{2}{3}x^{\frac{3}{2}} + C$; $2x^{\frac{1}{2}} + C$; $\frac{3}{7}x^{\frac{7}{3}} + C$; $\frac{3}{2}x^{\frac{2}{3}} + C$.
(d) $-x^{-1} + C = -(1/x) + C$; $-\frac{1}{3}x^{-3} = -1/(3x^3 + C$; $\ln(-x) + C$
(e) $\frac{2}{3}x^{\frac{3}{2}} + C$; $2x^{\frac{1}{2}} + C = 2\sqrt{x} + C$; $-2x^{-\frac{1}{2}} + C$.
(f) $\frac{3}{2}x^2 + C$; $\frac{1}{6}x^3 + C$; $-1/(3x) + C$; $x^{\frac{3}{4}} + C$.
(g) $\mathrm{e}^x + C$; $-\mathrm{e}^{-x} + C$; $\frac{5}{2}\mathrm{e}^{2x} + C$; $-2\mathrm{e}^{-\frac{1}{2}x} + C$; $-\frac{3}{2}\mathrm{e}^{-2x}$.
(h) $\sin x + C$; $\frac{1}{3}\sin 3x + C$; $-\cos x + C$; $-\frac{1}{3}\cos x + C$.
(i) $x - \frac{3}{2}x^2 + C$; $x + x^2 - x^3 + C$; $\frac{3}{5}x^5 - \frac{4}{3}x^3 + 5x + C$.
(j) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + C$; $x - \frac{4}{3}x^3 + C$; $\frac{1}{3}x^3 + x^2 + x + C$; $\frac{1}{2}x^2 - \ln x + C$; $\frac{1}{4}x^4 + \frac{1}{5}x^5 + C$.
(k) $x + \ln x + C$ for $x > 0$; $2x - 2\sqrt{x} + C$; $\ln x - 2x^{-1} - \frac{1}{2}x^{-2} + C$.
(l) $\mathrm{e}^x - \mathrm{e}^{-x} + C$; $\mathrm{e}^{2x} - \mathrm{e}^{3x} + C$; $2\mathrm{e}^{\frac{1}{2}x} + x + C$; $-\frac{1}{2}\mathrm{e}^{-2x}$; $x + \frac{1}{4}\mathrm{e}^{-4x} + C$.
(m) $\sin 2x + C$; $-6\cos\frac{1}{2}x - 12\sin\frac{1}{3}x + C$; $2x - \frac{1}{2}\cos 2x + C$.

14.2. (a) $\frac{1}{4}(x+1)^4 + C$; $\frac{1}{12}(3x+1)^4 + C$: $\frac{1}{12}(3x-8)^4 + C$.
(b) $-\frac{1}{5}(1-x)^5 + C$; $-\frac{2}{9}(8-3x)^{\frac{3}{2}} + C$; $-\frac{3}{4}(1-x)^{\frac{4}{3}} + C$.
(c) $-\frac{1}{2}(2x+1)^{-1} + C$; $-2(1-x)^{\frac{1}{2}} + C$; $-\frac{1}{3}(3x+1)^{-2} + C$; $-\frac{1}{3}(1-x)^{\frac{3}{4}} + C$.
(d) $\frac{2}{3}\sin(3x-2) + C$; $3\cos(1-x) + C$; $\frac{2}{3}\cos(2-3x) + C$.

14.3. (a) $\ln|1+x| + C$; $\ln|x-1| + C$; $\ln|3x-2| + C$; $\frac{2}{5}\ln|5x-4| + C$.
(b) $-\ln|1-x| + C$; $-\frac{1}{5}\ln|4-5x| + C$.
(c) $x - \ln|x+1| + C$.
(d) Using

$$\frac{x+1}{x-1} = 1 + \frac{2}{x-1},$$

the required antiderivative is $x + \ln|1-x| + C$.

14.4. (a) $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$; $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$; $-\frac{1}{4}\cos 2x + C$.
(b) $\frac{3}{2}x + \frac{3}{8}\sin 4x + C$; $\frac{1}{2}x - \frac{1}{12}\sin 6x + C$; $-\frac{1}{8}\cos 4x + C$.

(c) Using

$$\cos^4 x = \frac{1}{4}(1+\cos 2x)^2 = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x,$$

the antiderivative of $\cos^4 x$ is

$$\tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

14.5. (a) By the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^x) = \frac{\mathrm{d}(x)}{\mathrm{d}x}\mathrm{e}^x + x\frac{\mathrm{d}(\mathrm{e}^x)}{\mathrm{d}x} = \mathrm{e}^x + x\mathrm{e}^x.$$

By rearranging the terms:

$$x\mathrm{e}^x = \frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^x) - \mathrm{e}^x = \frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^x - \mathrm{e}^x).$$

Hence the antiderivatives of $x\mathrm{e}^x$ are

$$x\mathrm{e}^x - \mathrm{e}^x + C.$$

(b) Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^x) = 2x\mathrm{e}^x + x^2\mathrm{e}^x,$$

it follows that

$$x^2\mathrm{e}^x = \frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^x) - 2x\mathrm{e}^x = \frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^x) - 2\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^x - \mathrm{e}^x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x),$$

Therefore, the antiderivatives of $x^2\mathrm{e}^x$ are given by

$$\mathrm{e}^x(x^2 - 2x + 2) + C.$$

14.6. In each case the signed area between the curve defined by $y = f(x)$ and the x axis between $x = a$ and $x = b$ is

$$\mathcal{A} = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$.
(a) For $y = x$ between $x = 0$ and $x = 2$, choose $F(x) = \frac{1}{2}x^2$. Then

$$\mathcal{A} = F(2) - F(0) = \frac{1}{2}2^2 - 0 = 2.$$

(b) For $y = x$ between $x = -1$ and $x = 1$, choose $F(x) = \frac{1}{2}x^2$. Then

$$\mathcal{A} = F(1) - F(-1) = \frac{1}{2} - \frac{1}{2} = 0.$$

The graph of $y = x$ is a straight line through the origin between $x = -1$ and $x = 1$: the areas above and below the x axis cancel
(c) For $y = -x^2$ between $x = 0$ and $x = 1$, choose $F(x) = -\frac{1}{3}x^3$. Then

$$\mathcal{A} = F(1) - F(0) = -\tfrac{1}{3}.$$

(d) For $y = \cos x$ between $x = -\pi$ and $x = \pi$, choose $F(x) = \sin x$. Then

$$\mathcal{A} = F(\pi) - F(-\pi) = \sin\pi - \sin(-\pi) = 0.$$

(e) For $y = \cos x - 1$ between $x = 0$ and $x = 2\pi$, choose $F(x) = \sin x - x$. Then

$$\mathcal{A} = F(2\pi) - F(0) = \sin 2\pi - 2\pi - 0 = -2\pi.$$

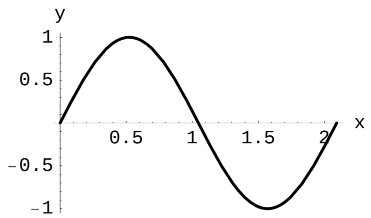

Figure 1: Problem 14.6g: $y = sin3x$ for $0 \leq x \leq \frac{2}{3}\pi$.

(f) For $y = x^{-1}$ between $x = -2$ and $x = -1$, choose $F(x) = \ln(-x)$. Then

$$\mathcal{A} = \ln 1 - \ln 2 = -\ln 2.$$

(g) For $y = \sin 3x$ between $x = 0$ and $x = \frac{2}{3}\pi$, choose $F(x) = -\frac{1}{3}\cos 3x$. Then

$$\mathcal{A} = F(\tfrac{2}{3}\pi) - F(0) = -\tfrac{1}{3}\cos 2\pi + \tfrac{1}{3} = -\tfrac{1}{3} + \tfrac{1}{3} = 0.$$

(h) For $y = 1/(1-x) = -1/(x-1)$ between $x = 2$ and $x = 3$, choose $F(x) = -\ln(x-1)$. Then

$$\mathcal{A} = F(3) - F(2) = -\ln 2 + \ln 1 = -\ln 2.$$

14.7. (a) The function $y = -3$ is always negative, and an antiderivative is $F(x) = -3x$. Hence the geometrical area A is given by

$$A = |\mathcal{A}| = |F(1) - F(0)| = |-3 - 0| = 3.$$

(b) The function $y = x^3$ is positive for $x > 0$ and negative for $x < 0$. An antiderivative is $F(x) = \frac{1}{4}x^4$. Then the geometric area is given by

$$A = |F(0) - F(-1)| + |F(1) - F(0)| = |0 - \frac{1}{4}| + [\frac{1}{4} - 0] = \frac{1}{2}.$$

(c) In the interval $-1 \leq x \leq 3$, $y = 4 - x^2$ is positive for $-1 < x < 2$ and negative for $2 < x < 3$. Choose the antiderivative $F(x) = 4x - \frac{1}{3}x^3$. Then the geometric area is

$$\begin{aligned} A &= |F(2) - F(-1)| + |F(3) - F(2)| = [8 - \tfrac{8}{3} + 4 - \tfrac{1}{3}] - |12 - 9 - 8 + \tfrac{8}{3}| \\ &= 9 + |-\tfrac{7}{3}| \\ &= \tfrac{34}{3}. \end{aligned}$$

(d) In the interval $0 \leq x \leq 2\pi$, $y = \cos x$ is positive for $0 < x < \frac{1}{2}\pi$ and $\frac{3}{2}\pi < x < 2\pi$, and negative for $\frac{1}{2}\pi < x < \frac{3}{2}\pi$. Choose the antiderivative $F(x) = \sin x$. Then the geometric area is

$$\begin{aligned} A &= |F(\tfrac{1}{2}\pi) - F(0)| + |F(\tfrac{3}{2}\pi) - F(\tfrac{1}{2}\pi)| + |F(2\pi) - F(\tfrac{3}{2}\pi)| \\ &= |\sin\tfrac{1}{2}\pi - 0| + |\sin\tfrac{3}{2}\pi - \sin\tfrac{1}{2}\pi| + |\sin 2\pi - \sin\tfrac{3}{2}\pi| \\ &= 1 + |-1 - 1| + 1 = 4. \end{aligned}$$

14.8. (a) The antiderivative of 0 is any constant A, and the antiderivative of A is $At + B$ where B is any constant. Therefore the most general function which satisfies

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0 \text{ is } x = At + B.$$

(b) The antiderivative of t is $\frac{1}{2}t^2 + A$, and the antiderivative of $\frac{1}{2}t^2 + A$ is $\frac{1}{6}t^3 + At + B$, where A and B are constants. Therefore the most general solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = t \text{ is } x = \tfrac{1}{6}t^3 + At + B.$$

(c) The antiderivative of $\sin t$ is $-\cos t + A$, and the antiderivative of this function is $-\sin t + At + B$, where A and B are constants. Therefore the most general solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \sin t \text{ is } x = -\sin t + At + B.$$

(d) The most general solution of

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} = 0 \text{ is } x = At^2 Bt + C.$$

(e) The most general solution of

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} = \cos t \text{ is } x = -\sin t + \tfrac{1}{2}t^2 + Bt + C.$$

(f) The most general solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = g \text{ is } x = \tfrac{1}{2}gt^2 + At + B.$$

(g) The most general solution of

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = w_0 \text{ is } y = \tfrac{1}{24}w_0x^4 + Ax^3 + Bx^2 + Cx + D.$$

Chapter 15: The definite and indefinite integral

15.1. (a) For $y = x^3$, $-1 \le x \le 2$, the signed area is given by

$$\begin{aligned} \mathcal{A} &= \lim_{n\to\infty} \sum_{x=-1}^{x=2} x^3 \delta x = \int_{-1}^{2} x^3 \mathrm{d}x \\ &= \left[\frac{1}{4}x^4\right]_{-1}^{2} = \left[4 - \frac{1}{4}(-1)^4\right] = \frac{15}{3}. \end{aligned}$$

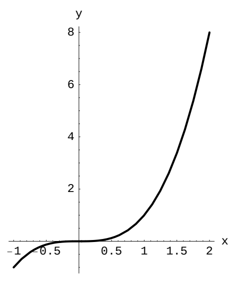

Figure 2: Problem 15.1a

(b) For $f(x) = x^5$, $-1 \le x \le 1$,

$$\begin{aligned} \mathcal{A} &= \lim_{n\to\infty} \sum_{x=-1}^{x=1} x^5 \delta x = \int_{-1}^{1} x^5 \mathrm{d}x = \left[\frac{1}{6}x^6\right]_{-1}^{1} \\ &= \frac{1}{6}\left[x^6\right]_{-1}^{1} = \frac{1}{6}[1 - (-1)^6] = 0. \end{aligned}$$

(c) For $y = \sin x$, $-\pi \le x \le 0$,

$$\begin{aligned}\mathcal{A} &= \lim_{n\to\infty} \sum_{x=\pi}^{x=0} \sin x\delta x = \int_{-\pi}^{0} \sin x \mathrm{d}x \\ &= [-\cos x]_{-\pi}^{0} = -[\cos x]_{-\pi}^{0} = -[1-(-1)] = -2.\end{aligned}$$

(d)For $y = \mathrm{e}^{-2x}$, $0 \le x \le 1$,

$$\begin{aligned}\mathcal{A} &= \lim_{n\to\infty} \sum_{x=0}^{x=1} \mathrm{e}^{-2x}\delta x = \int_{0}^{1} \mathrm{e}^{-2x} \mathrm{d}x = [-\frac{1}{2}\mathrm{e}^{-2x}]_0^1 \\ &= -\frac{1}{2}[\mathrm{e}^{-2x}]_0^1 = -\frac{1}{2}[\mathrm{e}^{-2}-1] = 0.432\ldots.\end{aligned}$$

15.2. We require the general antiderivative for each integrand.

(a) $\int x^{\frac{1}{2}}\mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}} + C$.

(b) $\int (x+1)^{\frac{1}{2}}\mathrm{d}x = \frac{2}{3}(x+1)^{\frac{3}{2}} + C$.

(c) $\int \mathrm{e}^{\frac{1}{2}x}\mathrm{d}x = 2\mathrm{e}^{\frac{1}{2}x} + C$.

(d) $\int \sin x \mathrm{d}x = -\cos x + C$.

(e) $\int (\cos x - 2\sin 2x)\mathrm{d}x = \sin x + \cos 2x + C$.

(f) $\int t^{-\frac{1}{2}}\mathrm{d}t = 2t^{\frac{1}{2}} + C$.

(g) $\int \cos 2u \mathrm{d}u = \frac{1}{2}\sin 2u + C$.

(h) $\int 3\mathrm{e}^{-\frac{1}{2}y}\mathrm{d}y = -6\mathrm{e}^{-\frac{1}{2}y} + C$.

(i) $\int (1+3t^2-2t)\mathrm{d}t = t + t^3 - t^2 + C$.

(j) $\int (1+4\cos 4w)\mathrm{d}w = w + \sin 4w + C$.

(k) $\int (-x)^{\frac{1}{2}}\mathrm{d}x = \frac{2}{3}x(-x)^{\frac{1}{2}} + C$ or $-\frac{2}{3}(-x)^{\frac{3}{2}}$.

15.3. All these integrals use formula (15.4):

$$\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b = F(b) - F(a).$$

(a)
$$\int_{-1}^{1} x^3 \mathrm{d}x = \left[\frac{1}{4}x^4\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0.$$

(b)
$$\int_{-1}^{1} x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

(c)
$$\int_{0}^{2} \mathrm{d}x = [x]_0^2 = 2.$$

(d)
$$\int_{0}^{4} x^{\frac{1}{2}} \mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_{0}^{4} = \frac{16}{3}.$$

(e)
$$\int_{-1}^{1} (1-3x+2x^2)\mathrm{d}x = \left[x - \frac{3}{2}x^2 + \frac{2}{3}x^3\right]_{-1}^{1} = \frac{10}{3}.$$

(f) $$\int_1^2 (x^{-3}+x^{-2})\mathrm{d}x = \left[-x^{-1}-\frac{1}{2}x^{-2}\right]_1^2 = \frac{7}{8}.$$

(g) $$\int_1^2 x^{-2}\mathrm{d}x = \left[-x^{-1}\right]_1^2 = \frac{1}{2}.$$

(h) $$\int_{-2}^{-1} x^{-1}\mathrm{d}x = [\ln|x|]_{-2}^{-1} = -\ln 2.$$

(i) $$\int_{-2}^{-1} (-x)^{\frac{1}{2}}\mathrm{d}x = \left[\frac{2}{3}x\surd(-x)\right]_{-2}^{-1} = -\frac{2}{3}+\frac{4\surd 2}{3}.$$

(j) $$\int_0^1 \mathrm{e}^{-3x}\mathrm{d}x = \left[-\frac{1}{3}\mathrm{e}^{-3x}\right]_0^1 = \frac{1}{3}(1-\mathrm{e}^{-3}).$$

(k) $$\int_0^{\frac{1}{4}\pi} \sin 4x\mathrm{d}x = \left[-\frac{1}{4}\cos 4x\right]_0^{\frac{1}{4}\pi} = \frac{1}{2}.$$

(l) $$\int_0^{2\pi} \sin\frac{1}{2}x\mathrm{d}x = \left[-2\cos\frac{1}{2}x\right]_0^{2\pi} = 4.$$

(m) $$\int_0^{2\pi} \cos\frac{1}{2}x\mathrm{d}x = \left[2\sin\frac{1}{2}x\right]_0^{2\pi} = 0.$$

15.4. In each case the antiderivative $F(x)$ of $f(x)$ is required, and then the definite integral is given by

$$\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b = F(b)-F(a).$$

(a) $$\int_0^1 x(x^2+x+1)\mathrm{d}x = \left[\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^1 = \frac{13}{12}.$$

(b) $$\int_{-1}^1 (x-1)(x+1)\mathrm{d}x = \int_{-1}^1 (x^2-1)\mathrm{d}x = \left[\frac{1}{3}x^3-x\right]_{-1}^1 = -\frac{4}{3}.$$

(c) $$\int_0^2 x(x^2-1)\mathrm{d}x = \left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]_0^2 = 2.$$

(d) $$\int_1^2 \frac{x+x^2}{x^3}\mathrm{d}x = \int_1^2 (x^{-2}+x^{-1})\mathrm{d}x = \left[-\frac{1}{x}+\ln x\right]_1^2 = \frac{1}{2}+\ln 2.$$

(e) $$\int_1^2 \frac{t(t+1)}{t^{\frac{1}{2}}}\mathrm{d}t = \int_1^2 (t^{\frac{3}{2}} + t^{\frac{1}{2}})\mathrm{d}t = \left[\frac{2}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}}\right]_1^2 = \frac{44\sqrt{2}-16}{15}.$$

(f) $$\int_1^4 \frac{\sqrt{u}-1}{u}\mathrm{d}u = \int_1^4 (u^{-\frac{1}{2}} - u^{-1})\mathrm{d}u = \left[2u^{\frac{1}{2}} - \ln u\right]_1^4 = 2 - 2\ln 2.$$

(g) $$\int_{-1}^0 \frac{\mathrm{d}w}{2w+3} = \left[\frac{1}{2}\ln(2w+3)\right]_{-1}^0 = \frac{1}{2}\ln 3.$$

(h) $$\int_{-2}^{-1} \frac{x}{x-1}\mathrm{d}x = \int_{-2}^{-1}\left[1 + \frac{1}{x-1}\right]\mathrm{d}x = [x + \ln|x-1|]_{-2}^{-1} = 1 + \ln 2 - \ln 3.$$

(i) $$\int_0^{\pi} \cos^2 3t\mathrm{d}t = \int_0^{\pi} \frac{1}{2}(1+\cos 6t)\mathrm{d}t = \left[\frac{1}{2}t + \frac{1}{12}\sin 6t\right]_0^{\pi} = \frac{1}{2}\pi.$$

15.5. These are all improper integrals of the various types discussed in Section 15.6.

(a) $$\int_1^{\infty} \mathrm{e}^{-3t}\mathrm{d}t = \left[-\frac{1}{3}\mathrm{e}^{-3t}\right]_1^{\infty} = \frac{1}{3}\mathrm{e}^{-3}.$$

(b) $$\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}v}\mathrm{d}v = \left[-2\mathrm{e}^{-\frac{1}{2}v}\right]_0^{\infty} = 2.$$

(c) $$\int_1^{\infty} \frac{\mathrm{d}x}{x^3} = \left[-\frac{1}{2}x^{-2}\right]_1^{\infty} = \frac{1}{2}.$$

(d) $$\int_0^{\infty} \frac{\mathrm{d}x}{(2x+3)^2} = \left[-\frac{1}{2(2x+3)}\right]_0^{\infty} = \frac{1}{6}$$

(e) $$\int_0^1 \frac{\mathrm{d}s}{s^{\frac{1}{4}}} = \left[\frac{4}{3}s^{\frac{3}{4}}\right]_0^1 = \frac{4}{3}.$$

(f) $$\int_1^2 \frac{\mathrm{d}t}{(t-1)^{\frac{1}{2}}} = \left[2(t-1)^{\frac{1}{2}}\right]_1^2 = 2.$$

(g) Using the formula (15.11) or the method illustrated in Example 15.12,

$$\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}t}\cos 2t\mathrm{d}t = \frac{3}{13}.$$

(h) Using the formula (15.11) or the method illustrated in Example 15.12,

$$\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}t}\cos 2t\mathrm{d}t = \frac{2}{17}.$$

15.6. The mean of $f(t)$ over an interval $0 \le t \le T$ is the quantity

$$\frac{1}{T}\int_0^T f(t)\mathrm{d}t.$$

(a) For $f(t) = t$ over $0 \le t \le 1$, its mean is

$$\int_0^1 t\mathrm{d}t = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}.$$

(b) For $f(t) = t$ over $-1 \le t \le 1$, its mean is

$$\frac{1}{2}\int_{-1}^1 t\mathrm{d}t = \frac{1}{2}\left[\frac{1}{2}t^2\right]_{-1}^1 = 0.$$

(c) For $f(t) = \sin t$ over $0 \le t \le \pi$, its mean is

$$\frac{1}{\pi}\int_0^\pi \sin t\mathrm{d}t = \frac{1}{\pi}\left[-\cos t\right]_0^\pi = \frac{2}{\pi}.$$

(d) For $f(t) = \sin t$ over $0 \le t \le 2\pi$, its mean is

$$\frac{1}{2\pi}\int_0^{2\pi} \sin t\mathrm{d}t = 0.$$

(e) The mean of $f(t) = t^{-2}$ over $1 \le t \le T$ is

$$\frac{1}{T-1}\int_1^T t^{-2}\mathrm{d}t = \frac{1}{T-1}\left[-t^{-1}\right]_1^T = \frac{1}{T}.$$

(f) The mean of $f(t) = \mathrm{e}^{-t}\cos t$ over $0 \le t \le 2\pi$ is

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-t}\cos t\mathrm{d}t = \frac{1}{2\pi}\frac{1}{2}\left[\mathrm{e}^{-t}(\cos t + \sin t)\right]_0^{2\pi} = -\frac{\mathrm{e}^{-2\pi}-1}{4\pi}.$$

(g) The mean of $f(t) = \mathrm{e}^{-2t}\sin t$ over $0 \le t < \infty$ is defined by the limiting process:

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T \mathrm{e}^{-2t}\sin t\mathrm{d}t\right] = \lim_{T\to\infty}\left[\frac{1}{T}\frac{1}{5}\left\{1-\mathrm{e}^{-2T}[\cos T + 2\sin T]\right\}\right] = 0.$$

(h) The mean of $f(t) = 1 - \mathrm{e}^{-t}$ over $0 \le t < \infty$ is

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T (1-\mathrm{e}^{-t})\mathrm{d}t\right] = \lim_{T\to\infty}\left[\frac{1}{T}(T-1+\mathrm{e}^{-T})\right] = 1.$$

(The limiting process is necessary since $\int_0^\infty (1-\mathrm{e}^{-t})\mathrm{d}t$ is infinite.)
(i) The mean of $f(t) = t^{-1}$ over $1 \le t < \infty$ is

$$\lim_{T\to\infty}\left[\frac{1}{T-1}\int_1^T t^{-1}\mathrm{d}t\right] = \lim_{T\to\infty}\frac{\ln T}{T-1} = 0.$$

(See the remarks under (h).)

15.7. (a) Since $\sin^4 t = \sin^4(-t)$ for all t, $\sin^4 t$ is an even function (see (1.12)), so that by (15.17)

$$\int_{-\pi}^\pi \sin^4 t\mathrm{d}t = 2\int_0^\pi \sin^4 t\mathrm{d}t,$$

since the interval is symmetrical about the origin.
(b) Since

$$\frac{(-t)^3}{(1+(-t)^4)} = -\frac{t^3}{(1+t^4)},$$

this function is odd, so that by (15.15), since the interval is also symmetrical about the origin,

$$\int_{-1}^{1} \frac{t^3}{1+t^4} \mathrm{d}t = 0.$$

(c) Since $(t \cos t)/(1+t^2)$ is an odd function and the interval of integration is $-\pi \leq x \leq \pi$,

$$\int_{-\pi}^{\pi} \frac{t \cos t}{1+t^2} \mathrm{d}t = 0.$$

(d) Since $(-t)^2 \sin[(-t)^3] = t^2 \sin[-(t^3)] = -t^2 \sin(t^3)$, this function is odd. Since the interval of integration is $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$,

$$\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} t^2 \sin(t^3) \mathrm{d}t = 0.$$

15.8. (a) Below is a simple program in *Mathematica* using the algorithm (15.2) for the numerical integration of e^{-x}/x over $1 \leq x \leq 2$:

```
f[x_] = Exp[-x]/x;
a = 1; b = 2; n = 100; h = (b - a)/n;
h*Sum[f[a + r*h], {r, 0, n - 1}] // N
```

The program uses 100 steps and gives the answer 0.17199. If the number of steps is increased to 101, then the estimate becomes 0.17197 to five decimal places, which gives a difference error of 0.00002. However, the correct value is 0.17048 to five decimal places so the numerical estimate is only correct to two decimal places. For an accuracy to four decimal places we require about 30000 steps using this algorithm.
(b) As in (a), a program is

```
f[x_] = Sin[x^2];
a = 0; b = Pi; n = 100; h = (b - a)/n;
h*Sum[f[a + r*h], r, 0, n - 1] // N
```

For 100 steps the estimate is 0.77894 and to 101 steps 0.77889 to five decimal places. The true value to five decimal places is 0.77265.
(c) Use the method above with $f(x) = \cos(\mathrm{e}^{-x})$. To five decimal places the value of the definite integral is 0.79383.

15.9. (a) If $x = 1$, then $\mathrm{e}^{-x^2} = \mathrm{e}^{-x} = \mathrm{e}^{-1}$. Also for $x > 1$, $-x^2 < -x$ so that $\mathrm{e}^{-x^2} < \mathrm{e}^{-x}$.
If $b > 1$, then, given $E < 1$,

$$\int_{b}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x < \int_{b}^{\infty} \mathrm{e}^{-x} \mathrm{d}x = \mathrm{e}^{-b} < E,$$

if $b > -\ln E$.
(b) As in Problem 15.8a, use the program

```
f[x_] = Exp[-x^2];
error = 0.001; a = 0; b = -Log[error]; n = 1000; h = (b - a)/n;
h*Sum[f[a + r*h], {r, 0, n - 1}] // N
```

Here the E(error) $= 0.001$, and $b = -\ln E$. The program gives the estimate

$$\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x \approx 0.88968 \text{ (true value is } \tfrac{1}{2}\sqrt{\pi} \text{ (see Example 32.11)).}$$

To five decimal places the the integral takes the value 0.88623.

15.10. The general formula for differentiation of an integral with respect to varable limits is given by (15.20):

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(t)\mathrm{d}t = f(v(x))\frac{\mathrm{d}v(x)}{\mathrm{d}x} - f(u(x))\frac{\mathrm{d}u(x))}{\mathrm{d}x}.$$

(a)
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^2\mathrm{d}t = x^2.$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sin^5 t\mathrm{d}t = \sin^5 x.$$

(c)
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\mathrm{e}^t}{1+t}\mathrm{d}t = \frac{\mathrm{e}^x}{1+x}.$$

(d)
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\mathrm{e}^x} t\ln t\mathrm{d}t = \mathrm{e}^x\ln(\mathrm{e}^x)\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} = x\mathrm{e}^{2x}.$$

(e)
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sqrt{x}}^{\sqrt{(x+1)}} \sin(t^2)\mathrm{d}t = \frac{\sin(x+1)}{2\sqrt{(x+1)}} - \frac{\sin x}{2\sqrt{x}}.$$

15.11. (a) We are given

$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}.$$

Case $x > 1$:

$$I(x) = \int_0^x f(t)\mathrm{d}t = \int_0^1 t\mathrm{d}t + \int_1^x 0\mathrm{d}t = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}.$$

Case $-1 \le x \le 1$:

$$I(x) = \int_0^x f(t)\mathrm{d}t = \int_0^x t\mathrm{d}t = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2.$$

Case $x < -1$:

$$I(x) = \int_0^x f(t)\mathrm{d}t = -\int_x^0 f(t)\mathrm{d}t = -\int_x^{-1} 0\mathrm{d}t - \int_{-1}^0 t\mathrm{d}t = \frac{1}{2}.$$

(b) We are given

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2-x & \text{if } 1 \le x \le \frac{1}{2} \\ 0 & \text{if } x > \frac{3}{2} \end{cases}.$$

Case $x > \frac{3}{2}$:

$$I(x) = \int_0^x f(t)\mathrm{d}t = \int_0^1 t\mathrm{d}t + \int_1^{\frac{3}{2}} (2-t)\mathrm{d}t + \int_{\frac{3}{2}} 0\mathrm{d}t = \frac{1}{2} + \frac{3}{8} + 0 = \frac{7}{8}.$$

Case $1 \le x \le \frac{3}{2}$:

$$I(x) = \int_0^x f(t)\mathrm{d}t = \int_0^1 t\mathrm{d}t + \int_1^x (2-t)\mathrm{d}t = \frac{1}{2} + 2x - \frac{1}{2}x^2 - \frac{3}{2} = -1 + 2x - \frac{1}{2}x^2.$$

Case $0 \le x < 1$:

$$I(x) = \int_0^x t\mathrm{d}t = \frac{1}{2}x^2.$$

15.12. For $t < t_0$:

$$Q(t) = \int_0^t I_0 \mathrm{d}u = I_0 t.$$

For $t > t_0$:

$$\begin{aligned} Q(t) &= \int_0^{t_0} I_0\mathrm{d}u + \int_{t_0}^t I_0 \mathrm{e}^{-R(u-t_0)/L}\mathrm{d}u \\ &= I_0 t_0 + I_0 \left[-\frac{L}{R}\mathrm{e}^{-R(u-t_0)/L}\right]_{t_0}^t \\ &= I_0\left[t_0 + \frac{L}{R}\left(1 - \mathrm{e}^{-R(t-t_0)/L}\right)\right] \end{aligned}$$

15.13. The function $f(x)$ is defined by

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \end{cases}.$$

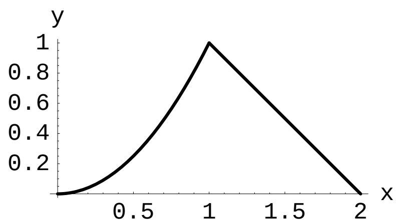

Figure 3: Problem 15.13

The area A is given by

$$\begin{aligned} A &= \int_0^2 f(x)\mathrm{d}x = \int_0^1 x^2\mathrm{d}x + \int_1^2 (2-x)\mathrm{d}x \\ &= \left[\frac{1}{3}x^3\right]_0^1 - \left[\frac{1}{2}(2-x)^2\right]_1^2 \\ &= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \end{aligned}$$

15.14.

$$\begin{aligned} \int_0^2 \frac{\mathrm{d}x}{|x-1|^{\frac{2}{3}}} &= \int_0^1 \frac{\mathrm{d}x}{(1-x)^{\frac{2}{3}}} + \int_1^2 \frac{\mathrm{d}x}{(x-1)^{\frac{2}{3}}} \\ &= [-3(1-x)]_0^1 + [3(x-1)]_1^2 \\ &= 3 + 3 = 6 \end{aligned}$$

Chapter 16: Applications involving the integral as a sum

16.1. Given the resistance $R(x) = 100x + 1000x^2$, the work done, δW, in compressing it through a short distance δx is, approximately,

$$\delta W \approx \text{resistance} \times \text{distance} = R\delta x = (100x + 1000x^2)\delta x.$$

The total work done

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{x=0}^{0.01} R(x)\delta x = \int_0^{0.01} (100x + 1000x^2)\mathrm{d}x \\ &= \left[50x^2 + \frac{1000}{3}x^3\right]_0^{0.01} = 0.00533\ldots. \end{aligned}$$

16.2. With $v(t) = 20 - 10t$, the displacement d which takes place between $t = 2$ and $t = 4$ is

$$d(t) = \lim_{n\to\infty} \sum_{t=2}^{4} v(t)\delta t = \int_2^4 v(t)\mathrm{d}t = \int_2^4 (20 - 10t)\mathrm{d}t = \left[20t - 5t^2\right]_2^4 = -20.$$

We require the general indefinite integral of $v = 20 - 10t$:

$$x(t) = \int v\mathrm{d}t = \int (20 - 10t)\mathrm{d}t = 20t - 5t^2 + C.$$

Since $x(2) = 3$,

$$3 = 40 - 20 + C, \text{ or } C = -17.$$

Therefore $x(t) = 20t - 5t^2 - 17$, so that $x(4) = -17$.

16.3. From (16.1) the volume V of a solid of revolution about the x axis formed by the profile $y = f(x)$ between $x = a$ and $x = b$ is given by

$$V = \int_a^b \pi y^2 \mathrm{d}x = \int_a^b [f(x)]^2 \mathrm{d}x.$$

(a) Profile $y = \mathrm{e}^{-x}$, $0 \le x \le 1$:

$$V = \pi \int_0^1 \mathrm{e}^{-2x}\mathrm{d}x = \pi \left[-\frac{1}{2}\mathrm{e}^{-2x}\right]_0^1 = \frac{\pi(\mathrm{e}^2 - 1)}{2\mathrm{e}^2}.$$

(b) Profile $y = 1/x$, $1 \le x \le 2$:

$$V = \pi \int_1^2 \frac{\mathrm{d}x}{x^2} = \frac{1}{2}\pi.$$

(c) Profile $y = x(1 - x)$, $0 \le x \le 1$:

$$V = \pi \int_0^1 x^2(1 - x)^2\mathrm{d}x = \pi \int_0^1 (x^2 - 2x^3 + x^4)\mathrm{d}x = \frac{\pi}{30}.$$

(d) Profile $y = \sin x$, $0 \le x \le \pi$:

$$V = \pi \int_0^\pi \sin^2 x\mathrm{d}x = \pi \int_0^\pi \frac{1}{2}(1 - \cos 2x)\mathrm{d}x = \frac{1}{2}\pi^2.$$

(e) Profile $y = x^3$, $-1 \le x \le 1$:

$$V = \pi \int_{-1}^1 x^6\mathrm{d}x = \frac{2\pi}{7}.$$

(f) Profile $y = x(1-x)$, $0 \le x \le 2$:

$$V = \pi \int_0^2 x^2(1-x)^2 \mathrm{d}x = \pi \int_0^2 (x^2 - 2x^3 + x^4)\mathrm{d}x = \frac{16\pi}{15}.$$

(g) Profile $y = 1/x$, $1 \le x < \infty$:

$$V = \pi \int_1^\infty \frac{\mathrm{d}x}{x^2} = \pi.$$

(h) Profile $y = x^{\frac{1}{4}}$, $0 \le x \le 1$.

$$V = \pi \int_0^1 x^{\frac{1}{2}} \mathrm{d}x = \frac{2\pi}{3}.$$

16.4. A sphere of radius R can be viewed as a surface of revolution about the x axis with a profile $y = \surd(R^2 - x^2)$. The volume of the sphere is

$$\begin{aligned} V &= \pi \int_{-R}^R y^2 \mathrm{d}x = \pi \int_R^R (R^2 - x^2)\mathrm{d}x \\ &= \pi \left[R^2 x - \frac{1}{3}x^3\right]_{-R}^R = 2\pi \left[R^3 - \frac{1}{3}R^3\right] = \frac{4}{3}\pi R^3. \end{aligned}$$

16.5. (a) The required profile for the ellipsoid is

$$y = \frac{b}{a}\surd(a^2 - x^2).$$

Hence the volume of the ellipsoid is

$$\begin{aligned} V &= \pi \int_{-a}^a \left(\frac{b}{a}\right)^2 (a^2 - x^2)\mathrm{d}x = \pi \left(\frac{b}{a}\right)^2 \left[a^2 x - \frac{1}{3}x^3\right]_{-a}^a \\ &= \frac{4}{3}\pi a b^2. \end{aligned}$$

(b) The volume of the ellipsoid is

$$V = \pi \int_{-a}^a y^2 \mathrm{d}x,$$

where

$$y = \frac{b}{a}\surd(a^2 - x^2).$$

Change the scale of the y coordinate by writing $y = (b/a)y'$ so that $y' = \surd(a^2 - x^2)$ is the profile of a semi-circle of radius a. Hence

$$V = \pi \left(\frac{b}{a}\right)^2 \int_{-a}^a y'^2 \mathrm{d}x = \left(\frac{b}{a}\right)^2 V_s,$$

where $V_s = \frac{4}{3}\pi a^3$ is the volume of a sphere of radius a. Therefore

$$V = \left(\frac{b}{a}\right)^2 \frac{4}{3}\pi a^3 = \frac{4}{3}\pi a b^2.$$

A similar argument can be devised if the x coordinate is scaled.

16.6. In this problem the line $y = \frac{1}{2}x$ or $x = 2y$ is rotated about the y axis between $y = 1$ and $y = 2$ to create a truncated cone. Its volume is given by

$$V = \pi \int_1^2 x^2 \mathrm{d}y = 4\pi \int_1^2 y^2 \mathrm{d}y = 4\pi \left[\frac{1}{3}y^3\right]_1^2 = \frac{28}{3}\pi.$$

16.7. Consider an element of the beam of incremental length δx, distance x from the wall at A. Its mass is $m\delta x$, and its moment about A is $mgx\delta x$, where g is the acceleration due to gravity. The total moment of the beam is the sum of these elements between $x = 0$ and $x = L$, which in the limit becomes the integral

$$\text{moment} = \int_0^L mgx\mathrm{d}x = \frac{1}{2}mgL^2.$$

16.8. An increment of width δx has mass given

$$\begin{aligned}\delta m &= \text{(width)} \times \text{(cross-sectional area)} \times \text{(density)}\\ &= 500 \times [4 \times 10^{-4}(1 + 0.4x^2)]\delta x.\end{aligned}$$

The moment M required to support the beam about A is the sum of the increments $xg\delta m$, which in the limit is the integral

$$\begin{aligned}M &= \int_0^1 500 \times 4 \times 10^{-4}(1 + 0.4x^2)xg\mathrm{d}x = g\int_0^1 [0.2x + 0.08x^3]\mathrm{d}x.\\ &= 0.12g = 1.18,\end{aligned}$$

assuming SI units with $g = 9.81\mathrm{ms}^{-2}$.

The result will not be affected by different cross-sections or a bent axis provided that each element δx remains at distance x from the wall.

16.9. Let δx be the width of an increment of the tube. Then the mass of solute in this increment is

$$\delta m = c(x) \times 0.1 \times \delta x.$$

The total mass of solute is, with $c(x) = 0.04\mathrm{e}^{-\frac{1}{4}x}$,

$$\begin{aligned}m &= \int_0^{10} \mathrm{d}m = \int_0^{10} 0.1c(x)\mathrm{d}x = \int_0^{10} 0.004\mathrm{e}^{-\frac{1}{4}x}\mathrm{d}x\\ &= 0.004\left[-4\mathrm{e}^{-\frac{1}{4}x}\right]_0^{10}\\ &= 0.015\mathrm{gm},\end{aligned}$$

to two significant figures.

16.10. Consider a horizontal slice of thickness δh of the water clock. The volume of this slice is approximately, with radius $r(h) = 0.39h^{\frac{1}{4}}$,

$$\delta V = \text{(cross-sectionsl area)} \times \text{(thickness)} = \pi[r(h)]^2\delta h = \pi(0.39)^2h^{\frac{1}{2}}\delta h.$$

In the limit, as $\delta h \to 0$,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \pi(0.39)^2h^{\frac{1}{2}}.$$

Since we are given that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -0.003h^{\frac{1}{2}},$$

it follows that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t}\Big/\frac{\mathrm{d}V}{\mathrm{d}h} = -0.003/(\pi(0.39)^2) = -0.00628$$

which is a constant. This means that the water level height h falls at a constant rate. The clock 'stops' when $h = 0$. Hence it runs for a time T where

$$T = -\int_{0.5}^0 \frac{\mathrm{d}h}{0.00628} = 79.6 \text{ hours.}$$

The clock will run for about 80 hours given the accuracy of the data.

16.11. The heat generated is

$$\begin{aligned} H &= \int_0^{2\pi/\omega} Ri^2 \mathrm{d}t = \int_0^{2\pi/\omega} Ri_0^2 \cos^2 \omega t \mathrm{d}t \\ &= \frac{1}{2} Ri_0^2 \int_0^{2\pi/\omega} (1 + \cos 2\omega t) \mathrm{d}t \\ &= Ri_0^2 \pi/\omega. \end{aligned}$$

If the cycle runs from $t = t_0$ to $t = t_0 + 2\pi/\omega$, then the heat generated is

$$\begin{aligned} H_0 &= \int_{t_0}^{t_0+2\pi/\omega} Ri^2 \mathrm{d}t = \int_{t_0}^{t_0+2\pi/\omega} Ri_0^2 \cos^2 \omega t \mathrm{d}t \\ &= \frac{1}{2} Ri_0^2 \int_{t_0}^{t_0+2\pi/\omega} (1 + \cos 2\omega t) \mathrm{d}t \\ &= Ri_0^2 \pi/\omega = H, \end{aligned}$$

since the integral of $\cos 2\omega t$ over *any* interval of length $2\pi/\omega$ is zero.

16.12. The line $y = -x$ and the parabola $y = x(x - 1)$ are shown in the figure: the parabola cuts the x axis at $x = 0$ and at $x = 1$ (note that the scales differ). Also $x(x - 1) \geq -x$ in the interval $0 \leq x \leq 2$. Therefore the geometric area is

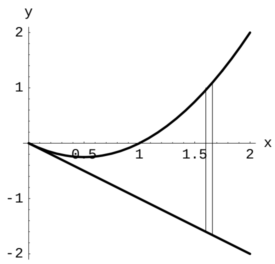

Figure 4: Problem 16.12

$$A = \int_0^2 |x(x - 1) + x| \mathrm{d}x = \int_0^2 x^2 \mathrm{d}x = \frac{8}{3}.$$

16.13. For $x > 0$, $x^3 > -x$, whilst for $x < 0$, $-x > x^3$. Hence, for $x > 0$ an element of area is

$$\delta A_1 \approx (x^3 + x)\delta x.$$

Hence

$$A_1 = \int_0^1 (x^3 + x) \mathrm{d}x = \left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^1 = \frac{3}{4}.$$

For $x < 0$, the area is

$$A_2 = \int_{-1}^0 (-x - x^3) \mathrm{d}x = \left[-\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{-1}^0 = \frac{3}{4}.$$

Hence the required geometric area is

$$A = A_1 + A_2 = \frac{3}{2}.$$

16.14. The incremental formula for sectorial area is $\delta A \approx \frac{1}{2} r^2 \delta\theta$, which gives the area

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta,$$

for the curve $r = f(\theta)$ between the angles $\theta = \alpha$ and $\theta = \beta$.
(a) Curve $r = \theta$, $0 \le \theta \le 2\pi$:

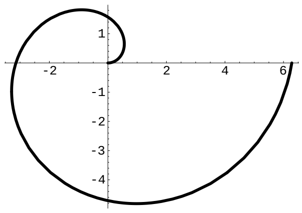

Figure 5: Problem 16.14a

Sectorial area:

$$A = \frac{1}{2}\int_0^{2\pi} \theta^2 \mathrm{d}\theta = \frac{1}{2}\left[\frac{1}{3}\theta^3\right]_0^{2\pi} = \frac{4}{3}\pi^3.$$

(b) Curve $r = 2\cos\theta$, $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$:

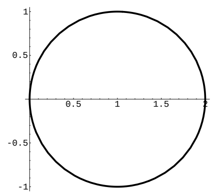

Figure 6: Problem 16.14b

Sectorial area:

$$A = \frac{1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 4\cos^2\theta \mathrm{d}\theta = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (1 + \cos 2\theta)\mathrm{d}\theta = \pi.$$

(c) Curve $r = \mathrm{e}^{\theta/2\pi}$, $0 \le \theta \le \pi$:

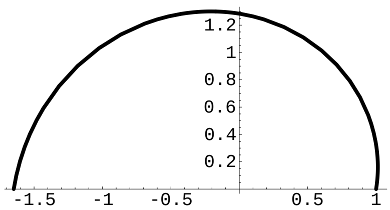

Figure 7: Problem 16.14c

Sectorial area:

$$A = \frac{1}{2}\int_0^{\pi} \mathrm{e}^{\theta/\pi} \mathrm{d}\theta = \frac{1}{2}\pi(\mathrm{e} - 1).$$

(d) Curve $r = \sin 2\theta$, $0 \le \theta \le \frac{1}{2}\pi$:
Sectorial area:

$$A = \frac{1}{2}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta \mathrm{d}\theta = \frac{1}{4}\int_0^{\frac{1}{2}\pi} (1 - \cos 4\theta)\mathrm{d}\theta = \frac{\pi}{8}.$$

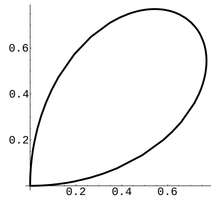

Figure 8: Problem 16.14d

16.15. Consider a strip of width δy and length L on the end of the trough at depth y below the top of the trough. The element of force on the strip is

$$\delta F \approx \text{(pressure)} \times \text{(increment of area)} = (\rho g y) \times (L\delta y) = \rho g y L \delta y.$$

The total force on the end of the trough is the limit as $\delta y \to 0$ of the sum of these elements, which is the integral

$$F = \int_0^H \rho g y L \mathrm{d}y = \frac{1}{2}\rho g L H^2.$$

The moment of the strip about the bottom of the end of the trough is

$$\delta M = (\delta F) \times (H - y) = \rho g y L (H - y)\delta y.$$

Hence the total moment is

$$M = \int_0^H \rho g L (yH - y^2)\mathrm{d}y = \frac{1}{6}\rho g L H^3.$$

16.16. Let the cone be generated by rotating the profile $y = (R/H)x$ about the x axis between $x = 0$ and $x = H$. Take a section of the cone of thickness δx at distance x from the origin. The mass δm of this disc is

$$\delta m = \rho \pi y^2 \delta x = \rho\pi \left(\frac{R}{H}\right)^2 x^2 \delta x,$$

where ρ is the density of the cone. The mass of the cone is the limit of the sum of these elements:

$$m = \rho\pi \left(\frac{R}{H}\right) \int_0^H x^2 \mathrm{d}x = \frac{1}{3}\rho\pi R^2 H. \qquad \text{(i)}$$

Let the centre of mass be at distance $\overline{x}$ from the origin: by symmetry the centre of mass will be on the x axis. Then

$$m\overline{x} = \rho\pi \left(\frac{R}{H}\right) \int_0^H x x^2 \mathrm{d}x = \frac{1}{4}\rho\pi R^2 H^2 = \frac{3}{4} H m,$$

by (i). Hence $\overline{x} = \frac{3}{4}H$.

16.17. The figure shows the strip of width δx: the origin is at the centre of the rectangle with the axes parallel to the sides as shown. The axis of rotation is the y axis. The mass of the rectangle is $m = \rho a b$, where ρ is its density (mass per unit area). The mass of the strip is $\rho b \delta x$, and the moment of inertia of the strip about the y axis is $\rho b x^2 \delta x$.

Therefore the moment of inertia of the whole rectangle about the y axis is

$$I = \int_{\frac{1}{2}a}^{\frac{1}{2}a} \rho b x^2 \mathrm{d}x = \rho b \left[\frac{1}{3} b x^3\right]_{-\frac{1}{2}a}^{\frac{1}{2}a} = \frac{1}{12}\rho b a^3 = \frac{1}{12} m a^2.$$

16.18. The figure shows the triangle with suitable elements of area in both cases

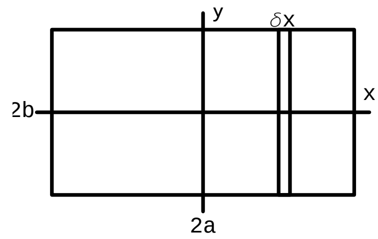

Figure 9: Problem 16.17

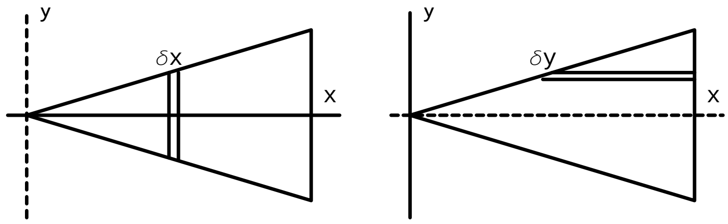

Figure 10: Problem 16.18

If ρ is the density (mass per unit area) then the mass of the triangle is $m = \frac{1}{2}\rho BH$.
(a) From the first figure, the moment of inertia δI of the strip of width δx about the y axis is

$$\delta I \approx x^2 2\rho y \delta x = \rho \frac{B}{H} x^3 \delta x.$$

where $y = Bx/(2H)$. Hence the moment of inertia is

$$I = \rho \frac{B}{H} \int_0^H x^3 \mathrm{d}x = \frac{1}{4}\rho BH^3 = \frac{1}{2} mH^2.$$

(b) From the second figure, the moment of inertia δI of the strip of width δy about the x axis is

$$\delta I \approx y^2 \rho (H - x)\delta y = \rho y^2 \left[H - \frac{H}{B} y\right] \delta y.$$

Therefore the moment of inertia is

$$\begin{aligned} I &= 2\rho \left(\frac{H}{B}\right) \int_0^{\frac{1}{2}B} y^2 (B - 2Hy) \mathrm{d}y \\ &= \frac{2\rho H}{B} \left[\frac{1}{3} B y^3 - \frac{1}{2} y^4\right]_0^{\frac{1}{2}B} \\ &= \frac{1}{48} HB^3 = \frac{1}{24} mB^2. \end{aligned}$$

16.19. The trapezium rule for numerical integration of $f(x)$ over $a \leq x \leq b$ is (see (16.14))

$$\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{N} \{\frac{1}{2} f(x_0) + (f(x_1) + f(x_2) + \cdots + f(x_{N-1})\} + \frac{1}{2} f(x_N),$$

where $x_0 = a$ and $x_N = b$.
(a) In this problem $f(x) = \mathrm{e}^{\frac{1}{2}x}$, $a = 0$ and $b = 1$. The exact value of the integral is

$$I = \int_0^1 \mathrm{e}^{\frac{1}{2}x} \mathrm{d}x = 2\sqrt{\mathrm{e}} - 2 = 1.2974\ldots.$$

Using the trapezium rule, the approximations to the integral for increasing values of N are

N	2	3
approximation, A_N	1.304	1.300
error, $100\|I - A_N\|/I$	0.52%	0.23%

With $N = 2$, the error is $100|I - A_N| = 0.52\%$, which is within the 1% accuracy required.

(b) In this example $f(x) = \sin x$, $a = 0$ and $b = \pi$. The exact value of the integral is

$$I = \int_0^\pi \sin x \mathrm{d}x = 2.$$

Using the trapezium rule, the approximations to the integral for increasing values of N are

N	2	3	4	10
approximation, A_N	1.571	1.814	1.896	1.984
error, $100\|I - A_N\|/i$	21.5%	9.3%	5.2%	0.8%

$N = 10$ steps are required to reduce the error below 1%.

(c) In this case $f(x) = \cos x$, $a = -\frac{1}{2}\pi$ and $b = \frac{1}{2}\pi$. The exact value of the integral is

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos x \mathrm{d}x = 2.$$

Since the cosine function is the sine function translated $\frac{1}{2}\pi$ to the left, and the interval is $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$, the data and errors in the table in (b) will be exactly the same for $f(x) = \cos x$.

16.20. The trapezium rule is given in Problem 16.19.

(a) With $f(x) = \sin^{\frac{1}{2}} x$, $a = 0$ and $b = \frac{1}{2}\pi$, approximations for increasing N are given in the Table:

N	5	10	20	50
approximation, A_N	1.162	1.852	1.197	1.197

To two decimal places the answer is 1.20. Numerical integration using *Mathematica* gives the answer 1.19814.

(b) With $f(x) = \mathrm{e}^{-x^2}$, $a = 0$ and $b = 1$, approximations for increasing N are:

N	3	6	10	20
approximation, A_N	0.740	0.745	0.746	0.747

To two decimal places the answer is 0.75.

(c) With $f(x) = \mathrm{e}^x/(1 + x^3)$, $a = 1$ and $b = 2$, a sample of approximations for increasing N are:

N	5	10	20	50
approximation, A_N	1.0482	1.0485	1.0482	1.0482

To 3 decimal places the answer is 1.048.

(d) With $f(x) = \sin x/x$, $a = 1$ and $b = 2$, approximations for increasing N are:

N	3	5	10	50
approximation, A_N	0.6593	0.6589	0.6592	0.6593

To 3 decimal places the answer is 0.659.

16.21. The formula for Simpson's rule is given in the question. Since N must be even, put $N = 2K$ and let the step-length be $h = (b - a)/N = (b - a)/(2K)$. If $y = f(x)$ over the interval $a \leq x \leq b$ is the function to be integrated, then Simpson's rule can be written as

$$I = \int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}[f(a) + 4f(a + h) + 2f(a + 2h) + 4f(a + 3h) + \cdots + 4f(a + (2K - 1)h) + f(b)].$$

It is easy to compose a short program in *Mathematica* or another programming language to simulate Simpson's rule.

For $f(x) = \mathrm{e}^{x^2}$, $a = 0$ and $b = 1$, $N = 4$ and $K = 2$, Simpson's rule gives $I \approx 0.746855$, whilst for $N = 6$ and $K = 3$, $I \approx 0.746830$. These answers can be compared with the results in Problem 16.20b where the trapezium rule was used for the same integral.

16.22. From the figure an element of arc-length δs can be approximated by the chord, which by Pythagoras's theorem is

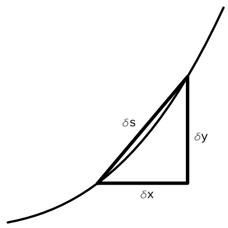

Figure 11: Problem 16.22

$$\delta s = \surd[(\delta x)^2 + (\delta y)^2]^{\frac{1}{2}} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

In the limit $\delta x \to 0$, arc-length s between $x = a$ to $x = b$ is given by the integral

$$s = \int_a^b \left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{1}{2}} \mathrm{d}x.$$

Simpson's rule given in Problem 16.21 has been used to compute the lengths of the curves. (Note that the number of coordinates called upon equals $2K$.)
(a) For $y = \sin x$, $0 \le x \le 1$, the length of the curve is

$$s = \int_0^1 \surd(1 + \cos^2 x)\mathrm{d}x.$$

A sample of numerical approximations is given in the table:

K	2	4	6	8
approximation, s_K	1.31148	1.31145	1.31144	1.31144

This gives the length to 4 decimal places.
(b) For $y = x^2$, $0 \le x \le 2$, the length of the curve is

$$s = \int_0^2 \surd(1 + 4x^2)\mathrm{d}x.$$

A sample of approximations is given in the table:

K	2	4	6	10
approximation, s_K	4.65020	4.64683	4.64678	4.64678

This gives the length as 4.6468 to 4 decimal places.
(c) For $y = \mathrm{e}^x$, $-1 \le x \le 1$, the length of the curve is

$$s = \int_{-1}^1 \surd(1 + \mathrm{e}^{2x})\mathrm{d}x.$$

A list of approximations s_K for increasing K is given in the table:

K	2	4	6	10
approximation, s_K	3.19695	3.19625	3.19621	3.19620

The length is 3.1962 to 4 decimal places.
(d) For the semicircle $y = (1-x^2)^{\frac{1}{2}}$, $-1 \le x \le 1$, the length of the curve is

$$s = \int_{-1}^{1} \left[1 + \frac{x^2}{1-x^2}\right]^{\frac{1}{2}} \mathrm{d}x = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{(1-x^2)}},$$

which is π.

16.23. Since $x = r\cos\theta$ and $y = r\sin\theta$,

$$\delta x \approx \delta(r\cos\theta) = \delta r\cos\theta - r\sin\theta\delta\theta, \quad \delta y = \delta(r\sin\theta) = \delta r\sin\theta + r\cos\theta\delta\theta.$$

Hence

$$\begin{aligned} \delta s &\approx [(\delta x)^2 + \delta y)^2]^{\frac{1}{2}} \\ &= [(\delta r\cos\theta - r\sin\theta\delta\theta)^2 + (\delta r\sin\theta + r\cos\theta\delta\theta)^2]^{\frac{1}{2}} \\ &= [r^2(\delta\theta)^2 + (\delta r)^2]^{\frac{1}{2}} = \left[r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2\right]^{\frac{1}{2}} \delta\theta. \end{aligned}$$

In the limit as $\delta\theta \to 0$, the length of the curve defined by $r = f(\theta)$ is the integral

$$s = \int_{\alpha}^{\beta} \{[f(\theta)]^2 + [f'(\theta)]^2\}^{\frac{1}{2}} \mathrm{d}\theta.$$

The limits for the cardioid $r = a(1+\cos\theta)$ are $\alpha = -\pi$ and $\beta = \pi$. Hence the length of the curve is

$$\begin{aligned} s &= \int_{-\pi}^{\pi} [a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{\frac{1}{2}} \mathrm{d}\theta = 2\int_{-\pi}^{\pi} a\cos\tfrac{1}{2}\theta\mathrm{d}\theta \\ &= 4a[\sin\tfrac{1}{2}\theta]_{-\pi}^{\pi} = 8a. \end{aligned}$$

Chapter 17: Systematic techniques for integration

17.1. (See Section 17.1.) Put $ax + b = u$; then $\mathrm{d}x = \mathrm{d}u/a$.

(a) Put $3x = u$ so $\mathrm{d}x = \frac{1}{3}u$: $$\int \sin 3x\mathrm{d}x = \int \tfrac{1}{3}\sin u\mathrm{d}u = -\tfrac{1}{3}\cos u = -\tfrac{1}{3}\cos 3x + C;$$

(b) Put $4x = u$ so $\mathrm{d}x = \frac{1}{4}u$: $$\int \cos 4x\mathrm{d}x = \frac{1}{4}\sin 4x + C;$$

(c) Put $-3x = u$ so $\mathrm{d}x = -\frac{1}{3}u$: $$\int \mathrm{e}^{-3x}\mathrm{d}x = -\tfrac{1}{3}\mathrm{e}^{-3x} + C;$$

(d) Put $1 + x = u$: $$\int (1+x)^{10}\mathrm{d}x = \tfrac{1}{11}(1+x)^{11} + C;$$

(e) Put $1 - x = u$: $$\int (1-x)^9\mathrm{d}x = -\tfrac{1}{10}(1-x)^{10} + C;$$

(f) Put $3-2x=u$: $$\int(3-2x)^5\mathrm{d}x=-\tfrac{1}{12}(3-2x)^6+C;$$

(g) Put $1+2x=u$: $$\int(1+2x)^n\mathrm{d}x=\frac{1}{2(n+1)}(1+2x)^{n+1}+C;$$

(h) Put $x-1=u$: $$\begin{aligned}\int x(x-1)^4\mathrm{d}x&=\int(x^5-4x^4+3x^3-4x^2+x)\mathrm{d}x\\&=\tfrac{1}{30}x^2(5x^4-24x^3+45x^2-40x+15)+C;\end{aligned}$$

(i) Put $1-x=u$ so $\mathrm{d}x=-\mathrm{d}u$: $$\int(1-x)^{\frac{1}{2}}\mathrm{d}x=-\int u^{\frac{1}{2}}\mathrm{d}u=\frac{2}{3}u^{\frac{3}{2}}+C=-\frac{2}{3}(1-x)^{\frac{3}{2}}+C;$$

(j) Put $2x-3=u$ so $\mathrm{d}x=\frac{1}{2}\mathrm{d}u$: $$\int(2x-3)^{-\frac{1}{2}}\mathrm{d}x=(2x-3)^{\frac{1}{2}}+C;$$

(k) Put $3x+2=u$: $$\int\frac{\mathrm{d}x}{(3x+2)^2}=-\frac{1}{3(3x+2)}+C;$$

(l) Put $1-x=u$: $$\int\frac{\mathrm{d}x}{(1-x)^4}=-\frac{1}{3(1-x)^3}+C;$$

(m) Put $1+x=u$: $$\int\frac{\mathrm{d}x}{1+x}=\int\frac{\mathrm{d}u}{u}=\ln|u|+C=\ln|1+x|+C;$$

(n) Put $3x+2=u$: $$\int(2x+3)^{-\frac{1}{2}}\mathrm{d}x=\tfrac{1}{2}\ln(2x+3)+C;$$

(o) Put $1-x=u$: $$\int\frac{x}{(1-x)^2}\mathrm{d}x=\ln|1-x|+\frac{1}{1-x}+C;$$

(p) Put $1-x=u$: $$\int\frac{1+x}{1-x}\mathrm{d}x=\int\frac{u-2}{u}\mathrm{d}u=-x-2\ln|1-x|+C;$$

(q) Put $x-1=u$. Then
$$\begin{aligned}\int\frac{\mathrm{d}x}{(x-1)^{\frac{1}{2}}}&=\int[u^{\frac{1}{2}}+u^{-\frac{1}{2}}]\mathrm{d}u=\frac{2}{3}u^{\frac{3}{2}}+2u^{\frac{1}{2}}+C\\&=\frac{2}{3}(x-1)^{\frac{3}{2}}+2(x-1)^{\frac{1}{2}}+C\end{aligned}$$

(r) Put $1-2x=u$: $$\int\cos(1-2x)\mathrm{d}x=-\tfrac{1}{2}\sin(1-2x)+C;$$

(s) Put $2x-3=u$: $$\int\sin(2x-3)\mathrm{d}x=-\cos(2x-3)+C.$$

17.2. (a) Let $u = 2t - 5$. Then $\mathrm{d}u/\mathrm{d}t = 2$ and

$$\int (2t-5)^5 \mathrm{d}t = \int u^5 \frac{1}{2} \mathrm{d}u = \frac{1}{12} u^6 + C = \frac{1}{12}(2t-5)^6 + C;$$

(b) Let $u = \frac{1}{2}(3t-1)$. Then $\mathrm{d}u/\mathrm{d}t = \frac{3}{2}$ and

$$\int \sin \frac{1}{2}(3t-1) \mathrm{d}t = \int \sin u \frac{2}{3} \mathrm{d}u = -\frac{2}{3} \cos u + C = -\frac{2}{3} \cos \frac{1}{2}(3t-1) + C;$$

(c)
$$\int \frac{1}{(2w+1)^2} \mathrm{d}w = -\frac{1}{2(2w+1)} + C;$$

(d)
$$\int \mathrm{e}^{-3r} \mathrm{d}r = -\frac{1}{3} \mathrm{e}^{-3r} + C;$$

(e) Let $u = -t$. Then $\mathrm{d}u/\mathrm{d}t = -1$ and

$$\int (-t)^{\frac{1}{2}} \mathrm{d}t = -\int u^{\frac{1}{2}} \mathrm{d}u = -\frac{2}{3} u^{\frac{3}{2}} + C = -\frac{2}{3}(-t)^{\frac{3}{2}} + C;$$

(f) Let $u = 1 - s$. Then $\mathrm{d}u/\mathrm{d}s = -1$, and

$$\int \frac{s}{(1-s)^3} \mathrm{d}s = \int [-u^{-3} + u^{-2}] \mathrm{d}u = \frac{1}{2u^2} - \frac{1}{u} + C = \frac{1}{2(1-s)^2} - \frac{1}{1-s} + C;$$

(g) Use the substitution $u = \omega t - \phi$. Then

$$\int \cos(\omega t - \phi) \mathrm{d}t = \frac{1}{\omega} \sin(\omega t - \phi) + C;$$

17.3. (See Section 17.2.) (a) Let $u = x^2$ so that $\mathrm{d}u/\mathrm{d}x = 2x$, and $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$. Then

$$\int x\mathrm{e}^{-x^2} \mathrm{d}x = \int \frac{1}{2} \mathrm{e}^{-u} \mathrm{d}u = -\frac{1}{2} \mathrm{e}^{-u} + C = \frac{1}{2} \mathrm{e}^{-x^2} + C;$$

(b) As in (a), let $u = x^2$. Then,

$$\int x \sin(x^2) \mathrm{d}x = -\frac{1}{2} \cos(x^2) + C;$$

(c) Let $u = x^2$. Then,

$$\int x \cos(x^2) \mathrm{d}x = \frac{1}{2} \sin(x^2) + C;$$

(d) Let $u = x^2 + 3$. Then $u = x^2 + 3$ so that $\mathrm{d}u/\mathrm{d}x = 2x$, and

$$\int x \cos(x^2 + 3) \mathrm{d}x = \int \frac{1}{2} \cos u \mathrm{d}u = \frac{1}{2} \sin u + C = \frac{1}{2} \sin(x^2 + 3) + C;$$

(e) Let $u = 1 - 3x^2$ so that $\mathrm{d}u/\mathrm{d}x = -6x$. Then

$$\int x \cos(1 - 3x^2) \mathrm{d}x = -\frac{1}{6} \int \cos u \mathrm{d}u = -\frac{1}{6} \sin u + C = -\frac{1}{6} \sin(1 - 3x^2) + C;$$

(f) Use the substitution $u = x^2 - 1$ so that $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$. Then

$$\int x(x^2-1)^4 \mathrm{d}x = \frac{1}{2} \int u^4 \mathrm{d}u = \frac{1}{10} u^5 + C = \frac{1}{10}(x^2-1)^5 + C;$$

(g) Use the substitution $u = 3x^2 + 4$. Then

$$\int x(3x^2+4)^3 \mathrm{d}x = \frac{1}{6}\int u^3 \mathrm{d}u = \frac{1}{24}u^4 + C = \frac{1}{24}(3x^2+4)^4 + C;$$

(h) Use the substitution $u = 1 + 2x^2$, so that $\mathrm{d}u/\mathrm{d}x = 4x$. Hence

$$\int \frac{x}{1+2x^2}\mathrm{d}x = \frac{1}{4}\int \frac{1}{u}\mathrm{d}u = \frac{1}{4}\ln u + C = \frac{1}{4}\ln|1+2x^2) + C;$$

(i) Let $u = 1 - x^2$. Then $\mathrm{d}u/\mathrm{d}x = -2x$, and

$$\begin{aligned}\int x^3(1-x^2)^3\mathrm{d}x &= -\frac{1}{2}\int(1-u)u^3\mathrm{d}u = -\frac{1}{2}\int(u^3-u^4)\mathrm{d}u \\ &= -\frac{1}{8}u^4 + \frac{1}{10}u^5 + C \\ &= -\frac{1}{8}(1-x^2)^4 + \frac{1}{10}(1-x^2)^5 + C;\end{aligned}$$

(j) Use the substitution $u = 1 + x^2$. Then

$$\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+x^2) + C;$$

(k) Use the substitution $u = 3x^2 - 2$. Then

$$\int \frac{x}{3x^2-2}\mathrm{d}x = \frac{1}{6}\int \frac{\mathrm{d}u}{u}\mathrm{d}u = \frac{1}{6}\ln|3x^2-2| + C.$$

17.4. (a) Let $u = \sin x$ so that $\mathrm{d}u/\mathrm{d}x = \cos x$. Then $\cos x\mathrm{d}x = \mathrm{d}u$, and

$$\int \sin x\cos x\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C;$$

If the substitution $v = \cos x$ is used instead, then the answer becomes

$$-\frac{1}{2}\cos^2 x + C_1.$$

Since $\sin^2 x = 1 - \cos^2 x$, the two forms of the solution represent the same family of solutions.
(b) Choose the substitution $u = \sin x$ (not $\cos x$). Then $\cos x\mathrm{d}x = \mathrm{d}u$ and

$$\int \sin^2 x\cos x\mathrm{d}x = \int u^2\mathrm{d}u = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C;$$

(c) Let $u = \sin 2x$ so that $\mathrm{d}u/\mathrm{d}x = 2\cos 2x$ and $\cos 2x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$

$$\int \sin^2 2x\cos 2x\mathrm{d}x = \frac{1}{2}\int u^2\mathrm{d}u = \frac{1}{6}u^3 + C = \frac{1}{6}\sin^3 2x + C;$$

(d) Let $u = \cos x$ so that $\mathrm{d}u/\mathrm{d}x = -\sin x$. Using this substitution

$$\int \cos^2 x\sin x\mathrm{d}x = -\int u^2\mathrm{d}u = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 x + C;$$

(e) Let $u = \cos 3x$ so that $\mathrm{d}u/\mathrm{d}x = -3\sin 3x$. Therefore, using this substitution

$$\int \cos^2 3x\sin 3x\mathrm{d}x = -\frac{1}{3}\int u^2\mathrm{d}u = -\frac{1}{9}u^3 + C = -\frac{1}{9}\cos^3 3x + C;$$

(f) Let $u = \sin x$. Using this substitution

$$\int \sin^3 x \cos x \mathrm{d}x = \frac{1}{4}\sin^4 x + C;$$

(g) Let $u = \sin 2x$ so that $\mathrm{d}u/\mathrm{d}x = 2\cos 2x$. Using this substitution

$$\int \cot 2x \mathrm{d}x = \int \frac{\cos 2x}{\sin 2x}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|\sin 2x| + C;$$

(h)
$$\int \tan \frac{1}{2}x \mathrm{d}x = -2\ln|\cos(\tfrac{1}{2}x)| + C;$$

(i) Use the substitution $u = \cos x$. Hence $\mathrm{d}u/\mathrm{d}x = -\sin x$. Therefore

$$\begin{aligned}\int \frac{\sin^3 x}{\cos x}\mathrm{d}x &= -\int \frac{1-u^2}{u}\mathrm{d}u = -\int \left[\frac{1}{u} - u\right]\mathrm{d}u = -\ln|u| + \frac{1}{2}u^2 + C \\ &= -\ln|\cos x| + \frac{1}{2}\cos^2 x + C;\end{aligned}$$

(j) Let $u = \cos x$. Then $\mathrm{d}u/\mathrm{d}x = -\sin x$, and

$$\int \sin^3 x \mathrm{d}x = -\int (1-u^2)\mathrm{d}u = -u + \frac{1}{3}u^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C$$

(k) Let $u = \cos x$. Then $\mathrm{d}u/\mathrm{d}x = -\sin x$ and

$$\begin{aligned}\int \tan^3 x \mathrm{d}x &= -\int \frac{1-u^2}{u^3}\mathrm{d}u = -\int \left[\frac{1}{u^3} - \frac{1}{u}\right]\mathrm{d}u = \frac{1}{2u^2} + \ln|u| + C \\ &= \frac{1}{2}\sec^2 x + \ln|\cos x| + C;\end{aligned}$$

(l) Let $u = \sin x$. Then $\mathrm{d}u/\mathrm{d}x = \cos x$, and

$$\int \cos^3 x \mathrm{d}x = \int (1-u^2)\mathrm{d}u = u - \tfrac{1}{3}u^3 + C = \sin x - \tfrac{1}{3}\sin^3 x + C.$$

17.5. Remember that the limits change with the substitution.
(a) Use the substitution $u = 1 + x$.

$$\int_1^{-1} (1+x)^7 \mathrm{d}x = \int_0^2 u^7 \mathrm{d}u = \frac{1}{8}[u^8]_0^2 = 32;$$

(b) Let $u = 1 - \frac{1}{2}x$. Then $\mathrm{d}u/\mathrm{d}x = -\frac{1}{2}$ so that

$$\int_{-1}^{1} (1 - \frac{1}{2}x)^7 \mathrm{d}x = -2\int_{\frac{3}{2}}^{\frac{1}{2}} u^7 \mathrm{d}u = 2\int_{\frac{1}{2}}^{\frac{3}{2}} u^7 \mathrm{d}u = \frac{2}{8}[u^8]_{\frac{1}{2}}^{\frac{3}{2}} = \frac{205}{32};$$

(c) Let $u = x^2 - 1$. Then $\mathrm{d}u/\mathrm{d}x = 2x$ and the integral becomes

$$\int_0^1 x(1-x^2)^3 \mathrm{d}x = -\frac{1}{2}\int_{-1}^{0} u^3 \mathrm{d}u = \frac{1}{8};$$

(d) Let $u = 2x + 3$. Then $\mathrm{d}u/\mathrm{d}x = 2$ and the integral is

$$\int_0^1 \frac{x\mathrm{d}x}{2x+3}\mathrm{d}x = \frac{1}{4}\int_3^5 \left[1 - \frac{3}{u}\right]\mathrm{d}u = \frac{1}{2} - \frac{3}{4}\ln[5/3];$$

(e) Use the substitution $u = 1 + x$. Then

$$\int_{-3}^{-2} \frac{\mathrm{d}x}{1+x} = [\ln|1+x|]_{-3}^{-2} = -\ln 2;$$

(f) Use the substitution $u = 3x - 2$. Then

$$\int_{3}^{4} \frac{\mathrm{d}x}{2-3x} = \frac{1}{3}[\ln 7 - \ln 10];$$

(g) Let $u = x^2 - 1$. Then $\mathrm{d}u/\mathrm{d}x = 2x$, so that the integral becomes

$$\int_{0}^{1} x^3(1-x^2)^3 \mathrm{d}x = -\frac{1}{2}\int_{-1}^{0}(u+1)u^3\mathrm{d}u = \frac{1}{40};$$

(h) Use the substitution $u = \cos t$. Then $\mathrm{d}u/\mathrm{d}t = -\sin t$, and

$$\int_{0}^{\frac{1}{4}\pi} \tan t \mathrm{d}t = -\int_{1}^{1/\sqrt{2}} \frac{\mathrm{d}u}{u} = \int_{1/\sqrt{2}}^{1} \frac{\mathrm{d}u}{u} = [\ln u]_{1/\sqrt{2}}^{1} = \frac{1}{2}\ln 2;$$

(i) Let $u = \sin 3w$. Then $\mathrm{d}u/dw = 3\cos 3w$, and

$$\int_{\pi/12}^{\pi/6} \cot 3w \mathrm{d}w = \frac{1}{3}\int_{1/\sqrt{2}}^{1} \frac{\mathrm{d}u}{u} = \left[\frac{1}{3}\ln u\right]_{1/\sqrt{2}}^{1} = \frac{1}{6}\ln 2;$$

(j) Let $v = \sin u$. Then

$$\int_{0}^{\frac{1}{2}\pi} \sin u \cos u \mathrm{d}u = \int_{0}^{1} v \mathrm{d}v = \frac{1}{2};$$

(k) Let $v = u + \frac{1}{2}\pi$. The integral becomes

$$\int_{0}^{\pi} (\sin v)^{\frac{1}{2}} \cos v \mathrm{d}v = -\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos u)^{\frac{1}{2}} \sin u \mathrm{d}u = 0$$

since the integrand is an odd function of u, and the interval is equally disposed about $u = 0$. (Alternatively, substitute $\sin v = u$.)

(l) Use the substitution $u = \sin\theta$. Then $\mathrm{d}u/\mathrm{d}\theta = \cos\theta$, and

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^3\theta \mathrm{d}\theta = \int_{-1}^{1}(1-u^2)\mathrm{d}u = \frac{4}{3};$$

(m) Let $u = 2t$. Then

$$\int_{0}^{\frac{1}{2}\pi} \sin 2t \mathrm{d}t = \frac{1}{2}\int_{0}^{\pi} \sin u \mathrm{d}u = 1;$$

(n) Use the substitution $u = \omega t + \phi$. Then

$$\int_{-\pi/(2\omega)}^{\pi/(2\omega)} \cos(\omega t + \phi)\mathrm{d}t = \frac{2}{\omega}\cos\phi.$$

17.6. (a) Put $\sin^2 t = \frac{1}{2}(1 - \cos t)$:

$$\int_{0}^{\pi} \sin^2 t \mathrm{d}t = \frac{1}{2}\int_{0}^{\pi}(1-\cos 2t)\mathrm{d}t = \frac{1}{2}[t - \frac{1}{2}\sin 2t]_{0}^{\pi} = \frac{1}{2}\pi;$$

(b)

$$\int_{0}^{\pi} \cos^2 t \mathrm{d}t = \frac{1}{2}\int_{0}^{\pi}(1+\cos 2t)\mathrm{d}t = \frac{1}{2}\pi;$$

(c)
$$\int_0^{\frac{1}{2}\pi} \sin^2 2t\mathrm{d}t = \frac{1}{2}\int_0^{\frac{1}{2}\pi}(1-\cos 4t)\mathrm{d}t = \frac{1}{4}\pi;$$

(d)
$$\int_0^{\frac{1}{2}\pi} \cos^2 \frac{1}{2}t\mathrm{d}t = \frac{1}{2}\int_0^{\frac{1}{2}\pi}(1+\cos t)\mathrm{d}t = \frac{1}{4}\pi + \frac{1}{2};$$

(e) Since $\sin 3t\cos 3t = \frac{1}{2}\sin 6t$,

$$\int_{-\pi}^{\pi} \sin^2 3t\cos 3t\mathrm{d}t = \frac{1}{2}\int_{-\pi}^{\pi}\sin 6t\sin 3t\mathrm{d}t = \frac{1}{4}\int_{-\pi}^{\pi}[\cos 3t - \cos 9t] = 0;$$

a product formula from Appendix B(d) has also been used;
(f) Use the identity

$$\cos^4 u = \frac{1}{4}(1+\cos 2u)^2 = \frac{1}{4}(1+2\cos 2u+\cos^2 2u) = \frac{1}{8}(3+4\cos 2u+\cos 4u).$$

Hence
$$\int_0^{\pi}\cos^4 u\mathrm{d}u = \frac{1}{8}\int_0^{\pi}(3+4\cos 2u+\cos 4u)\mathrm{d}u = \frac{3}{8}\pi.$$

17.7. (a) For the substitution $x = \mathrm{e}^u$, $\mathrm{d}x/\mathrm{d}u = \mathrm{e}^u = x$. Therefore

$$\int \frac{\ln x}{x}\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

(b) (i) For $u = (1-x^2)$, $\mathrm{d}u/\mathrm{d}x = -2x$. Then

$$\int x(1-x^2)^{\frac{1}{2}}\mathrm{d}x = -\frac{1}{2}\int u^{\frac{1}{2}}\mathrm{d}u = -\frac{1}{3}u^{\frac{3}{2}} + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.$$

(ii) For the alternative substitution $x = \sin u$, $\mathrm{d}x/\mathrm{d}u = \cos u$. Therefore

$$\int x(1-x^2)^{\frac{1}{2}}\mathrm{d}x = \int \cos^2 u\sin u\mathrm{d}u = -\frac{1}{3}\cos^3 u + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C,$$

using the result from Problem 17.4(d).
(c) Using the substitution $u = \mathrm{e}^x$, $\mathrm{d}u/\mathrm{d}x = \mathrm{e}^x = u$. Therefore

$$\int \frac{1}{\mathrm{e}^x + \mathrm{e}^{-x}}\mathrm{d}x = \int \frac{\mathrm{d}u}{1+u^2} = \arctan(u) + C = \arctan(\mathrm{e}^x) + C.$$

(d) (i) For $x = \sin u$, $\mathrm{d}x/\mathrm{d}u = \cos u$. Then

$$\int \frac{\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}} = \int \frac{\cos u\mathrm{d}u}{(1-\sin^2 u)^{\frac{1}{2}}} = \int \mathrm{d}u = u + C = \arcsin x + C.$$

(ii) For the alternative substitution $x = \cos u$, $\mathrm{d}x/\mathrm{d}u = -\sin u$. Therefore

$$\int \frac{\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}} = -\int \mathrm{d}u = -u + C = -\arccos x + C.$$

The results are the same since $\arcsin x = \frac{1}{2}\pi - \arccos x$.
(e) Using the substitution $u = \tan x$, $\mathrm{d}u/\mathrm{d}x = \sec^2 x = 1 + \tan^2 x = 1 + u^2$. Then

$$\begin{aligned}\int \tan^2 x\mathrm{d}x &= \int \frac{u^2\mathrm{d}u}{1+u^2} = \int\left[1 - \frac{1}{1+u^2}\right]\mathrm{d}u \\ &= u - \arctan u + C = \tan x - x + C\end{aligned}$$

(f) Using the substitution $x = 1/u$, $\mathrm{d}x/\mathrm{d}u = -1/u^2 = -x^2$. Then

$$\int \frac{\mathrm{d}x}{x^2(1+x^2)} = -\int \frac{u^2\mathrm{d}u}{1+u^2} = u - \arctan u + C = \frac{1}{x} - \arctan(1/x) + C.$$

(g) This integral is a standard form (see Appendix E). Let $x = \tan u$ so that $\mathrm{d}x/\mathrm{d}u = \sec^2 u = 1 + \tan^2 u = 1 + x^2$. Therefore

$$\int \frac{\mathrm{d}x}{1+x^2} = \int \mathrm{d}u = u + C = \arctan x + C.$$

(h) Using the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\mathrm{d}x = [1/\cos^2 x]\mathrm{d}x$. Therefore

$$\int \frac{\mathrm{d}x}{\cos^2 x} = \int \mathrm{d}u = u + C = \tan x + C.$$

(i) Let $t = u^2$ so that $\mathrm{d}t = 2u\mathrm{d}u$. Then

$$\int \frac{\mathrm{d}t}{t^{\frac{1}{2}}(1+t)} = 2\int \frac{\mathrm{d}u}{1+u^2} = 2\arctan u + C = 2\arctan(t^{\frac{1}{2}}) + C.$$

(j) Using the substitution $t = 1/u$, it follows that $\mathrm{d}t = -t^2\mathrm{d}u$. Therefore

$$\int \frac{1}{t^2}\sin\left(\frac{1}{t}\right)\mathrm{d}t = -\int \sin u\mathrm{d}u = \cos u + C = \cos\left(\frac{1}{t}\right) + C.$$

(k) Let $x = \sin u$. Then $\mathrm{d}x = \cos u\mathrm{d}u$, and

$$\begin{aligned}\int (1-x^2)^{\frac{1}{2}}\mathrm{d}x &= \int \cos^2 u\mathrm{d}u = \frac{1}{2}\int (1+\cos 2u)\mathrm{d}u \\ &= \frac{1}{2}u + \frac{1}{4}\sin 2u + C = \frac{1}{2}u + \frac{1}{2}\sin u\cos u + C \\ &= \frac{1}{2}\arcsin x + \frac{1}{2}x(1-x^2)^{\frac{1}{2}} + C\end{aligned}$$

(l) This is a standard integral in Appendix E. Let $x = \tan u$, so that $\mathrm{d}x = \sec^2 u\mathrm{d}u$. Then

$$\begin{aligned}\int \frac{\mathrm{d}x}{(1+x^2)^{\frac{1}{2}}} &= \int \frac{\sec^2 u}{\sec u}\mathrm{d}u = \int \frac{\mathrm{d}u}{\cos u} \\ &= \int \frac{\cos u}{\cos^2 u}\mathrm{d}u = \int \frac{\mathrm{d}v}{1-v^2} \quad (v = \sin u) \\ &= \frac{1}{2}\int \left[\frac{1}{1+v} + \frac{1}{1-v}\right]\mathrm{d}v = \frac{1}{2}[\ln(1+v) - \ln(1-v)] + C \\ &= \frac{1}{2}\ln\left[\frac{1+\sin u}{1-\sin u}\right] + C = \frac{1}{2}\ln\left[\frac{(1+x^2)^{\frac{1}{2}} + x}{(1+x^2)^{\frac{1}{2}} - x}\right] + C \\ &= \ln[x + (1+x^2)^{\frac{1}{2}}] + C.\end{aligned}$$

An alternative substitution is $x = \sinh u$ which leads to the alternative answer $\sinh^{-1} x + C$ with less working.

17.8. (See partial fractions, Section 1.14.)
(a) Using partial fractions, let

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \text{ or } 1 = A(x+2) + B(x-2).$$

Put $x = 2$; then $1 = 4A$. Put $x = -2$; then $1 = -4B$. Hence $A = \frac{1}{4}$ and $B = -\frac{1}{4}$. The integral becomes

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2-4} &= \frac{1}{4}\int\left[\frac{1}{x-2}-\frac{1}{x+2}\right]\mathrm{d}x\\ &= \frac{1}{4}[\ln(x-2)-\ln(x+2)+C] = \frac{1}{4}\ln\left[\frac{x-2}{x+2}\right]+C.\end{aligned}$$

(b) Using partial fractions

$$\frac{1}{x(x+2)} = \frac{1}{2x} - \frac{1}{2(x+2)}.$$

Hence

$$\int \frac{\mathrm{d}x}{x(x+2)} = \int\left[\frac{1}{2x}-\frac{1}{2(x+2)}\right]\mathrm{d}x = \frac{1}{2}[\ln x - \ln|x+2|]+C.$$

(c) Using partial fractions

$$\frac{1}{x^2(x-1)} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}.$$

Hence

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2(x-1)} &= \int\left[\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}\right]\mathrm{d}x\\ &= \ln|x-1| - \ln|x| + \frac{1}{x} + C\end{aligned}$$

(d) Using partial fractions

$$\frac{x}{(2x+1)(x+1)} = \frac{1}{x+1} - \frac{1}{2x+1}.$$

Hence

$$\begin{aligned}\int \frac{x}{(2x+1)(x+1)}\mathrm{d}x &= \int\left[\frac{1}{x+1}-\frac{1}{2x+1}\right]\mathrm{d}x\\ &= \ln|x+1| - \frac{1}{2}\ln|2x+1| + C\end{aligned}$$

(e) Using partial fractions

$$\frac{x+1}{4x^2-9} = \frac{5}{12(2x-3)} + \frac{1}{12(2x+3)}.$$

Hence, its integral is

$$\begin{aligned}\int \frac{x+1}{4x^2-9}\mathrm{d}x &= \int\left[\frac{5}{12(2x-3)}+\frac{1}{12(2x+3)}\right]\mathrm{d}x\\ &= \frac{5}{24}\ln|2x-3| + \frac{1}{24}\ln|2x+3| + C\end{aligned}$$

(f) In terms of partial fractions

$$\frac{1}{x(x^2+1} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \text{ or } 1 = A(x^2+1)+(Bx+C)x.$$

Put $x = 0$ in this identity; then $A = 1$. Now put $x = 1$ and $x = -1$ leading to

$$1 = 2 + B + C, \qquad 1 = 2 + B - C.$$

Therefore $B = -1$ and $C = 0$. The integral of the function becomes

$$\int \frac{\mathrm{d}x}{x(x^2+1)} = \int\left[\frac{1}{x}-\frac{x}{x^2+1}\right]\mathrm{d}x = \ln|x| - \frac{1}{2}\ln[x^2+1]+C$$

(g) Factorizing; $2x^2 + 3x + 1 = (2x + 1)(x + 1)$. Hence the integral can be written

$$\int \frac{x\mathrm{d}x}{2x^2 + 3x + 1} = \int \frac{x\mathrm{d}x}{(2x + 1)(x + 1)} = \ln|x + 1| - \frac{1}{2}\ln|2x + 1| + C,$$

as in Problem 17.8d.
(h) Using partial fractions

$$\int \frac{\mathrm{d}x}{x^2(2x + 1)} = \frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x + 1} + C.$$

(i) Let $u = \sin x$. Then $\mathrm{d}u/\mathrm{d}x = \cos x$. Hence, changing the variable

$$\begin{aligned}\int \frac{\mathrm{d}x}{\cos x} &= \int \frac{\mathrm{d}u}{1 - u^2} = \frac{1}{2}\int \left[\frac{1}{1 + u} + \frac{1}{1 - u}\right]\mathrm{d}u \\ &= \frac{1}{2}[\ln(1 + u) - \ln(1 - u)] + C \\ &= \frac{1}{2}\ln\left[\frac{1 + \sin x}{1 - \sin x}\right] + C\end{aligned}$$

(j) Let $u = \cos x$. Then

$$\int \frac{\mathrm{d}x}{\sin x} = -\int \frac{\mathrm{d}u}{1 - u^2} = -\frac{1}{2}[\ln(1 + u) - \ln(1 - u)] + C = -\frac{1}{2}\ln\left[\frac{1 - \cos x}{1 + \cos x}\right] + C,$$

as in the previous problem.

17.9. (Use the method of Example 17.16.)
(a) Let $u = x^3 - 1$, so that $\mathrm{d}u/\mathrm{d}x = 3x^2$. Then

$$x^2(x^3 - 1)^5 = \frac{1}{3}u^5\mathrm{d}u/\mathrm{d}x = g(u)\mathrm{d}u/\mathrm{d}x,$$

where $g(u) = \frac{1}{3}u^5$. Then

$$I = \int x^2(x^3 - 1)\mathrm{d}x = \frac{1}{3}\int u^5\mathrm{d}u = \frac{1}{18}u^6 + C = \frac{1}{18}(x^3 - 1)^6 + C.$$

(b) Let $u = x^2 - 2x + 3$, so that $\mathrm{d}u/\mathrm{d}x = 2x^2 - 2$. Then

$$(x - 1)(x^2 - 2x + 3)^{-1} = \left(\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x}\right)u^{-1} = g(u)\frac{\mathrm{d}u}{\mathrm{d}x},$$

where $g(u) = \frac{1}{2}u^{-1}$. Then

$$I = \int \frac{x - 1}{x^2 - 2x + 3}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2 - 2x + 3) + C.$$

(c) Let $u = \ln x$, so that $\mathrm{d}u/\mathrm{d}x = 1/x$. Then

$$\frac{1}{x(\ln x)^2} = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{1}{u^2},$$

and

$$\int \frac{\mathrm{d}x}{x(\ln x)^2} = \int \frac{\mathrm{d}u}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

(d) Let $u = 3x^{\frac{3}{2}} + 2$, so that $\mathrm{d}u/\mathrm{d}x = \frac{9}{2}x^{\frac{1}{2}}$, and $g(u) = \frac{2}{9}u^{\frac{1}{2}}$. Then

$$\int x^{\frac{1}{2}}(3x^{\frac{3}{2}} + 2^{\frac{1}{2}})\mathrm{d}x = \int \frac{2}{9}u^{\frac{1}{2}}\mathrm{d}u = \frac{4}{27}u^{\frac{3}{2}} + C = \frac{4}{27}(3x^{\frac{3}{2}} + 2)^{\frac{3}{2}} + C.$$

(e) Let $u = \mathrm{e}^x + \mathrm{e}^{-x}$, so that $\mathrm{d}u/\mathrm{d}x = \mathrm{e}^x - \mathrm{e}^{-x}$, and $g(u) = 1/u$. Then

$$\int \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}\mathrm{d}x = \int \frac{\mathrm{d}u}{u} = \ln u + C = \ln(\mathrm{e}^x + \mathrm{e}^{-x}) + C.$$

(f) Let $u = x^{\frac{1}{2}} + 1$, and $g(u) = 2/u$. Then

$$\int \frac{\mathrm{d}x}{x^{\frac{1}{2}}(x^{\frac{1}{2}} + 1)} = 2\int \frac{\mathrm{d}u}{u} = 2\ln u + C = 2\ln(x^{\frac{1}{2}} + 1) + C.$$

(g) Let $u = x^3 + 1$. Then

$$\int \frac{x^2\mathrm{d}x}{x^3 + 1} = \frac{1}{3}\int \frac{\mathrm{d}u}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3 + 1| + C.$$

17.10. The integration by parts formula is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + C.$$

(a) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \mathrm{e}^{-x}$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \qquad v = \int \mathrm{e}^{-x}\mathrm{d}x = -\mathrm{e}^{-x}.$$

Hence

$$\int x\mathrm{e}^{-x}\mathrm{d}x = -x\mathrm{e}^{-x} - \int(-\mathrm{e}^{-x})1\mathrm{d}x + C = -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + C.$$

(b) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \mathrm{e}^{3x}$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \qquad v = \int \mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x}.$$

Hence

$$\int x\mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}x\mathrm{e}^{3x} - \frac{1}{3}\int \mathrm{e}^{3x}\mathrm{d}x + C = \frac{1}{3}x\mathrm{e}^{3x} - \frac{1}{9}\mathrm{e}^{3x} + C.$$

(c) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \mathrm{e}^{-3x}$. Then

$$\int x\mathrm{e}^{-3x}\mathrm{d}x = -\frac{1}{3}x\mathrm{e}^{-3x} - \frac{1}{9}\mathrm{e}^{-3x} + C.$$

(d) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \cos x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \qquad v = \int \cos x\mathrm{d}x = \sin x.$$

Hence

$$\int x\cos x\mathrm{d}x = x\sin x - \int \sin x\mathrm{d}x + C = x\sin x + \cos x + C.$$

(e) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \sin x$. Then

$$\int x\sin x\mathrm{d}x = -x\cos x + \int \cos x\mathrm{d}x + C = -x\cos x + \sin x + C.$$

(f) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \cos\frac{1}{2}x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \qquad v = \int \cos\frac{1}{2}x\mathrm{d}x = 2\sin\frac{1}{2}x.$$

Hence

$$\int \cos\frac{1}{2}x\mathrm{d}x = 2x\sin\frac{1}{2}x - 2\int \sin\frac{1}{2}x\mathrm{d}x + C = 2x\sin\frac{1}{2}x + 4\cos\frac{1}{2}x + C.$$

(g) $$\int x\sin 2x\mathrm{d}x = -\frac{1}{2}x\cos 2x + \int \cos 2x\mathrm{d}x + C = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

(h) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = (1-x)^{10}$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \qquad v = \int (1-x)^{10}\mathrm{d}x = -\frac{1}{11}(1-x)^{11} + C.$$

Hence

$$\begin{aligned}\int x(1-x)^{10}\mathrm{d}x &= -\frac{1}{11}x(1-x)^{11} + \frac{1}{11}\int (1-x)^{11}\mathrm{d}x + C \\ &= -\frac{1}{11}x(1-x)^{11} + \frac{1}{132}(1-x)^{12} + C.\end{aligned}$$

(i) Let $u = \ln x$ and $\mathrm{d}v/\mathrm{d}x = x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}, \qquad v = \int x\mathrm{d}x = \frac{1}{2}x^2.$$

Hence

$$\int x\ln x\mathrm{d}x = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{1}{x}\mathrm{d}x = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\mathrm{d}x = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

(j) Let $u = \ln x$ and $\mathrm{d}v/\mathrm{d} = x^n$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}, \qquad v = \int x^n\mathrm{d}x = \frac{1}{n+1}x^{n+1}.$$

Hence

$$\begin{aligned}x^n\ln x\mathrm{d}x &= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}\mathrm{d}x + C \\ &= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C.\end{aligned}$$

(k) Let $u = \ln x$ and $\mathrm{d}v/\mathrm{d}x = 1/x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}, \qquad v = \int \frac{\mathrm{d}x}{x} = \ln x.$$

Hence

$$\int \frac{\ln x}{x}\mathrm{d}x = (\ln x)^2 - \int \frac{\ln x}{x}\mathrm{d}x + 2C.$$

or

$$\int \frac{\ln x}{x}\mathrm{d}x = \frac{1}{2}(\ln x)^2 + C.$$

17.11. (a) Let $u = (\ln x)^2$ and $\mathrm{d}v/\mathrm{d}x = 1$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\ln x}{x}, \qquad v = x.$$

Hence

$$\begin{aligned}\int (\ln x)^2\mathrm{d}x &= x(\ln x)^2 - 2\int \frac{\ln x}{x}x\mathrm{d}x + C \\ &= x(\ln x)^2 - 2x\ln x + 2\int 1\mathrm{d}x + C \\ &= x(\ln x)^2 - 2x\ln x + 2x + C,\end{aligned}$$

where a further integration by parts been used for $\int \ln x \mathrm{d}x$.
(b) Let $u = \arcsin x$ and $\mathrm{d}v/\mathrm{d}x = 1$. Then

$$\begin{aligned}\int \arcsin x \mathrm{d}x &= x \arcsin x - \int x \frac{\mathrm{d}}{\mathrm{d}x}[\arcsin x]\mathrm{d}x + C \\ &= x \arcsin x - \int \frac{x\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}}\mathrm{d}x \\ &= x \arcsin x + (1-x^2)^{\frac{1}{2}} + C\end{aligned}$$

(c) Let $u = \arccos x$ and $\mathrm{d}v/\mathrm{d}x = 1$. Then

$$\begin{aligned}\int \arccos x \mathrm{d}x &= x \arccos x - \int x \frac{\mathrm{d}}{\mathrm{d}x}[\arccos x]\mathrm{d}x \\ &= x \arccos x + \int \frac{x}{(1-x^2)^{\frac{1}{2}}}\mathrm{d}x \\ &= x \arccos x - (1-x^2)^{\frac{1}{2}}.\end{aligned}$$

(d) Let $u = \arctan x$ and $\mathrm{d}v/\mathrm{d}x = 1$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2}, \qquad v = x.$$

Hence

$$\int \arctan x \mathrm{d}x = x \arctan x - \int \frac{x\mathrm{d}x}{(1+x^2)} + C = x \arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

17.12. (a) Let $u = \sin x$ and $\mathrm{d}v/\mathrm{d}x = \mathrm{e}^x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x, \qquad v = \mathrm{e}^x.$$

$$I = \int \mathrm{e}^x \sin x \mathrm{d}x = \mathrm{e}^x \sin x - \int \mathrm{e}^x \cos x \mathrm{d}x + 2C.$$

In the integral on the right, now put $w = \cos x$ and $\mathrm{d}z/\mathrm{d}x = \mathrm{e}^x$. Then by integrating by parts for a second time

$$I = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - \int \mathrm{e}^x \sin x \mathrm{d}x + 2C = \mathrm{e}^x(\sin x - \cos x) - I + 2C.$$

Therefore

$$I = \frac{1}{2}\mathrm{e}^x(\sin x - \cos x) + C.$$

(b) Let $u = \sin x$ and $\mathrm{d}v/\mathrm{d}x = \mathrm{e}^{-x}$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x, \qquad v = -\mathrm{e}^x.$$

$$I = \int \mathrm{e}^{-x} \sin x \mathrm{d}x = -\mathrm{e}^{-x} \sin x + \int \mathrm{e}^{-x} \cos x \mathrm{d}x + 2C.$$

In the integral on the right, now put $w = \cos x$ and $\mathrm{d}z/\mathrm{d}x = \mathrm{e}^{-x}$. Then by integrating by parts for a second time

$$I = -\mathrm{e}^{-x} \sin x - \mathrm{e}^{-x} \cos x - \int \mathrm{e}^{-x} \sin x \mathrm{d}x + 2C = -\mathrm{e}^{-x}(\sin x - \cos x) - I + 2C.$$

Therefore

$$I = -\frac{1}{2}\mathrm{e}^{-x}(\sin x + \cos x) + C.$$

(c) These integrals can be integrated by parts using the alternative choices for u and v. Let $u = \mathrm{e}^{-x}$ and $\mathrm{d}v/\mathrm{d}x = \cos x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\mathrm{e}^{-x}, \qquad v = \sin x.$$

Hence

$$I = \int \mathrm{e}^{-x} \cos x \mathrm{d}x = \mathrm{e}^{-x} \sin x + \int \mathrm{e}^{-x} \sin x \mathrm{d}x + 2C.$$

In the integral on the right, put $w = \mathrm{e}^{-x}$ and $\mathrm{d}z/\mathrm{d}x = \sin x$. Then by integration by parts for a second time

$$I = \mathrm{e}^{-x} \sin x - \mathrm{e}^{-x} \cos x - \int \mathrm{e}^{-x} \cos x \mathrm{d}x = \mathrm{e}^{-x} \sin x - \mathrm{e}^{-x} \cos x - I + 2C.$$

Therefore

$$I = \frac{1}{2}\mathrm{e}^{-x}(\sin x - \cos x) + C.$$

17.13. (a) The integration by parts formula for definite integrals is

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = [uv]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x.$$

(a) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \cos x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1, \qquad v = \sin x.$$

Hence

$$\int_0^{\frac{1}{2}\pi} x \cos x \mathrm{d}x = [x \sin x]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \sin x \mathrm{d}x = \frac{1}{2}\pi + [\cos x]_0^{\frac{1}{2}\pi} = \frac{1}{2}\pi - 1.$$

(b) Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \cos 2x$. Then

$$\int_0^{\pi} x \cos 2x = \frac{1}{2}[x \sin 2x]_0^{\pi} - \frac{1}{2}\int_0^{\pi} \sin 2x \mathrm{d}x = \frac{1}{4}[\cos 2x]_0^{\pi} = 0.$$

(c) In the following there are two successive integrations by parts, leading to:

$$\begin{aligned}\int_0^{\pi} x^2 \cos x \mathrm{d}x &= [x^2 \sin x]_0^{\pi} - \int_0^{\pi} 2x \sin x \mathrm{d}x \\ &= 0 + [2x \cos x]_0^{\pi} - \int_0^{\pi} 2 \cos x \mathrm{d}x \\ &= -2\pi - [2 \sin x]_0^{\pi} = -2\pi.\end{aligned}$$

(d) From Problem 17.12b,

$$\int_0^{\infty} \mathrm{e}^{-x} \sin x \mathrm{d}x = -\frac{1}{2}[\mathrm{e}^{-x}(\cos x + \sin x)]_0^{\infty} = \frac{1}{2}.$$

(e) From Problem 17.12c,

$$\int_0^{\infty} \mathrm{e}^{-x} \cos x \mathrm{d}x = \frac{1}{2}[\mathrm{e}^{-x}(\sin x - \cos x)]_0^{\infty} = \frac{1}{2}.$$

(f) Let $u = \ln x$ and $\mathrm{d}v/\mathrm{d}x = 1/x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}, \qquad v = \ln x$$

. Therefore

$$\int_1^2 \frac{\ln x}{x}\mathrm{d}x = [(\ln x)^2]_1^2 - \int_1^2 \frac{\ln x}{x}\mathrm{d}x = (\ln 2)^2 - \int_1^2 \frac{\ln x}{x}\mathrm{d}x.$$

Hence

$$\int_1^2 \frac{\ln x}{x}\mathrm{d}x = \frac{1}{2}(\ln 2)^2.$$

(g) Let $u = \arcsin x$ and $\mathrm{d}v/\mathrm{d}x = 1$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{(1-x^2)^{\frac{1}{2}}}, \text{ (see Appendix D) } v = x.$$

Hence

$$\begin{aligned}\int_0^1 \arcsin x\mathrm{d}x &= [x \arcsin x]_0^1 - \int_0^1 \frac{x}{(1-x^2)^{\frac{1}{2}}}\mathrm{d}x \\ &= \frac{1}{2}\pi + [(1-x^2)^{\frac{1}{2}}]_0^1 = \frac{1}{2}\pi - 1.\end{aligned}$$

(h) Let $u = \arccos x$ and $\mathrm{d}v/\mathrm{d}x = 1$. By a method similar to that given in (g),

$$\int_{-1}^1 \arccos x\mathrm{d}x = \pi.$$

(i) Let $u = \arctan x$ and $\mathrm{d}v/\mathrm{d}x = 1$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2}, \text{ (see Appendix D) } v = x.$$

Hence

$$\begin{aligned}\int_0^1 \arctan x\mathrm{d}x &= [x \arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{4}\pi - \frac{1}{2}[\ln(1+x^2)]_0^1 \\ &= \frac{1}{4}\pi - \frac{1}{2}\ln 2.\end{aligned}$$

(j)
$$\int_1^2 \ln x\mathrm{d}x = [x \ln x]_1^2 - \int_1^2 x\frac{1}{x}\mathrm{d}x = 2\ln 2 - [x]_1^2 = 2\ln 2 - 1.$$

17.14. Let $u = x^k$ and $\mathrm{d}v/\mathrm{d}x = \mathrm{e}^x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = kx^{k-1}, \qquad v = \mathrm{e}^x.$$

Therefore, for $k \geq 1$

$$F(k) = \int_0^1 x^k \mathrm{e}^x \mathrm{d}x = [x^k \mathrm{e}^x]_0^1 - \int_0^1 kx^{k-1}\mathrm{e}^x\mathrm{d}x = \mathrm{e} - kF(k-1).$$

Repeating the formula

$$\begin{aligned}F(4) &= \mathrm{e} - 4F(3) = \mathrm{e} - 4[\mathrm{e} - 3F(2)] = -3\mathrm{e} + 12F(2) \\ &= -3\mathrm{e} + 12[\mathrm{e} - 2F(1)] \\ &= 9\mathrm{e} - 24[\mathrm{e} - F(0)] = -15\mathrm{e} + 24F(0),\end{aligned}$$

where

$$F(0) = \int_0^1 \mathrm{e}^x\mathrm{d}x = [\mathrm{e}^x]_0^1 = \mathrm{e} - 1.$$

Therefore

$$F(4) = -15\mathrm{e} + 24(\mathrm{e} - 1) = 9\mathrm{e} - 24.$$

17.15. Let $u = \cos^{k-1} x$ and $\mathrm{d}v/\mathrm{d}x = \cos x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -(k-1)\cos^{k-2} x \sin x, \qquad v = \sin x.$$

Therefore, for $k \geq 2$, we may integrate by parts:

$$\begin{aligned} F(k) &= \int_0^{\frac{1}{2}\pi} \cos^k x \mathrm{d}x = [\cos^{k-1} x \sin x]_0^{\frac{1}{2}\pi} - (k-1)\int_0^{\frac{1}{2}\pi} \cos^{k-2} x \sin^2 x \mathrm{d}x \\ &= 0 - (k-1)\int_0^{\frac{1}{2}\pi} \cos^{k-2} x (1 - \cos^2 x)\mathrm{d}x \\ &= -(k-1)F(k-2) - (k-1)F(k). \end{aligned}$$

Hence

$$F(k) = \frac{k-1}{k} F(k-2)$$

as required. The first two integrals in the sequence are

$$F(0) = \int_0^{\frac{1}{2}\pi} \mathrm{d}x = \frac{1}{2}\pi, \quad F(1) = \int_0^{\frac{1}{2}\pi} \cos x \mathrm{d}x = [\sin x]_0^{\frac{1}{2}\pi} = 1.$$

Using the reduction formula

$$F(2) = \frac{1}{2}F(0) = \frac{1}{4}\pi, \quad F(3) = \frac{2}{3}F(1) = \frac{2}{3},$$

$$F(4) = \frac{3}{4}F(2) = \frac{3}{16}\pi, \quad F(5) = \frac{4}{5}F(3) = \frac{8}{15}.$$

17.16. (a) Integrating by parts

$$\begin{aligned} F(k) = \int_1^2 (\ln x)^k \mathrm{d}x &= [x(\ln x)^k]_1^2 - \int_1^2 xk\frac{(\ln x)^{k-1}}{x}\mathrm{d}x \\ &= 2(\ln 2)^k - kF(k-1). \end{aligned}$$

Therefore

$$\int_1^2 (\ln x)^3 \mathrm{d}x = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6.$$

(b) Let $u = x^k$ and $\mathrm{d}v/\mathrm{d}x = \sin x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = kx^{k-1}, \qquad v = -\cos x.$$

Therefore,

$$\begin{aligned} F(k) &= \int_0^{\pi} x^k \sin x \mathrm{d}x = -[x^k \cos x]_0^{\pi} + k\int_0^{\pi} x^{k-1}\cos x \mathrm{d}x \\ &= 2\pi^k + k[x^{k-1}\sin x]_0^{\pi} - k(k-1)\int_0^{\pi} x^{k-2}\sin x \mathrm{d}x \\ &= 2\pi^k - k(k-1)F(k-2) \end{aligned}$$

Special cases are

$$F(2) = 2\pi^2 - F(0) = 2\pi^2 - \int_0^{\pi} \sin x \mathrm{d}x = 2\pi^2 - 4, \quad F(3) = \pi(\pi^2 - 6)$$

$$F(4) = \pi^4 - 12\pi^2 + 48, \quad F(5) = \pi(\pi^4 - 20\pi^2 + 120.$$

(c) Let $u = \sin^{k-1} x$ and $\mathrm{d}v/\mathrm{d}x = \sin x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (k-1)\sin^{k-2} x \cos x, \qquad v = -\cos x.$$

Therefore, for $k \geq 2$,

$$\begin{aligned}
F(k) &= \int_0^{\frac{1}{2}\pi} \sin^k x \mathrm{d}x \\
&= -[\sin^{k-1} x \cos x]_0^{\frac{1}{2}\pi} + \int_0^{\frac{1}{2}\pi} (k-1)\sin^{k-2} x \cos x \cos x \mathrm{d}x \\
&= (k-1)\int_0^{\frac{1}{2}\pi} \sin^{k-2} x(1-\sin^2 x)\mathrm{d}x \\
&= (k-1)F(k-2) - (k-1)F(k).
\end{aligned}$$

Finally

$$F(k) = \frac{(k-1)}{k}F(k-2).$$

The first few integrals in the sequence are

$$F(2) = \frac{1}{4}\pi, \quad F(3) = \frac{2}{3}, \quad F(4) = \frac{3}{16}\pi, \quad F(5) = \frac{8}{15}.$$

17.17. (a) Let $c = a^{-1}$ and use the change of variable $x = u^{-1}$. Then

$$F(a^{-1}) = \int_1^{a^{-1}} \frac{\mathrm{d}x}{x} = \int_1^a u\frac{(-\mathrm{d}u)}{u^2} = -\int_1^a \frac{\mathrm{d}u}{u} = -F(a).$$

(b) Use the change of variable $x = au$:

$$\begin{aligned}
F(ab) &= \int_1^{ab} \frac{\mathrm{d}x}{x} = \int_{1/a}^b \frac{\mathrm{d}u}{u} \\
&= \int_1^b \frac{\mathrm{d}u}{u} - \int_1^{1/a} \frac{\mathrm{d}u}{u} = F(b) + F(a).
\end{aligned}$$

using (a).

(c) Let $x = u/b$. Then

$$\begin{aligned}
F(a/b) &= F(a) + F(1/b) \quad \text{(by (b))} \\
&= F(a) - F(b) \quad \text{(by (a))}
\end{aligned}$$

(d) Let $x = u^n$:

$$F(a^n) = \int_1^{a^n} \frac{\mathrm{d}x}{x} = \int_1^a \frac{nu^{n-1}\mathrm{d}u}{u^n} = nF(a).$$

17.18. Let $u = x$ and $\mathrm{d}v/\mathrm{d}x = \mathrm{e}^x$. Then $\mathrm{d}u/\mathrm{d}x = 1$ and choose $v = \mathrm{e}^x + A$. Integrate by parts, choosing $v = \mathrm{e}^x + A$:

$$\int x\mathrm{e}^x \mathrm{d}x = x(\mathrm{e}^x + A) - \int (\mathrm{e}^x + A)\mathrm{d}x + C,$$

where C is arbitrary. Also

$$\int (\mathrm{e}^x + A)\mathrm{d}x = \int \mathrm{e}^x \mathrm{d}x + Ax + B,$$

where B is arbitrary. Therefore

$$\int x\mathrm{e}^x \mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + D,$$

where $D = B + C$ is arbitrary, and this matches Example (17.7).

The formula for integration by parts given in (17.7) is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + C.$$

Suppose that we now replace v by the indefinite integral $v + A$ where A is any constant. Then (17.7) is replaced by

$$\begin{aligned}\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x &= u(v+A) - \int (v+A)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + C\\ &= uv + uA - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x - uA - B + C\\ &= uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + D.\end{aligned}$$

Here, $D = B + C$, where B is the arbitrary constant of integration arising in

$$\int A\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = Au + B.$$

So D is arbitrary, and the formula is equivalent to eqn (17.7).

17.19. Any indefinite integral can only be found to within an arbitrary constant. Hence an indefinite integral evaluated by two different methods can lead to answers which apparently differ by a constant.

17.20. (a) *Circular disc, mass m, radius a about a diameter.* Take an origin at the centre of the disc with x axis along the diameter. Take an increment of width δy parallel to the diameter. The density per unit area of the disc, assumed to be uniform, is $m/\pi a^2$. Hence the moment of inertia of the increment about the x axis is

$$\frac{m}{\pi a^2}[(2\surd(a^2 - y^2)\delta y]y^2.$$

The total moment of inertia I is therefore the sum of these increments

$$I = \frac{2m}{\pi a^2}\int_{-a}^{a} y^2\surd(a^2 - y^2)\mathrm{d}y.$$

Use the substitution $y = a\sin t$. Then

$$\begin{aligned}I &= \frac{2m}{\pi a^2}a^4\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin^2 t\cos^2 \mathrm{d}t = \frac{2ma^2}{4\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin^2 2t\mathrm{d}t\\ &= \frac{ma^2}{4\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(1-\cos 4t)\mathrm{d}t\\ &= \frac{1}{4}ma^2.\end{aligned}$$

(b) *Uniform sphere, mass m, radius a, about a diameter.* Let the centre of the sphere be the origin with the x axis along the diameter. Consider a circular slice of thickness δx distance x from the origin and cut perpendicular to the diameter. The density of the sphere is $\rho = 3m/(4\pi a^3)$. The circular disc has radius $\surd(a^2 - x^2)$ and mass approximately $\rho\pi(a^2 - x^2)\delta x$. Its moment of inertia about the diameter of the sphere is (see Example 16.10) is $\frac{1}{2}\rho\pi(a^2-x^2)^4\delta x$. Hence the moment of inertia of the whole sphere is

$$\begin{aligned} I &= \frac{1}{2}\int_{-a}^{a}\rho\pi(a^2-x^2)^2)\mathrm{d}x = \frac{1}{2}\rho\pi\left[2a^5-\frac{4}{3}a^5+\frac{2}{3}a^5\right] \\ &= \frac{8}{15}\rho\pi a^5 = \frac{8}{15}\pi a^5.\frac{3m}{4\pi a^3} \\ &= \frac{2}{5}ma^2 \end{aligned}$$

(c) *Spherical shell, mass m, radius a about a diamater.* Let the origin be at the centre of the shell with x axis along the diameter. Take a thin section of the shell perpendicular to the diameter distance x from the origin. This problem involves area rather than volume: the density of the shell is $\rho = m/(4\pi a^2)$. Imagine the shell as surface generated by rotating a circle about the x axis. The thickness of the shell is δx along the diameter, but its mass is

$$(\text{density})\times(\text{surface area}) = (\text{density})\times(\text{circumference})\times(\delta s) = 2\pi\surd(a^2-x^2)\delta s,$$

where δs is an increment of arc-length of the circle. Since all points on the section are equidistant from the axis, the total moment of inertia is

$$\begin{aligned} I &= \int_{-a}^{a} 2\pi\rho\surd(a^2-x^2)(a^2-x^2)\frac{a}{\surd(a^2-x^2)}\mathrm{d}x \\ &= 2\pi\rho a\int_{-a}^{a}(a^2-x^2)\mathrm{d}x = \frac{8}{3}\pi\rho a^4 \\ &= \frac{2}{3}ma^2 \end{aligned}$$

(d) *Rectangle, mass m, side-lengths 2a and 2b about a diagonal.* Let the diagonal be the x axis, and the line through the centre perpendicular to the diagonal be the y axis. Take a strip of width δy parallel to a diagonal as shown in the figure. Let d be the distance of one of the opposite corners from the diagonal.

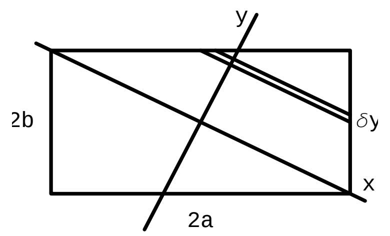

Figure 12: Problem 17.20d

The moment of inertia of the rectangle will be twice that of one of the triangles touching the diagonal. If y is the distance of the strip from the diagonal then, by similar triangles, the length s of the strip is given by

$$\frac{d-y}{d} = \frac{s}{2\surd(a^2+b^2)}, \quad \text{or } s = \frac{2}{d}(d-y)\surd(a^2+b^2).$$

Hence

$$\begin{aligned} I &= 4\frac{m}{2ab}\int_0^d \frac{1}{d}(d-y)\surd(a^2+b^2)y^2\mathrm{d}y = \frac{2m}{abd}\surd(a^2+b^2)\frac{1}{12}d^4 \\ &= \frac{md^3}{6ab}\surd(a^2+b^2). \end{aligned}$$

From the figure $d = 2ab/\sqrt{(a^2+b^2)}$. Hence

$$I = \frac{4}{3}\frac{a^2b^2}{a^2+b^2}.$$

(e) *Cone, mass m, base radius a, height h about its axis.* Let vertex of the cone be the origin and let the axis of the cone be the x axis. Consider a circular section of the cone of thickness δx cut perpendicular to the axis at a distance x from the vertex. Its radius is $x\tan\alpha$ where $\tan\alpha = a/h$. The moment of inertia of the disc about the axis is $\frac{1}{2}\rho x^4 \tan^4\alpha\delta x$, where the density $\rho = 3m/(\pi a^2 h)$. Hence the total moment of inertia is

$$I = \frac{1}{2}\int_0^h \rho(\tan^4\alpha)x^4 \mathrm{d}x = \frac{1}{2}\rho(\tan^4\alpha)\frac{h^5}{5} = \frac{3}{10}ma^2.$$

17.21. Let

$$F(t) = \int \mathrm{e}^{-at}\cos bt\mathrm{d}t = A\mathrm{e}^{-at}\cos bt + b\mathrm{e}^{-at}\sin bt + C.$$

Then differentiating both sides, we have

$$\mathrm{e}^{-at}\cos bt = (-aA + bB)\mathrm{e}^{-at}\cos bt + (-Ab - aB)\sin bt.$$

Hence equating like terms on both sides

$$1 = -aA + bB, \qquad 0 = -bA - aB.$$

Solving these equations

$$A = \frac{-a}{a^2+b^2}, \qquad B = \frac{b}{a^2+b^2}.$$

which agrees with eqn (15.11).

17.22. Let

$$I(\alpha) = \int x^2\mathrm{e}^{-\alpha x}\mathrm{d}x + C = \int \frac{d^2}{\mathrm{d}\alpha^2}\mathrm{e}^{-\alpha x}\mathrm{d}x + C,$$

with C arbitrary. Interchange integration and differentiation:

$$\begin{aligned} I(\alpha) &= \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left[\int \mathrm{e}^{-\alpha x}\mathrm{d}x\right] + C = \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left[-\frac{1}{\alpha}\mathrm{e}^{-\alpha x}\right] + C \\ &= \mathrm{e}^{-\alpha x}\left[-\frac{2}{\alpha^3} - \frac{2x}{\alpha^2} - \frac{x^2}{a}\right] + C. \end{aligned}$$

17.23. Let $x = a/u$. Then $\mathrm{d}x/du = -a/u^2$, and

$$\int \frac{\mathrm{d}x}{x\sqrt{(x^2-a^2)}} = -\frac{1}{a}\int \frac{\mathrm{d}u}{\sqrt{(1-u^2)}} = -\frac{1}{a}\arcsin u + C = -\frac{1}{a}\arcsin(a/x) + C$$

using a standard integral from Appendix E.

(b) Let $x = a/u$. Then, as in (a),

$$\begin{aligned} \int \frac{\mathrm{d}x}{x\sqrt{(a^2-x^2)}} &= -\frac{1}{a}\int \frac{\mathrm{d}u}{\sqrt{(u^2-1)}} \\ &= -\frac{1}{a}\ln[u + \sqrt{(u^2-1)}] + C \\ &= -\frac{1}{a}\ln\left[\frac{a+\sqrt{(a^2-x^2)}}{x}\right] + C. \end{aligned}$$

(c) Let $u = (a\tan x)/b$. Then $\mathrm{d}u/\mathrm{d}x = (a\sec^x)/b$. Using this substitution

$$\begin{aligned}\int \frac{\mathrm{d}x}{a^2\cos^2 x + b^2\sin^2 x} &= \frac{1}{ab}\int \frac{\mathrm{d}u}{u^2+1} \\ &= \int \frac{1}{ab}\arctan u + C = \frac{1}{ab}\arctan\left[\frac{a}{b}\tan x\right] + C.\end{aligned}$$

(d) Using the substitution $u = \tan\frac{1}{2}x$,

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{2\cos^2\frac{1}{2}x\mathrm{d}u}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \int \frac{\mathrm{d}u}{u} = \ln|u| + C = \ln|\tan\frac{1}{2}x| + C.$$

(e) Using the substitution $u = \tan\frac{1}{2}x$,

$$\begin{aligned}\int \frac{\mathrm{d}x}{3+5\cos x} &= \int \frac{2\cos^2\frac{1}{2}x\mathrm{d}u}{3+5(2\cos^2\frac{1}{2}x - 1)} \\ &= \int \frac{\mathrm{d}u}{4-u^2} = \frac{1}{4}\int\left[\frac{1}{2-u} + \frac{1}{2+u}\right]\mathrm{d}u \\ &= \frac{1}{4}\ln\left[\frac{2+u}{2-u}\right] + C = \frac{1}{4}\ln\left[\frac{2+\tan\frac{1}{2}x}{2-\tan\frac{1}{2}x}\right] + C\end{aligned}$$

(f) Using the substitution $u = \tanh\frac{1}{2}x$,

$$\begin{aligned}\int \frac{\mathrm{d}x}{5\cosh x + 4\sinh x} &= \int \frac{2\mathrm{d}u}{10 - 5\mathrm{sech}^2\frac{1}{2}x + 8\tanh\frac{1}{2}x} \\ &= \int \frac{\mathrm{d}u}{5u^2+8u+5} = \frac{2}{3}\arctan[\frac{1}{3}(4+5u)] + C \\ &= \frac{2}{3}\arctan[\frac{1}{3}(4+5\tanh\frac{1}{2}x)] + C\end{aligned}$$

(g) Express the integral in the form

$$I = \int \sec x\mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\mathrm{d}x.$$

Now use the substitution

$$u = \sec x + \tan x, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec x\tan x + \sec^2 x,$$

so that

$$I = \int \frac{\mathrm{d}u}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

(h) Using the substitution $x = u^2$,

$$\begin{aligned}\int_0^4 \frac{\mathrm{d}x}{1+\sqrt{x}} &= \int_0^2 \frac{2u\mathrm{d}u}{1+u} = \int_0^2\left[2 - \frac{2}{1+u}\right]\mathrm{d}u \\ &= [2u - 2\ln(1+u)]_0^2 = 4 - 2\ln 3\end{aligned}$$

(i) Using the substitution $x = u^3 - 1$,

$$\begin{aligned}\int x(1+x)^{\frac{1}{3}}\mathrm{d}x &= \int 3u^3(u^3-1)\mathrm{d}u = \frac{3}{7}u^7 - \frac{3}{4}u^4 + C \\ &= \frac{1}{28}(1+x)^{\frac{4}{3}}(12x-9) + C\end{aligned}$$

(j) Let $u = x - 1/x$ so that $\mathrm{d}u/\mathrm{d}x = 1 + 1/x^2$. Then

$$\begin{aligned}
\int_1^2 \frac{(x^2+1)\mathrm{d}x}{x\sqrt{[x^4+7x^2+1]}} &= \int_0^{\frac{3}{2}} \frac{\mathrm{d}u}{\sqrt{[x^2+7+x^{-2}]}} \\
&= \int_0^{\frac{3}{2}} \frac{\mathrm{d}u}{\sqrt{[u^2+9]}} = [\ln[u+\sqrt{(u^2+9)}]]_0^{\frac{3}{2}} \\
&= \ln[(1+\sqrt{5})/2]
\end{aligned}$$

(k) Let $u = 1 + x^{\frac{1}{2}}$, so that $\mathrm{d}u/\mathrm{d}x = \frac{1}{2}x^{-\frac{1}{2}}$. Then

$$\begin{aligned}
\int_0^4 \sqrt{(1+\sqrt{x})}\mathrm{d}x &= 2\int_1^3 [u^{\frac{1}{2}} - u^{\frac{3}{2}}]\mathrm{d}u = 2\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_1^3 \\
&= \frac{8}{15} - \frac{16}{5}\sqrt{3}.x
\end{aligned}$$

17.24. Use repeated integration by parts starting with

$$u_1 = p(x), \quad \frac{\mathrm{d}u}{\mathrm{d}x} = p'(x), \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x, \quad v = \mathrm{e}^x.$$

Therefore

$$\int \mathrm{e}^x p(x)\mathrm{d}x = \mathrm{e}^x p(x) - \int \mathrm{e}^x p'(x)\mathrm{d}x + C.$$

Repeat the integration by parts with $u_2 = p'(x)$, $v = \mathrm{e}^x$ so that

$$\int \mathrm{e}^x p(x)\mathrm{d}x = \mathrm{e}^x p(x) - \mathrm{e}^x p'(x) + \int \mathrm{e}^x p''(x)\mathrm{d}x + C.$$

Continue n times until $p^{(n)}(x)$ is reached. Since $p(x)$ is a polynomial of degree n, $p^{(n)}(x)$ must be a constant which means that

$$\int \mathrm{e}^x p^{(n)}(x)\mathrm{d}x = \mathrm{e}^x p^{(n)}(x),$$

apart from a constant. Finally

$$\int \mathrm{e}^x p(x)\mathrm{d}x = \mathrm{e}^x[p(x) - p'(x) + p''(x) - \cdots + (-1)^n p^{(n)}(x)] + C.$$

In the Problem

$$\begin{aligned}
&\int_0^1 \mathrm{e}^x(x^3 - 2x^2 + x - 2)\mathrm{d}x = \\
&\quad [\mathrm{e}^x\{(x^3 - 2x^2 + x - 2) - (3x^2 - 4x + 1) + (6x - 4) - 6\}]_0^1 \\
&= \quad 13 - 6\mathrm{e}
\end{aligned}$$

For the next case, the procedure is as in the first part except the $v = -\mathrm{e}^{-x}$: the signs now alternate in the opposite way. Hence

$$\int \mathrm{e}^{-x} p(x)\mathrm{d}x = \mathrm{e}^x[-p(x) + p'(x) - p''(x) + \cdots + (-1)^{n+1} p^{(n)}(x)] + C.$$

The definite integral takes the value

$$\int_0^1 \mathrm{e}^x p(x)\mathrm{d}x = \mathrm{e}[-p(1) + p'(1) - p''(1) + \cdots + (-1)^n p^{(n)}(1)] - \\ [-p(0) + p'(0) - \cdots + (-1)^{n+1} p^{(n)}(0)].$$

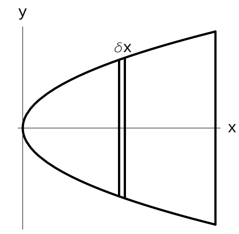

Figure 13: Problem 17.25

From the previous formula

$$\int_0^\infty e^x p(x)dx = p(0) - p'(0) + p''(0) - \cdots + (-1)^n p^{(n)}(0).$$

17.25. The plate is shown in the figure.

By symmetry the centroid must lie on the x axis. Take a strip of width δx parallel to the y axis. We first require the area A of the plate:

$$A = \int_0^h 2y dx = 2\int_0^h 2a^{\frac{1}{2}}x^{\frac{1}{2}}dx = 4a^{\frac{1}{2}}\frac{2}{3}[x^{\frac{3}{2}}]_0^h = \frac{8}{3}a^{\frac{1}{2}}h^{\frac{3}{2}}.$$

The distance $\overline{x}$ of the centroid from the origin is given by

$$A\overline{x} = 2\int_0^h y^2 dx = 8a\int_0^h x dx = 4ah^2.$$

Therefore

$$\overline{x} = \frac{4ah^2}{\frac{8}{3}a^{\frac{1}{2}}h^{\frac{3}{2}}} = \frac{3}{2}(ah)^{\frac{1}{2}}.$$

Chapter 18: Unforced linear differential equations with constant coefficients

18.1. The following equations are linear, unforced with constant coefficients: (b), (e), (f), (i), (j), (k).

18.2. All the solutions are given by (18.4).
(a) General solution of $x' + 5x = 0$ is $x(t) = Ae^{-5t}$.
(b) General solution of $x' - \frac{1}{2}x = 0$ is $x(t) = Ae^{\frac{1}{2}t}$.
(c) General solution of $x' - x = 0$ is $x(t) = Ae^{t}$.
(d) General solution of $x' + 3x = 0$ is $x(t) = Ae^{-3t}$.
(e) General solution of $3x' + 4x = 0$ is $x(t) = Ae^{-4t/3}$.
(f) General solution of $x' = 2x$ is $x(t) = Ae^{2t}$.
(g) General solution of $x' = 3x$ is $x(t) = Ae^{3t}$.
(h) The equation $(x'/x) = -3$ can be rewritten as $x' + 3x = 0$, which has the general solution $x = e^{-3t}$.
(i) The equation $(x' + 1)/(x + 1) = 1$ is the same as $x' - x = 0$, which has the general solution $x(t) = Ae^{t}$.

18.3. The general solution of these first-order equations is given by (18.4).
(a) The general solution of $x' + 2x = 0$ is $x(t) = Ce^{-2t}$. The initial condition is $x(0) = 3$. Hence $C = 3$ and the required solution is $x(t) = 3e^{-2t}$.
(b) The solution of $3x' - x = 0$ subject to $x(1) = 1$ is $x(t) = e^{(t-1)/3}$.
(c) The solution of $y' - 2y = 0$ with the condition $y(-3) = 2$ is $y(x) = 2e^{2t+6}$.

(d) The solution of $x' + x = 0$ with the condition $x(-1) = 10$ is $x(t) = 10\mathrm{e}^{-t-1}$.
(e) The solution of $2y' - 3y = 0$ subject to $y(0) = 1$ is $x(t) = \mathrm{e}^{3t/2}$.
(f) Since the slope is $5y$, it follows that

$$\frac{dy}{\mathrm{d}x} = 5y.$$

The general solution is $y = C\mathrm{e}^{5x}$. Hence, since the curve passes through $(1, -2)$, that is, $y(1) = -2$, the curve is given by $y = -2\mathrm{e}^{5t-5}$.

18.4. The equation for the current $x(t)$ in the circuit in Fig. 18.1 is

$$L\frac{\mathrm{d}x}{\mathrm{d}t} + Rx = E(t).$$

Assume that the applied voltage becomes zero at $t = 0$. Then we have to solve

$$L\frac{\mathrm{d}x}{\mathrm{d}t} + Rx = 0, \quad x(0) = I_0.$$

The general solution is $x = A\mathrm{e}^{-Rt/L}$. The initial condition gives $A = I_0$. The required solution is $x = I_0\mathrm{e}^{-Rt/L}$.

Let the current halve in time T. Then $\frac{1}{2}I_0 = I_0\mathrm{e}^{-RT/L}$ so that

$$T = \frac{L}{R}\ln 2,$$

which is independent of I_0.

18.5. Given that

$$\frac{\mathrm{d}A}{\mathrm{d}t} \propto A, \text{ or } \frac{\mathrm{d}A}{\mathrm{d}t} = -kA.$$

(a) The solution is $A = A_0\mathrm{e}^{-kt}$.
(b) The half-life $T = (\ln 2)/k$. We are given that 82.5% of uranium-232 remain after 20 years. Hence with t measured in years, the constant k is given by

$$\frac{82.5}{100}A_0 = A_0\mathrm{e}^{-20k}.$$

Solving this equation $k = 9.6 \times 10^{-3}$ (years)$^{-1}$. Finally, the half-life of uranium-232 is $T = (\ln 2)/k = 72$ years.

18.6. Let $N(t)$ be number of rabbits at time t with t measured in years. If the number of rabbits increases by δN in time δt, then

$$\delta N = 20(\frac{1}{2}N)\delta t = 10N\delta t.$$

Let $\delta t \to 0$, so that the differential equation for $N(t)$ is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 10N \text{ with initial value } N(0) = 100$$

The solution of this initial value problem is

$$N(t) = N(0)\mathrm{e}^{10t} = 100\mathrm{e}^{10t}.$$

At time $t = 4$, $N(4) = 100\mathrm{e}^{40} = 2.35 \times 10^{19}$.

If the rabbits only live for a year, then $N(t)$ satisfies

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 9N.$$

Hence $N(t) = 100\mathrm{e}^{36} = 4.31 \times 10^{17}$ which is still a very large number.

18.7. (a) For the differential equation

$$x'' - 3x' + 2x = 0,$$

its characteristic equation is

$$m^2 - 3m + 2 = 0, \text{ which has solutions } m_1 = 1,\ m_2 = 2.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{t} + B\mathrm{e}^{2t}.$$

(b) For the differential equation

$$x'' + x' - 2x = 0,$$

its characteristic equation is

$$m^2 + m - 2 = 0, \text{ which has solutions } m_1 = -2,\ m_2 = 1.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{-2t} + B\mathrm{e}^{t}.$$

(c) For the differential equation

$$x'' - x = 0,$$

its characteristic equation is

$$m^2 - 1 = 0, \text{ which has solutions } m_1 = -1,\ m_2 = 1.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{-t} + B\mathrm{e}^{t}.$$

(d) For the differential equation

$$x'' - 4x = 0,$$

its characteristic equation is

$$m^2 - 4 = 0, \text{ which has solutions } m_1 = -2,\ m_2 = 2.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{-2t} + B\mathrm{e}^{2t}.$$

(e) For the differential equation

$$3x'' - \frac{1}{4}x = 0,$$

its characteristic equation is

$$3m^2 - \frac{1}{4} = 0, \text{ which has solutions } m_1 = -1/(2\sqrt{3}),\ m_2 = 1/(2\sqrt{3}).$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{-t/(2\sqrt{3})} + B\mathrm{e}^{t/(2\sqrt{3})}.$$

(f) For the differential equation

$$x'' - 9x = 0,$$

its characteristic equation is

$$m^2 - 9 = 0, \text{ which has solutions } m_1 = -3,\ m_2 = 3.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{-3t} + B\mathrm{e}^{3t}.$$

(g) For the differential equation

$$x'' + 2x' - x = 0,$$

its characteristic equation is

$$m^2 + 2m - 1 = 0, \text{ which has solutions } m_1 = -1 - \sqrt{2},\ m_2 = -1 + \sqrt{2}.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{(-1-\sqrt{2})t} + B\mathrm{e}^{(-1+\sqrt{2})t}.$$

(h) For the differential equation

$$x'' - 2x' - 2x = 0,$$

its characteristic equation is

$$m^2 - 2m - 2 = 0, \text{ which has solutions } m_1 = 1 - \sqrt{3},\ m_2 = 1 + \sqrt{3}.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{(1-\sqrt{3})t} + B\mathrm{e}^{(1+\sqrt{3})t}.$$

(i) For the differential equation

$$2x'' + 2x' - x = 0,$$

its characteristic equation is

$$2m^2 + 2m - 1 = 0, \text{ which has solutions } m_1 = \frac{1}{2}(-1 - \sqrt{3}),\ m_2 = \frac{1}{2}(-1 + \sqrt{3}).$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{\frac{1}{2}(-1-\sqrt{3})t} + B\mathrm{e}^{\frac{1}{2}(-1+\sqrt{3})t}.$$

(j) For the differential equation

$$3x'' - x' - 2x = 0,$$

its characteristic equation is

$$3m^2 - m - 2 = 0, \text{ which has solutions } m_1 = -\tfrac{2}{3},\ m_2 = 1.$$

Hence, the general solution is

$$x(t) = A\mathrm{e}^{-\frac{2}{3}t} + B\mathrm{e}^{t}.$$

(k) For the differential equation

$$x'' + 4x' + 4x = 0,$$

its characteristic equation is

$$m^2 + 4m + 4 = 0, \text{ which has solutions } m_1 = m_2 = -2.$$

This is the special case of equal or coincident roots. Hence, the general solution is

$$x(t) = A\mathrm{e}^{-2t} + Bt\mathrm{e}^{-2t} = (A + Bt)\mathrm{e}^{-2t}.$$

(l) For the differential equation

$$x'' + 6x' + 9x = 0,$$

its characteristic equation is

$$m^2 + 6m + 9 = 0, \text{ which has solutions } m_1 = m_2 = -3.$$

This is the special case of equal or coincident roots. Hence, the general solution is

$$x(t) = A\mathrm{e}^{-3t} + Bt\mathrm{e}^{-3t} = (A + Bt)\mathrm{e}^{-3t}.$$

(m) For the differential equation

$$4x'' + 4x' + x = 0,$$

its characteristic equation is

$$4m^2 + 4m + 1 = 0, \text{ which has solutions } m_1 = m_2 = -\tfrac{1}{2}.$$

This is the special case of equal or coincident roots. Hence, the general solution is

$$x(t) = A\mathrm{e}^{-\frac{1}{2}t} + Bt\mathrm{e}^{-\frac{1}{2}t} = (A + Bt)\mathrm{e}^{-\frac{1}{2}t}.$$

(n) The differential equation $x'' = 0$ has the characteristic equation $m^2 = 0$ with solutions $m_1 = m_2 = 0$. Hence the general solution is $x(t) = A + Bt$. Alternatively the solution can be obtained by direct integration of $x'' = 0$.

18.8. The characteristic equation of

$$x'' + bx' + cx = 0$$

is

$$m^2 + bm + c = 0.$$

Suppose that $b^2 = 4c$ which means that the equation has equal roots $m_1 = m_2 = m_0$, say. One solution is $x = \mathrm{e}^{m_0 t}$. Let $x = t\mathrm{e}^{m_0 t}$. Then

$$x' = (1 + m_0 t)\mathrm{e}^{m_0 t}, \qquad x'' = m_0(2 + m_0 t)\mathrm{e}^{m_0 t}.$$

Therefore

$$\begin{aligned} x'' + bx' + cx &= [2m_0 + m_0^2 t + b(1 + m_0 t) + c]\mathrm{e}^{m_0 t} \\ &= \{[2m_0 + b] + t[m_0^2 + m_0 b + c]\}\mathrm{e}^{m_0 t} = 0, \end{aligned}$$

since m_0 satisfies the characteristic equation, and $m_0 = -\frac{1}{2}b$. Hence $x = t\mathrm{e}^{m_0 t}$ is a second independent solution.

18.9. (a) The characteristic equation of

$$x'' - 4x = 0$$

is $m^2 - 4 = 0$, which has solutions $m_1 = -2,\ m_2 = 2$.Hence the general solution is

$$x = A\mathrm{e}^{-2t} + B\mathrm{e}^{2t}.$$

For the initial conditions $x(0) = 1,\ x'(0) = 0$, the solution is

$$x = \tfrac{1}{2}\mathrm{e}^{-2t} + \tfrac{1}{2}\mathrm{e}^{2t}.$$

(b) The characteristic equation of

$$x'' + x' - 2x = 0$$

is $m^2 + m - 2 = 0$, which has solutions $m_1 = -2,\ m_2 = 1$. Hence the general solution is

$$x = A\mathrm{e}^{-2t} + B\mathrm{e}^{t}.$$

For the initial conditions $x(0) = 0,\ x'(0) = 2$, the solution is

$$x = -\tfrac{2}{3}\mathrm{e}^{-2t} + \tfrac{2}{3}\mathrm{e}^{t}.$$

(c) The characteristic equation of

$$y'' - 4y' + 4y = 0$$

is $m^2 - 4m + 4 = 0$, which has coincident solutions $m_1 = m_2 = 2$. Hence the general solution is

$$y = A\mathrm{e}^{2x} + Bx\mathrm{e}^{2x}.$$

For the initial conditions $y(0) = 0,\ y'(0) = -1$, the solution is

$$y = -x\mathrm{e}^{2x}.$$

(d) The characteristic equation of

$$y'' + 2y' + y = 0$$

is

$$m^2 + 2m + 1 = 0, \text{ which has coincident solutions } m_1 = m_2 = -1.$$

Hence the general solution is

$$y = A\mathrm{e}^{-x} + Bx\mathrm{e}^{-x}.$$

For the initial conditions $y(1) = 0,\ y'(1) = 1$, the solution is

$$y = -\mathrm{e}^{1-x}(x - 1).$$

(e) The solution of the initial value problem

$$x'' - 9x = 0, \qquad x(1) = 1, \quad x'(1) = 1,$$

is

$$x = \frac{1}{3}\mathrm{e}^{3-3t} + \frac{2}{3}\mathrm{e}^{-3+3t}.$$

(f) The characteristic equation of

$$x'' - 4x' = 0$$

is $m^2 - 4m = 0$, which has solutions $m_1 = 0,\ m_2 = 4$. Hence the general solution is

$$x = A + B\mathrm{e}^{4t}.$$

For the initial conditions $x(1) = 1,\ x'(1) = 1$, the solution is $x = 1$.

18.10. (See Section 18.4.) (a) The differential equation $x'' + x = 0$ has the characteristic equation $m^2 + 1 = 0$, which has the complex solutions $m_1 = \mathrm{i}$, $m_2 = -\mathrm{i}$. A complex basis is $(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{-\mathrm{i}t})$, so $(\cos t, \sin t)$ is a real basis. Therefore the general real solution is

$$x = A\cos t + B\sin t.$$

(b) The differential equation $x'' + 9x = 0$ has the characteristic equation $m^2 + 9 = 0$, which has the complex solutions $m_1 = 3\mathrm{i}$, $m_2 = -3\mathrm{i}$. The general real solution (compare (a)) is

$$x = A\cos 3t + B\sin 3t.$$

(c) The differential equation $x'' + \frac{1}{4}x = 0$ has the characteristic equation $m^2 + \frac{1}{4} = 0$, which has the complex solutions $m_1 = \frac{1}{2}\mathrm{i}$, $m_2 = -\frac{1}{2}\mathrm{i}$. The general real solution (compare (a)) is

$$x = A\cos \tfrac{1}{2}t + B\sin \tfrac{1}{2}t.$$

(d) The differential equation $x'' + \omega_0^2 x = 0$ has the characteristic equation $m^2 + \omega_0^2 = 0$, which has the complex solutions $m_1 = \mathrm{i}\omega_0$, $m_2 = -\mathrm{i}\omega_0$. The general real solution (compare (a)) is

$$x = A\cos \omega_0 t + B\sin \omega_0 t.$$

(e) The differential equation $x'' + 2x' + 2x = 0$ has the characteristic equation $m^2 + 2m + 2 = 0$, which has the complex solutions $m_1 = -1 + \mathrm{i}$, $m_2 = -1 - \mathrm{i}$. The complex solution basis is $(\mathrm{e}^{(-1+\mathrm{i})t}, \mathrm{e}^{(-1-\mathrm{i})t})$, from which a real solution basis is $(\mathrm{e}^{-t}\cos t, \mathrm{e}^{-t}\sin t)$. The general real solution is

$$x = \mathrm{e}^{-t}(A\cos t + B\sin t).$$

(f) The differential equation $y'' - 2y' + 2y = 0$ has the characteristic equation $m^2 - 2m + 2 = 0$, which has the complex solutions $m_1 = 1 + \mathrm{i}$, $m_2 = 1 - \mathrm{i}$. The general real solution is

$$y = \mathrm{e}^{t}(A\cos t + B\sin t),$$

derived from the complex form

$$x = C\mathrm{e}^{(\mathrm{i}+1)t} + D\mathrm{e}^{(-\mathrm{i}+1)t}.$$

(g) The differential equation $y'' + y' + y = 0$ has the characteristic equation $m^2 + m + 1 = 0$, which has the complex solutions $m_1 = -\frac{1}{2} + \mathrm{i}\frac{\surd 3}{2}$, $m_2 = -\frac{1}{2} - \mathrm{i}\frac{\surd 3}{2}$. The general real solution (compare (e)) is

$$y = \mathrm{e}^{-\frac{1}{2}t}(A\cos\tfrac{\surd 3}{2}t + \sin\tfrac{\surd 3}{2}t).$$

(h) The differential equation $2x'' + 2x' + x = 0$ has the characteristic equation $2m^2 + 2m + 1 = 0$, which has the complex solutions $m_1 = -\frac{1}{2} + \frac{1}{2}\mathrm{i}$, $m_2 = -\frac{1}{2} - \frac{1}{2}\mathrm{i}$. The general real solution (compare (e)) is

$$x = \mathrm{e}^{-\frac{1}{2}t}(A\cos\tfrac{1}{2}t + \sin\tfrac{1}{2}t).$$

(i) The general real solution (compare (e)) of $3x'' + 4x' + 2 = 0$ is

$$x = \mathrm{e}^{-\frac{2}{3}t}[A\cos(\tfrac{\surd 2}{3}t) + B\sin(\tfrac{\surd 2}{3}t)].$$

(j) The general real solution of $3x'' - 4x' + 2 = 0$ is (compare (e))

$$x = \mathrm{e}^{\frac{2}{3}t}[A\cos(\tfrac{\surd 2}{3}t) + B\sin(\tfrac{\surd 2}{3}t)].$$

18.11. (a) The characteristic equation of $x'' + x = 0$ is $m^2 + 1 = 0$, which has the complex solutions $m_1 = \mathrm{i}$, $m_2 = -\mathrm{i}$. The general real solution is $x = A\cos t + B\sin t$. For the initial conditions $x(0) = 0$ and $x'(0) = 1$, $A = 0$ and $1 = B$. Therefore the solution is $x = \sin t$.

(b) The characteristic equation of $x'' + 4x = 0$ is $m^2 + 4 = 0$, which has the complex solutions $m_1 = 2\mathrm{i}$, $m_2 = -2\mathrm{i}$. The general real solution is $x = A\cos 2t + B\sin 2t$. For the initial conditions $x(0) = 1$ and $x'(0) = 0$, the solution is $x = \cos 2t$.

(c) The characteristic equation of $x'' + \omega_0^2 x = 0$ is $m^2 + \omega_0^2 = 0$, which has the complex solutions $m_1 = \mathrm{i}\omega_0$, $m_2 = -\mathrm{i}\omega_0$. The general real solution is $x = A\cos\omega_0 t + B\sin\omega_0 t$. For the initial conditions $x(0) = a$ and $x'(0) = b$, $A = 0$ and $1 = B$. Therefore the solution is

$$x = a\cos\omega_0 t + b\omega_0^{-1}\sin\omega_0 t.$$

(d) • $k^2 > 1$. The characteristic equation of $x'' + 2kx' + x = 0$ is $m^2 + 2km + 1 = 0$, which has the real solutions

$$m_1 = -k + \surd(k^2 - 1), \qquad m_2 = -k - \surd(k^2 - 1).$$

The general solution is

$$x = A\mathrm{e}^{m_1 t} + B\mathrm{e}^{m_2 t}.$$

For the initial conditions $x(0) = 0$ and $x'(0) = b$, the solution is

$$x = \frac{b}{2\surd(k^2 - 1)}\left[\mathrm{e}^{[-k+\surd(k^2-1)]t} - \mathrm{e}^{[-k-\surd(k^2-1)]t}\right].$$

• $k^2 < 1$. The solutions of the characteristic equation are complex

$$m_1 = -k + \mathrm{i}\surd(1 - k^2), \qquad m_2 = -k - \mathrm{i}\surd(1 - k^2).$$

For the given initial conditions

$$x = \frac{b}{\surd(1-k^2)}\mathrm{e}^{-kt}\sin[\surd(1-k^2)t].$$

• $k^2 = 1$. The solution is $x = bt\mathrm{e}^{-kt}$.

18.12. The linearized pendulum equation is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\theta = 0.$$

The general solution of this equation is

$$\theta = A\cos[(g/l)^{\frac{1}{2}}t] + B\sin[(g/l)^{\frac{1}{2}}t].$$

The initial conditions for the pendulum are $\theta = \alpha$ and $\mathrm{d}\theta/\mathrm{d}t = 0$ at time $t = 0$. Hence

$$\theta = \alpha\cos[(g/l)^{\frac{1}{2}}t].$$

Thed pendulum oscillates with amplitude α.

18.13. The general solution is given in the previous answer. However, in this case the initial conditions are $\theta = 0$ and $\mathrm{d}\theta/\mathrm{d}t = v/l$ at time $t = 0$. Hence

$$\theta = \frac{v}{\sqrt{gl}}\sin[(g/l)^{\frac{1}{2}}t].$$

18.14. With friction included the linearized pendulum equation becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + K\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{l}\theta = 0.$$

The characteristic equation is

$$m^2 + Km + \frac{g}{l}\theta = 0,$$

which has the solutions

$$m_1 = \frac{1}{2}\left[-K + \surd(K^2 - 4(g/l))\right], \quad m_2 = \frac{1}{2}\left[-K - \surd(K^2 - 4(g/l))\right].$$

The friction is small so that we may assume that $K^2 < 4(g/l)$, which means that the roots are complex. Let $\omega = \frac{1}{2}\surd[4(g/l) - K^2]$. Then the general solution is

$$\theta = \mathrm{e}^{-\frac{1}{2}Kt}[A\cos\omega t + B\sin\omega t].$$

Initially, $\theta = 0$ and $\mathrm{d}\theta/\mathrm{d}t = v/l$. Hence

$$\theta = \frac{v}{\surd(gl)}\sin[(g/l)^{\frac{1}{2}}t].$$

The given data are $g = 9.7$, $l = 20$, $K = 0.066$ and $v = 1$. Hence $\omega = 0.70$ (all calculations are to 2 significant figures). Hence

$$\theta = 0.072\mathrm{e}^{-0.033t}\sin(0.70t).$$

18.15. The differential equation

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - y = 0$$

has the characteristic equation $m^3 - 1 = 0$. Hence $m^3 = 1$, which has the roots

$$m_1 = 1, \quad m_2 = \mathrm{e}^{\frac{2}{3}\pi\mathrm{i}} = -\tfrac{1}{2} + \tfrac{\mathrm{i}\sqrt{3}}{2}, \quad m_3 = \mathrm{e}^{\frac{4\pi}{3}\pi\mathrm{i}} = -\tfrac{1}{2} - \tfrac{\mathrm{i}\sqrt{3}}{2}.$$

Hence the general real solution is therefore

$$y = Ae^{x} + Be^{-\frac{1}{2}x}\cos[\tfrac{\sqrt{3}}{2}\pi x] + Ce^{-\frac{1}{2}x}\sin[\tfrac{\sqrt{3}}{2}\pi x].$$

18.16. The characteristic equation of

$$\frac{d^3y}{dx^3} + y = 0$$

is $m^3 + 1 = 0$. Its roots are

$$m_1 = -1, \quad m_2 = e^{\frac{1}{3}\pi i} = \tfrac{1}{2} + \tfrac{i\sqrt{3}}{2}, \quad m_3 = e^{-\frac{1}{3}\pi i} = \tfrac{1}{2} - \tfrac{i\sqrt{3}}{2}.$$

The general real solution is therefore

$$y = Ae^{-x} + Be^{\frac{1}{2}x}\cos[\tfrac{i\sqrt{3}}{2}\pi x] + Ce^{\frac{1}{2}x}\sin[\tfrac{\sqrt{3}}{2}\pi x].$$

18.17. The characteristic equation of

$$\frac{d^4y}{dx^4} - y = 0$$

is $m^4 - 1 = 0$. The four roots of this equation are

$$m_1 = 1, \quad m_2 = -1, \quad m_3 = i, \quad m_4 = -i.$$

Hence the general solution is therefore

$$y = Ae^{x} + Be^{-x} + C\cos x + D\sin x.$$

18.18. Let δu be an vertical increment of height in the column. Then the mass of the column above a height y is

$$\rho\int_y^H A(u)du,$$

where $A(u)$ is the cross-sectional area at height u. Let the cross-sectional area of the column be such that the pressure on any section is a constant P independent of u. Then on the section at height y

$$\text{pressure} \times \text{area} = \text{weight of statue} \; + \; \text{weight of column above.}$$

or

$$PA(y) = Mg + \rho g\int_y^H A(u)du.$$

Differentiate both sides with respect to y. Then

$$\frac{dA}{dy} = -\rho g A(y),$$

using (15.20). This an equation of type (18.4.) with general solution

$$A(y) = Be^{-\rho g y/P},$$

where B is a constant. The constant B is determined by the condition that the top of the column should just support the statue, that is, $PAH = Mg$. Hence $B = Mge^{\rho g H/P}$. Finally, the cross-sectional area is

$$A(y) = \frac{Mg}{P}e^{\rho g(H-y)/P}.$$

Note that the formula does not specify what the cross-sectional shape should be: the cross-section could square, circular or some other shape, or could vary up the column.

Chapter 19: Forced linear differential equations

19.1. (a) Try $x = p\mathrm{e}^{2t}$. Then

$$x' + x - 3\mathrm{e}^{2t} = 2p\mathrm{e}^{2t} + p\mathrm{e}^{2t} - 3\mathrm{e}^{2t} = \mathrm{e}^{2t}(3p - 3) = 0$$

for all t if $p = 1$. Therefore $x = \mathrm{e}^{2t}$ is a particular solution of the equation

$$x' + x = 3\mathrm{e}^{2t}.$$

(b) As a trial solution for the equation

$$x' - 3x = t^3 + 1$$

we must include a constant and every power of t up to and including t^3. Let

$$x = p + qt + rt^2 + st^3.$$

Substituting into the differential equation

$$\begin{aligned} x' - 3x - t^3 - 1 &= [q + 2rt + 3st^2] - [3p + 3qt + 3rt^2 + 3st^3] - t^3 - 1 \\ &= [q - 3p - 1] + [(2r - 3q)t + (3s - 3r)t^2 + (-3s - 1)t^3 \\ &= 0 \end{aligned}$$

for all t if

$$q - 3p - 1 = 0, \quad 2r - 3q = 0, \quad 3s - 3r = 0, \quad -3s - 1 = 0.$$

The solution of these linear equations is

$$p = -\tfrac{11}{27}, \quad q = -\tfrac{2}{9}, \quad r = -\tfrac{1}{3}, \quad s = -\tfrac{1}{3}.$$

Hence a particular solution is

$$x = -\tfrac{11}{27} - \tfrac{2}{9}t - \tfrac{1}{3}t^2 - \tfrac{1}{3}t^3.$$

(c) Try the solution $x = A + Bt + C\mathrm{e}^t$. Then

$$2x' + 3x - t - 3\mathrm{e}^t = (2B + 3A) + (3B - 1)t + (5C - 3)\mathrm{e}^t = 0$$

for all t if $A = -\frac{2}{9}$, $B = \frac{1}{3}$ and $C = \frac{3}{5}$. Hence a particular solution is

$$x = -\tfrac{2}{9} + \tfrac{1}{3}t + \tfrac{3}{5}\mathrm{e}^t.$$

(d) Try the solution $x = A\mathrm{e}^{2t}$. Then

$$x'' + x - 3\mathrm{e}^{2t} = 4A\mathrm{e}^{2t} + A\mathrm{e}^{2t} - 3\mathrm{e}^{2t} = (5A - 3)\mathrm{e}^{2t} = 0$$

for all t if $A = \frac{3}{5}$. Hence a particular solution is $x = \frac{3}{5}\mathrm{e}^{2t}$.
(e) A particular solution of $x'' - \frac{1}{4}x = 2\mathrm{e}^t + 3\mathrm{e}^{-t}$ is $x = \frac{8}{3}\mathrm{e}^t + 4\mathrm{e}^{-t}$.
(f) Try the constant solution $x = A$. Then

$$x'' - 2x' + x - 3 = A - 3 = 0$$

if $A = 3$. Hence a particular solution is $x = 3$.
(g) Since the forcing term is $3t^2 - t$, try the solution $x = At^2 + Bt + C$. It can be shown, as in (a) to (f), that

$$x'' + 4x' - x = 3t^2 - t$$

has the particular solution $x = -3t^2 - 23t - 98$.
(h) For the equation

$$x'' - x = 2\cos t$$

try $x = A\cos t$. Then

$$x'' - x - 2\cos t = -A\cos t - A\cos t - 2\cos t = (-2A - 2)\cos t = 0$$

for all t if $A = -1$. Hence a particular solution is $x = -\cos t$.
(i) For the equation

$$2x'' + 3x = 2\sin 3t,$$

confirm, as in (h), that a particular solution is $x = -\frac{2}{15}\sin 3t$.
(j) Try a solution which includes both a sine and cosine. Let $x = A\cos t + B\sin t$. Then

$$\begin{aligned} 2x'' + x' - \sin t + \cos t &= -2A\cos t - 2B\sin t - A\sin t + B\cos t - \sin t + \cos t \\ &= (-2A + B + 1)\cos t + (-2B - A - 1)\sin t = 0 \end{aligned}$$

for all t if

$$-2A + B + 1 = 0, \qquad A + 2B + 1 = 0.$$

Hence $A = \frac{1}{5}$ and $B = -\frac{3}{5}$, and a particular solution is $x = \frac{1}{5}\cos t - \frac{3}{5}\sin t$.
(k) Try $x = A\cos 2t + B\sin 2t$. Then

$$\begin{aligned} &x'' + 2x' + x - \cos 2t \\ &= [-4A\cos 2t - 4B\sin 2t] + 2[-2A\sin 2t + 2B\cos 2t] + [A\cos 2t + B\sin 2t] \\ &\quad - \cos 2t \\ &= [-3A + 4B - 1]\cos 2t + [-4A - 3B]\sin 2t = 0 \end{aligned}$$

for all t, if $-3A + 4B - 1 = 0$ and $-4A - 3B = 0$. Solving these equations for A and B, it follows that

$$x = -\tfrac{3}{25}\cos t + \tfrac{4}{25}\sin t.$$

(l) Try $x = A + Be^{2x}$. Then

$$\frac{d^2y}{dx^2} - y - 1 + 3e^{2x} = 4Ae^{2x} - A - Be^{2x} - 1 + 3e^{2x} = [-A - 1] + [4A - B + 3]e^{2x}.$$

Therefore $A = -1$ and $B = -1$, so $x = -1 - e^{2x}$ is a particular solution.
(m) Try $y = A\cos 2x + B\sin 2x$, equating to zero terms in $\cos 2x$ and $\sin 2x$, so confirming that

$$x = \tfrac{3}{4}[\cos 2x - \sin 2x]$$

is a particular solution.

19.2. (a) Replace the right-hand side by $3e^{2it}$, of which $3\cos 2t$ is the real part. Solve the complex equation X using the trial $X = Ae^{2it}$:

$$X'' - X - 3e^{2it} = -4Ae^{2it} - Ae^{2it} - 3e^{2it} = (-5A - 3)e^{2it} = 0$$

for all t if $A = -3/5$. Therefore a particular solution of this equation is $X = -\frac{3}{5}e^{2it}$. A particular solution of

$$x'' - x = 3\cos 2t$$

is therefore $x = \mathrm{Re}\,(X) = \mathrm{Re}\,[\frac{3}{5}e^{2it}] = -\frac{3}{5}\cos 2t$.
(b) Consider the equation

$$X'' + X = 2e^{3it}.$$

Let $X = Ae^{3it}$. Then

$$X'' + X - 2e^{3it} = -9Ae^{3it} + Ae^{3it} - 2e^{3it} = (-8A - 2)e^{3it} = 0$$

if $A = -\frac{1}{4}$. A particular solution of

$$x'' + x = 2\sin 3t$$

is therefore $x = \text{Im}\,(X) = \text{Im}\,[-\frac{1}{4}\text{e}^{3\text{i}t}] = -\frac{1}{4}\sin 3t$.
(c) Consider the equation

$$X'' + 2X' + X = 3\text{e}^{\text{i}t}.$$

Try $X = A\text{e}^{\text{i}t}$. Then

$$X'' + 2X' + X - 3\text{e}^{\text{i}t} = (-A + 2A\text{i} + A - 3)\text{e}^{\text{i}t} = (2A\text{i} - 3)\text{e}^{\text{i}t} = 0$$

if $A = -\text{i}\frac{3}{2}$. Hence a particular of

$$x'' + 2x' + x = 3\sin t$$

is $x = \text{Im}\,(X) = \text{Im}\,[-\text{i}\frac{3}{2}\text{e}^{\text{i}t}] = -\frac{3}{2}\cos t$.
(d) Consider the equation

$$X'' - X' - X = 3\text{e}^{\text{i}t}.$$

Let $X = A\text{e}^{\text{i}t}$. Then

$$X'' - X' - X - 3\text{e}^{\text{i}t} = (-A - A\text{i} - A - 3)\text{e}^{\text{i}t} = (-2A - A\text{i} - 3)\text{e}^{\text{i}t} = 0$$

if $A = -3/(2+\text{i}) = -\frac{6}{5} + \frac{3}{5}\text{i}$. Therefore a particular solution of

$$x'' - x' - x = 3\cos t$$

is $x = \text{Re}\,(X) = \text{Re}\,[(-\frac{6}{5} + \frac{3}{5}\text{i})\text{e}^{\text{i}t}] = -\frac{6}{5}\cos t - \frac{3}{5}\sin t$.
(e) In this problem we require the real part of the solution of

$$2X'' + X' + 2X = 2\text{e}^{2\text{i}t}.$$

Try $X = A\text{e}^{2\text{i}t}$ so that

$$2X'' + X' + 2X - 2\text{e}^{2\text{i}t} = (-8A + 2\text{i}A + 2A - 2)\text{e}^{2\text{i}t} = [(-6A + 2\text{i})A - 2)\text{e}^{2\text{i}t} = 0$$

for all t if $A = 1/(-3+\text{i}) = \frac{3}{10} + \frac{1}{10}\text{i}$. Hence a particular solution of

$$2x'' + x' + 2x = 2\cos t$$

is $x = \text{Re}\,(X) = -\frac{3}{10}\cos 2t + \frac{1}{10}\sin 2t$.
(f) A particular solution of

$$3X'' + 2X' + X = 2\text{e}^{2\text{i}t}$$

is $X = (-\frac{22}{137} - \frac{8}{137}\text{i})\text{e}^{2\text{i}t}$. Hence a particular solution of

$$3x'' + 2x' + x = 2\sin 2t$$

is $x = \text{Im}\,(X) = -\frac{8}{137}\cos 2t - \frac{22}{137}\sin 2t$.
(g) Consider the equation

$$X'' - 4X = \text{e}^{(-1+\text{i})t}.$$

Try $X = A\text{e}^{(-1+\text{i})t}$. Then

$$X'' - 4X - \text{e}^{(-1+\text{i})t} = [(-1+\text{i})^2 A - 4A - 1]\text{e}^{(-1+\text{i})t} = 0$$

for all t if $A = (-2-\text{i})/10$. Hence a particular solution of

$$x'' - 4x = \text{e}^{-t}\cos t$$

is $x = \text{Re}\,(X) = \text{Re}\,[(-\frac{1}{5} - \frac{1}{10}\text{i})\text{e}^{(-1+\text{i})t} = -\frac{1}{5}\text{e}^{-t}\cos t - \frac{1}{10}\sin t$.
(h) Consider the equation

$$X'' - 4X = 3\text{e}^{(1+2\text{i})t}.$$

Let $X = A\mathrm{e}^{(1+2\mathrm{i})t}$. Then

$$X'' - 4X - 3\mathrm{e}^{(1+2\mathrm{i})t} = [(1+2\mathrm{i})^2 A - 4A - 3]\mathrm{e}^{(1+2\mathrm{i})t} = 0$$

for all t if $A = -\frac{21}{65} - \frac{12}{65}\mathrm{i}$. Hence a particular solution of

$$x'' - 4x = 3\mathrm{e}^t \sin 2t$$

is $x = \mathrm{Im}\,(X) = \mathrm{e}^t[-\frac{12}{65}\cos 2t - \frac{21}{65}\sin 2t]$.
(i) Consider the equation

$$X'' + X' + 4X = 5\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}(3t+\phi)}.$$

Try $X = A\mathrm{e}^{\mathrm{i}(3t+\phi)}$: then the solution in the required form is the real part of X. Substituting X into the equation:

$$\begin{aligned} X'' + X' + 4X - 5\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}(3t+\phi)} &= [-9A + 3\mathrm{i}A + 4A - 5\mathrm{e}^{-\mathrm{i}\phi}]\mathrm{e}^{\mathrm{i}(3t+\phi)} \\ &= [(-5+3\mathrm{i})A - 5\mathrm{e}^{-\mathrm{i}\phi}]\mathrm{e}^{\mathrm{i}(3t+\phi)} = 0 \end{aligned}$$

for all t, if $(-5+3\mathrm{i})A = 5\mathrm{e}^{-\mathrm{i}\phi}$ or,

$$\begin{aligned} A &= \frac{1}{-5+3\mathrm{i}} = \frac{-(5+3\mathrm{i})}{34}\mathrm{e}^{-\mathrm{i}\phi} = -\frac{1}{34}(5+3\mathrm{i})(\cos\phi - \mathrm{i}\sin\phi) \\ &= -\frac{5}{34}\{(5\cos\phi + 3\sin\phi) + \mathrm{i}(3\cos\phi - 5\sin\phi)\} \\ &= -\frac{5}{34}\cos\phi\{(5 + 3\tan\phi) + \mathrm{i}(3 - 5\tan\phi)\} \\ &= -\frac{5}{34}\cos\phi\left\{(5 + \frac{9}{5}) + \mathrm{i}(3-3)\right\} = -\frac{5}{34}cos\phi\left(\frac{34}{5}\right) = -\cos\phi \\ &= -\frac{5}{\sqrt{34}}. \end{aligned}$$

Therefore

$$x = \mathrm{Re}\,(X) = -\frac{5}{\sqrt{34}}\cos(3t+\phi),$$

where $\tan\phi = \frac{3}{5}$.

19.3. (a) The characteristic equation of

$$x'' + x = 3\cos t$$

is $m^2 + 1 = 0$, with roots $m = \pm\mathrm{i}$. Hence a particular solution cannot be a multiple of $\cos t$. Try, instead, $x = At\sin t$ (see (19.6)): then

$$x'' + x - 3\cos t = 2A\cos t - At\sin t + At\sin t - 3\cos t = (2A-3)\cos t = 0$$

for all t, if $A = \frac{3}{2}$. Hence a particular solution is $x = \frac{3}{2}t\sin t$.
(b) Since $A\sin 2t$ satisfies the corresponding homogeneous equation, it cannot be also a particular solution of

$$x'' + 4x = 3\sin 2t.$$

As in (19.6), try $x = At\cos 2t$: then

$$\begin{aligned} x'' + 4x - 3\sin 2t &= -4A\sin 2t - 4At\cos 2t + 4At\cos 2t - 3\sin 2t \\ &= (-4A-3)\sin 2t = 0 \end{aligned}$$

for all t if $A = -\frac{3}{4}$. Hence a particular solution is $x = -\frac{3}{4}t\cos 2t$.

(c) A particular solution for the constant 1 is obviously $\frac{1}{4}$. Since $A\cos 2t$ satisfies the corresponding homogeneous equation, we must try $x = \frac{1}{4} + At\sin 2t$: therefore

$$\begin{aligned} x'' + 4x - 1 - 3\cos 2t &= 4A\cos 2t - 4At\sin 2t + 1 + 4At\sin 2t - 1 - 3\cos 2t \\ &= (4A - 3)\sin 2t = 0 \end{aligned}$$

for all t if $A = \frac{3}{4}$. Hence a particular equation is $x = \frac{1}{4} + \frac{3}{4}t\sin 2t$.
(d) A particular solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = 2\sin 3x$$

is $x = -\frac{1}{3}x\cos 3x$.
(e) Consider the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^x\cos x.$$

The corresponding homogeneous equation has the characteristic equation

$$m^2 - 2m + 2 = 0, \text{ which has the solutions } m_1 = 1 + \mathrm{i}, \quad m_2 = 1 - \mathrm{i},$$

which means that $A\mathrm{e}^x\cos x$ cannot be a particular solution of the given differential eqaution. Instead try $y = Ax\mathrm{e}^x\sin x$: then

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y - \mathrm{e}^x\cos x = (-1 + 2A)\mathrm{e}^x\cos x = 0$$

if $A = \frac{1}{2}$. Hence a particular solution is $y = \frac{1}{2}x\mathrm{e}^x\sin x$.

19.4. (a) Let $x = pt\mathrm{e}^t$. Then

$$x'' - x - \mathrm{e}^t = 2p\mathrm{e}^t + pt\mathrm{e}^t - pt\mathrm{e}^t - \mathrm{e}^t = (2p - 1)\mathrm{e}^t = 0$$

if $p = \frac{1}{2}$. Hence the equation has a particular solution $x = \frac{1}{2}t\mathrm{e}^t$.
(b) Let $x = pt^2\mathrm{e}^t$. Then

$$x'' - 2x' + x - \mathrm{e}^t = p(2 + 4t + t^2)\mathrm{e}^t - 2(2t + t^2)p\mathrm{e}^t + pt^2\mathrm{e}^t - \mathrm{e}^t = (2p - 1)\mathrm{e}^t = 0$$

if $p = \frac{1}{2}$. Therefore a particular solution is $x = \frac{1}{2}t^2\mathrm{e}^t$.
(c) Try $x = pt^3$. Then

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - t = 6pt - t = (6p - 1)t = 0$$

if $p = \frac{1}{6}$. Hence a particular solution is $x = \frac{1}{6}t^3$.
(d) Try $y = px^2 + qx + r$. Then

$$\frac{\mathrm{d}^2y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} - x = 2p + 2px + q - x = (2p + q) + (2p - 1)x = 0$$

for all x if $2p + q = 0$ and $2p - 1 = 0$. Therefore $p = \frac{1}{2}$ and $q = -1$. The constant term r can take any value since it is a solution of the homogeneous equation; let $r = 0$, say. Then a particular solution is $x = \frac{1}{2}x^2 - x$.
(e) (See also Problem 19(e)) Let $x = \mathrm{e}^t(p\cos t + q\sin t)$. Then

$$x'' - 2x' + 2x - \mathrm{e}^t\cos t = \mathrm{e}^t[(-1 + 2q)\cos t - 2p\sin t] = 0$$

for all t if $-1 + 2q = 0$ and $p = 0$. Therefore $q = \frac{1}{2}$ and a particular solution is $x = \frac{1}{2}t\mathrm{e}^t\sin t$.
(f) The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y = \mathrm{e}^x$$

has the particular solution $y = xe^x$.

19.5. In each case the solution of the differential equation is the sum of its complementary function (the general solution of the homogeneous equation) and a particular solution.
(a) $x'' + 9x = 3e^{2t}$. The characteristic equation for $x'' + 9x = 0$ is

$$m^2 + 9 = 0, \text{ which has the solutions } m_1 = 3,\ m_2 = -3.$$

The complementary function is therefore $x_c = Ae^{3t} + Be^{-3t}$. For a particular solution try $x_p = pe^{2t}$. Then

$$x_p'' + 9x_p - 3e^{2t} = (4p + 9p - 3)e^{2t} = (13p - 3)e^{2t} = 0$$

for all t if $p = \frac{3}{13}$. Hence the general solution is

$$x = x_c + x_p = Ae^{3t} + Be^{-3t} + \frac{3}{13}e^{2t}.$$

(b) $x'' - 4x = 2e^{-t}$. The characteristic equation for $x'' - 4x = 0$ is $m^2 - 4 = 0$. Hence the complementary function is

$$x_c = Ae^{2t} + B = e^{-2t}.$$

a particular solution try $x_p = pe^{-t}$: then

$$x_p'' - 4x_p - 2e^{-t} = (-3p - 2)e^{-t} = 0$$

if $p = -\frac{2}{3}$. The general solution is therefore

$$x = Ae^{2t} + Be^{-2t}\frac{2}{3}e^{-t}.$$

(c) $4x'' - x = 1 + 3\cos 2t$. The complementary function is

$$x_c = Ae^{\frac{1}{2}t} + Be^{-\frac{1}{2}t}.$$

For a particular solution try $x_p = p + q\cos 2t$: then

$$\begin{aligned} 4x_p'' - x_p - 1 - 3\cos 2t &= -16q\cos 2t - p - q\cos 2t - 1 - 3\cos 2t \\ &= (-p - 1) + (-17q - 3)\cos 2t = 0 \end{aligned}$$

if $p = -1$ and $q = -\frac{3}{17}$. The general solution is

$$x = Ae^{\frac{1}{2}t} + Be^{-\frac{1}{2}t} - 1 - \frac{3}{17}\cos 2t.$$

(d) $y'' + 2y' + 2x = 3$. The characteristic equation is

$$m^2 + 2m + 2 = 0, \text{ which has the complex roots } m_1 = -1 + i,\ m_2 = -1 - i.$$

Hence the complementary function can be expressed as

$$y_c = Ae^{-x}\cos x + Be^{-x}\sin x.$$

A particular solution is simply $x_p = \frac{3}{2}$. Therefore the general solution is

$$x = Ae^{-x}\cos x + Be^{-x}\sin x + \tfrac{3}{2}.$$

(e) $x'' - 2x' + 2x = 3\sin 2t$. The characteristic equation is $m^2 - 2m + 2 = 0$ which has the complex roots $m_1 = 1 + i$, $m_2 = 1 - i$. Hence the complementary function is $x_c = Ae^t\cos t + Be^t\sin t$. For a particular solution try $x_p = p\cos 2t + q\sin 2t$. Then $p = \frac{3}{5}$ and $-\frac{3}{10}$. Therefore

$$x = x_c + x_p = Ae^t\cos t + Be^t\sin t + \frac{3}{5}\cos 2t - \frac{3}{10}\sin 2t.$$

(f) $4x'' - 2x' - 2x = 3t^2$. The characteristic equation is $4m^2 - 2m - 2 = 0$ which has the roots $m_1 = 1,\ m_2 = -\frac{1}{2}$. The complementary function is therefore $x_c = A\mathrm{e}^{-\frac{1}{2}t} + B\mathrm{e}^t$. For the particular solution try $x_p = pt^2 + qt + r$:

$$\begin{aligned} 4x'' - 2x' - 2x - 3t^2 &= 4(2p) - 2(2pt + q) - 2(pt^2 + qt + r) - 3t^2 \\ &= (-2p - 3)t^2 + (-4p - 2q)t + (8p - 2q - 2r) = 0 \end{aligned}$$

for all t if $p = -\frac{3}{2}$, $q = -2p = 3$ and $r = 4p - q = -9$. Hence $x_p = -\frac{3}{2}t^2 + 3t - 9$. The general solution is

$$x = x_c + x_p = A\mathrm{e}^{-\frac{1}{2}t} + B\mathrm{e}^t - \tfrac{3}{2}t^2 + 3t - 9.$$

(g) $x'' + x' = 2 - 3\mathrm{e}^{-t}\cos t$. The characteristic equation is

$$m^2 + m = 0 \text{ with solutions } m_1 = 0,\ m_2 = -1.$$

Therefore the complementary function is $x_c = A + B\mathrm{e}^{-t}$. The absence of x in the equation means that the equation is a special case. For the particular solution try $x_p = pt + q\mathrm{e}^{-t}\cos t + r\mathrm{e}^{-t}\sin t$: then

$$x'' + x' - 2 + 3\mathrm{e}^{-t}\cos t = (p - 2) + (-q - r + 3)\mathrm{e}^{-1}\cos t + (-r + q)\mathrm{e}^{-t}\sin t$$

for all t if $p = 2$ and $q = r = \frac{3}{2}$. Hence $x_p = 2t + \frac{3}{2}\mathrm{e}^{-t}\cos t + \frac{3}{2}\mathrm{e}^{-t}\sin t$. The general solution is

$$x = x_c + x_p = A + B\mathrm{e}^{-t} + 2t + \tfrac{3}{2}\mathrm{e}^{-t}\cos t + \tfrac{3}{2}\mathrm{e}^{-t}\sin t.$$

(h) $2x'' + x' - x = \frac{1}{2}t + 3\mathrm{e}^{-t}$. The characteristic equation is

$$2m^2 + m - 1 = 0 \text{ which has the roots } m_1 = -1,\ m_2 = \tfrac{1}{2}.$$

Hence $x_c = A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{-t}$. Since the forcing term includes $3\mathrm{e}^{-t}$, this is a special case. Therefore try $x_p = p + qt + r\mathrm{e}^{-t}$: then

$$2x'' + x' - x - \tfrac{1}{2}t - 3\mathrm{e}^{-t} = (-3r - 3)\mathrm{e}^{-t} + (q - p) + (-q - \tfrac{1}{2})t = 0$$

if $p = -\frac{1}{2}$, $q = -\frac{1}{2}$ and $r = -1$. The general solution is

$$x = A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{-t} - \tfrac{1}{2}t - t\mathrm{e}^{-t}.$$

(i) $y'' + y = 1 + 2\mathrm{e}^{3x} + x^2$. The general solution is

$$y = A\cos x + B\sin x - 1 + \tfrac{1}{5}\mathrm{e}^{3x} + x^2.$$

(j) $y'' + 2y' + y = 3\cos 2x + \sin 2x$. The characteristic equation has the repeated root $m = -1$. Therefore the complementary function is $y_c = A\mathrm{e}^{-x} + Bx\mathrm{e}^{-x}$. For the particular solution try $y_p = p\cos 2x + q\sin 2x$: then

$$y'' + 2y' + y - 3\cos 2x - \sin 2x = (-3A + 4B - 3)\cos 2x + (-3B - 4A - 1)\sin 2x = 0$$

if $-3A + 4B - 3 = 0$ and $-3B - 4A - 1 = 0$. Therefore $A = -\frac{13}{25}$ and $B = \frac{9}{25}$ and the general solution is

$$A\mathrm{e}^{-x} + Bx\mathrm{e}^{-x} - \frac{13}{25}\cos 2x + \frac{9}{25}\sin 2x.$$

(k) $y'' + 4y' + 5y = \mathrm{e}^{-x}\sin x$. The characteristic equation is

$$m^2 + 4m + 5 - 0, \text{ which has the complex roots } m_1 = -2 + \mathrm{i},\ m_2 = -2 - \mathrm{i}.$$

The complementary function is $y_c = A\mathrm{e}^{-2x}\cos x + B\mathrm{e}^{-2x}\sin x$. For the particular solution try $y_p = \mathrm{e}^{-x}[p\cos x + q\sin x]$: then $p = -\frac{2}{5}$ and $q = \frac{1}{5}$. Therefore the general solution is

$$y = A\mathrm{e}^{-2x}\cos x + B\mathrm{e}^{-2x}\sin x - \frac{2}{5}\mathrm{e}^{-x}\cos x + \frac{1}{5}\mathrm{e}^{-x}\sin x.$$

19.6. From (19.15) and (19.16) the integrating factor of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + g(t)x = f(t)$$

(note that this must be written in the standard form) is $I(t) = \mathrm{e}^{\int g(t)\mathrm{d}t}$, and the general solution of the equation is

$$x(t) = \frac{1}{I(t)}\int I(t)f(t)\mathrm{d}t + \frac{C}{I(t)}.$$

(a) $x' - 3x = 0$. Then $I(t) = \mathrm{e}^{\int 3\mathrm{d}t} = \mathrm{e}^{3t}$. The general solution is $x = C\mathrm{e}^{-3t}$ since $f(t) = 0$.

(b) $x' + 2t = 3$. In this case $g(t) = 2$ and $f(t) = 3$. Then $I(t) = \mathrm{e}^{\int 2\mathrm{d}t} = \mathrm{e}^{2t}$. Therefore the general solution is

$$x = \mathrm{e}^{-2t}\left[\int 3\mathrm{e}^{2t}\mathrm{d}t + C\right] = \frac{3}{2} + C\mathrm{e}^{-2t}.$$

(c) $x' - 2t = t$. In this example $g(t) = -2t$ and $f(t) = t$. Then

$$I(t) = \mathrm{e}^{\int -2t\mathrm{d}t} = \mathrm{e}^{-t^2}.$$

The general solution is

$$x = \mathrm{e}^{t^2}[\textstyle\int(t\mathrm{e}^{-t^2})\mathrm{d}t + C] = \mathrm{e}^{t^2}[-\frac{1}{2}\mathrm{e}^{-t^2} + C] = -\frac{1}{2} + C\mathrm{e}^{t^2}.$$

(d) $x' - t^{-1}x = t + t\mathrm{e}^{-t}$. In this case $g(t) = -t^{-1}$ and $f(t) = t + t\mathrm{e}^{-t}$. Hence, the integrating factor is

$$I(t) = \mathrm{e}^{-\int t^{-1}\mathrm{d}t} = 1/t.$$

The general solution is

$$x = t[\int \frac{1}{t}(t + t\mathrm{e}^{-t})\mathrm{d}t + C] = t[-\int(1 + t\mathrm{e}^{-t})\mathrm{d}t + C] = t^2 - t\mathrm{e}^{-t} + Ct.$$

(e) $x' - t^{-1}x = t - 1$. Here $g(t) = -t^{-1}$ and $f(t) = t - 1$. As in (d), $I(t) = 1/t$. Hence

$$x = t[\int t^{-1}(t-1)\mathrm{d}t + C] = t^2 - t\ln t + Ct.$$

(f) $tx' - 2x + 3 = 0$. After a rearrangement to standard form, $g(t) = -2/t$ and $f(t) = -3/t$. The integrating factor is $I(t) = \mathrm{e}^{-\int(2/t)\mathrm{d}t} = -1/t^2$. Hence

$$x = -t^2\left[-\int \frac{3}{t^3}\mathrm{d}t + C\right] = \frac{3}{2} + At^2.$$

(g) $y' + (x+1)^{-1}y = \sin x$. Then $I(x) = \mathrm{e}^{\int[\mathrm{d}x/(x+1)]} = x + 1$. Hence

$$\begin{aligned} y &= \frac{1}{x+1}[\int(x+1)\sin x\mathrm{d}x + C] = \frac{1}{x+1}[-(x+1)\cos x + \int \cos x\mathrm{d}x + C] \\ &= -\cos x + \frac{\sin x}{x+1} + \frac{C}{x+1} \end{aligned}$$

(h) $3y' + x^{-1}y = x$. In this case $g(x) = 1/(3x)$ and $f(x) = x/3$. The integrating factor $I(x) = \mathrm{e}^{\frac{1}{3}\int x^{-1}\mathrm{d}x} = x^{\frac{1}{3}}$. The general solution is

$$\begin{aligned} y &= x^{-\frac{1}{3}}\left[\int x^{\frac{1}{3}}\frac{1}{3}x\mathrm{d}x + C\right] = x^{\frac{1}{3}}\left[\frac{1}{3}\int x^{\frac{4}{3}}\mathrm{d}x + C\right] \\ &= \frac{1}{7}x^2 + Ax^{-\frac{1}{3}} \end{aligned}$$

(i) $(x-1)y' - y = (x-1)^2$. In this case $g(x) = -1/(x-1)$ and $f(x) = (x-1)$. For $x > 1$, the integrating function is

$$I(x) = \mathrm{e}^{\int [\mathrm{d}x/(x-1)]} = \mathrm{e}^{-\ln(x-1)} = 1/(x-1).$$

(we do not need to consider the case $x < 1$ separately, since one integrating factor will do for all x). The general solution is

$$y = (x-1)[\textstyle\int \mathrm{d}x + C] = x(x-1) + C(x-1).$$

(j) $x' - t^{-1}x = \ln t$. The integrating factor is $I(t) = \mathrm{e}^{-\int t^{-1}\mathrm{d}t} = 1/t$. The general solution is

$$x = \tfrac{1}{2}t[\ln t]^2 + At.$$

(k) $tx' - x = 1 + t$. The general solution is

$$x = -1 + t\ln t + At.$$

(l) The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x+1}$$

can be rearranged into the standard form

$$y' - (x+1)^{-1}y = x(x+1)^{-1}.$$

The integrating factor is $I(x) = \mathrm{e}^{-\int (x+1)^{-1}\mathrm{d}x} = 1/(x+1)$ for all $x \neq -1$. Hence

$$\begin{aligned} y &= (x+1)\left[\int \frac{x}{(x+1)^2}\mathrm{d}x + C\right] \\ &= (x+1)\left[\int \left(\frac{1}{1+x} - \frac{1}{(1+x)^2}\right)\mathrm{d}x + C\right] \\ &= 1 + (1+x)\ln|1+x| + C(1+x) \end{aligned}$$

(m) $x' + x\cos t = \cos t$. The integrating factor is $I(t) = \mathrm{e}^{\int \cos t \mathrm{d}t} = \mathrm{e}^{\sin t}$. Hence, the general solution is

$$x = \mathrm{e}^{-\sin t}[\textstyle\int \cos t \mathrm{e}^{\sin t}\mathrm{d}t + C] = \mathrm{e}^{-\sin t}[\mathrm{e}^{\sin t} + C] = 1 + C\mathrm{e}^{-\sin t}.$$

(n) The equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-y}{1-x}$$

can be rearranged into $y' + [x(1-x)]^{-1}y = [x(1-x)]^{1}$. The integrating factor is

$$\begin{aligned} I(x) &= \exp[\int \frac{\mathrm{d}x}{x(1-x)} = \exp\left\{\int \left[\frac{1}{x} + \frac{1}{1-x}\right]\mathrm{d}x\right\} \\ &= \exp[\ln x - \ln(1-x)] \\ &= \frac{x}{1-x} \end{aligned}$$

for all $x \neq 0$ or 1. The general solution is

$$y = \left(\frac{1-x}{x}\right)\left[\int \frac{1}{(1-x)^2}\mathrm{d}x + C\right] = -\frac{1}{x} + C\left(\frac{1-x}{x}\right).$$

(o) $(1-t^2)x' + tx = t$. The integrating factor is

$$I(t) = \exp\left[\int \frac{t\mathrm{d}t}{1-t^2}\right] = \exp\left[-\frac{1}{2}\ln(1-t^2)\right] = (1-t^2)^{-\frac{1}{2}},$$

assuming $|t| < 1$. Therefore the general solution is

$$x = (1-t^2)^{\frac{1}{2}}\left[\int (1-t^2)^{-\frac{1}{2}} t\mathrm{d}t + C\right] = 1 + C(1-t^2)^{\frac{1}{2}}.$$

19.7. The integrating factor of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = f(x)$$

is $I(x) = \exp[\int x^{-1}\mathrm{d}x] = x$. Hence, by (19.16), the general solution is

$$y = \frac{1}{I(x)}\left[\int I(x)f(x)\mathrm{d}x + C\right] = \frac{1}{x}\int xf(x)\mathrm{d}x + \frac{C}{x}.$$

If $f(x) = \ln x$, then

$$\begin{aligned} y &= \frac{1}{x}\int x\ln x\mathrm{d}x + \frac{C}{x} \\ &= \frac{1}{x}\left[\frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{1}{x}\mathrm{d}x\right] + \frac{C}{x} \quad \text{(integrating by parts)} \\ &= \frac{1}{x}\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right] + \frac{C}{x} \\ &= \frac{1}{2}x\ln x - \frac{1}{4}x + \frac{C}{x}. \end{aligned}$$

For the condition $y(1) = 0$, $C = \frac{1}{4}$. Therefore $y = \frac{1}{2}x\ln x - \frac{1}{4}x + \frac{1}{4x}$.

19.8. (a) The integrating factor of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x)$$

is $I(x) = \mathrm{e}^{\int \mathrm{d}x} = \mathrm{e}^x$. Hence, by (19.16), the general solution is

$$y = \frac{1}{I(x)}\left[\int I(x)f(x)\mathrm{d}x + C\right] = \mathrm{e}^{-x}\int \mathrm{e}^x f(x)\mathrm{d}x + C\mathrm{e}^{-x}.$$

(b) Express the indefinite integral in the solution in (a) as a definite integral in the form

$$y = \mathrm{e}^{-x}\int_0^x \mathrm{e}^u f(u)\mathrm{d}u + C\mathrm{e}^{-x}.$$

The condition $y(0) = y_0$ implies $C = y_0$. Hence required solution is

$$y = \mathrm{e}^{-x}\int_0^x \mathrm{e}^u f(u)\mathrm{d}u + y_0\mathrm{e}^{-x}.$$

19.9. The equation for the temperature T in Newton cooling satisfies

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_0).$$

The general solution of the equation is

$$T = T_0 + C\mathrm{e}^{-kt}.$$

We are given that $T_0 = 40$ and $T(0) = 100$. Hence $100 = 40 + C$, or $C = 60$. With t measured in minutes, we also know that $T(3) = 85$. Hence $85 = 40 - 60\mathrm{e}^{-3k}$ so that

$$k = -\frac{1}{3}\ln[(85-40)/60] = -\frac{1}{3}\ln[9/12] = 0.0959.$$

Let t_1 be the time when the temperature reaches 60° Then

$$60 = 40 + 60\mathrm{e}^{-kt_1}, \text{ from which, } t_1 = -k^{-1}\ln(\tfrac{1}{3}) = 11.5 \text{ minutes},$$

approximately.

Chapter 20: Harmonic functions and the harmonic oscillator

20.1. A harmonic function is said to be in standard amplitude-phase form, if it is written as $C\cos(\omega t + \phi)$ where $C > 0$ and $-\pi < \phi \leq \pi$.
(a) $3\cos(3t + \frac{3}{2}\pi)$. In this problem $C = 3 > 0$ but the phase $\frac{3}{2}\pi$ has to be adjusted. We can add or subtract any multiple of 2π without affecting the value of the harmonic function. Here we must subtract 2π from the phase to give the standard form $3\cos(3t - \frac{1}{2}\pi)$.
(b) $3\cos(\omega t - 3\pi)$. Add 4π to the phase: standard form is $3\cos(\omega t + \pi)$.
(c) $2\sin 3t$. Use the identity $\sin A = \cos(A - \frac{1}{2}\pi)$: standard form is $2\cos(3t - \frac{1}{2}\pi)$.
(d) $3\sin(2t + \frac{1}{2}\pi)$. Standard form is $3\cos 2t$.
(e) $-3\cos(2t - \frac{1}{2}\pi) = 3\cos(2t - \frac{1}{2}\pi + \pi) = 3\cos(2t + \frac{1}{2}\pi)$ which is the standard form.
(f) $-4\cos(2t + \frac{1}{4}\pi) = 4\cos(2t + \frac{1}{4}\pi - \pi) = 4\cos(2t - \frac{3}{4}\pi)$ which is the standard form.
(g) $-\sin t = \sin(t + \pi) = cos(t + \pi - \frac{1}{2}pi) = \cos(t + \frac{1}{2}\pi)$.
(h) $3\cos 2t + 4\sin 2t = \surd[3^2 + 4^2]\cos(2t + \phi) = 5\cos(2t + \phi)$, where ϕ is defined by

$$\cos\phi = \tfrac{3}{5},\ \sin\phi = -\tfrac{4}{5}.$$

For a standard form we choose $\phi = -0.927\ldots$ in radians.
(i) $\cos 2t + \cos(2t - \pi) = \cos 2t + \cos 2t\cos\pi + \sin 2t\sin\pi = \cos 2t - \cos 2t = 0$, which is the standard form.
(j) $\cos(2t - \frac{3}{2}\pi) - \cos(2t + \frac{3}{2}\pi) = -2\sin(\frac{1}{2}4t)\sin(-\frac{3}{2}\pi) = -2\sin(2t)$ using the product formula, Appendix B. Then

$$-2\sin(2t) = 2\sin(2t + \pi) = 2\cos(2t + \pi - \tfrac{1}{2}\pi) = 2\cos(2t + \tfrac{1}{2}\pi),$$

which is the standard form.

20.2. If $x(t) = C_1\cos(\omega t + \phi_1)$, $y(t) = C_2\cos(\omega t + \phi_2)$, and $\phi_1 > \phi_2$, x is said to lead y by $\phi_1 - \phi_2$; if $\phi_1 < \phi_2$ then x lags y.
(a) $x = 4\cos 3t,\ y = 3\cos(3t - \frac{1}{2}\pi)$. Here $\phi_1 = 0$ and $\phi_2 = -\frac{1}{2}\pi$: hence $\phi_1 > \phi_2$ so that x leads y by $\frac{1}{2}\pi$.
(b) $x = 2\cos(2t + \frac{1}{4}\pi),\ y = 3\cos(2t + \frac{9}{2}\pi)$. y is not in standard form: replace it by the standard form $y = 3\cos(2t + \frac{1}{2}\pi)$. Then $\phi_1 = \frac{1}{4}\pi$ and $\phi_2 = \frac{1}{2}\pi$: hence $\phi_1 < \phi_2$ so that x lags y.
(c) $x = -3\cos 2t = 3\cos(2t + \pi),\ y = 4\cos 2t$. Here $\phi_1 = \pi$ and $\phi_2 = 0$: hence $\phi_1 > \phi_2$ which means that x leads y.
(d) $x = \cos 3t,\ y = \sin 3t = \cos(3t - \frac{1}{2}\pi)$. Here $\phi_1 = 0$ and $\phi_2 = -\frac{1}{2}\pi$: hence $\phi_1 > \phi_2$ which means that x leads y.
(e) $x = 2\cos 3t,\ \cos(3t - \frac{9}{4}\pi) = \cos(3t - \frac{1}{4}\pi)$. In this case $\phi_1 = 0$ and $\phi_2 = -\frac{1}{4}\pi$ so that $\phi_1 > \phi_2$ which means that x leads y.

20.3. If the equation is expressed in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0,$$

then (a) natural frequency is $\omega_0/(2\pi)$;

(b) the 'frequency' of the damped oscillation is $(\omega_0^2 - k^2)^{\frac{1}{2}}/(2\pi)$; (c) the 'amplitude' of the damped oscillation is $A\mathrm{e}^{-kt}$; this drops by a tenth in time T where $\mathrm{e}^{-kT} = \frac{1}{10}$. Hence $T = (\ln 10)/k$, and the number of cycles is approximately $T\omega_0/(2\pi)$.

(a) (i) $\omega_0/(2\pi) = \sqrt{2.5 \times 10^5}/(2\pi) = 79.58$ cycles/s;
(ii) damped frequency= $\sqrt{2.5 \times 10^5 - 10^2}/(2\pi) = 79.56$ cycles/s;
(iii) number of cycles $=\omega_0(\ln 10)/(20\pi) = 18.32$ cycles: 19 cycles needed.

(b) (i) $\omega_0/(2\pi) = \sqrt{4}/(2\pi) = 0.318$ cycles/s;
(ii) damped frequency=$\sqrt{4 - 0.25^2}/(2\pi) = 0.316$ cycles/s;
(iii) No. of cycles=$\omega_0(\ln 10)/(2\pi k) = (2\ln 10)/(2\pi \times 0.25) = 2.931$ cycles: 3 cycles needed

(c) (i) $\omega_0/(2\pi) = \sqrt{3}/(2\pi) = 0.276$ cycles/s;
(ii) damped frequency=$\sqrt{3 - 0.075^2}/(2\pi) = 0.275$ cycles/s;
(iii) No. of cycles=$\omega_0(\ln 10)/(2\pi k) = 8.463$ cycles: 9 cycles needed

(d) (i) $\omega_0/(2\pi) = 0.711$ cycles/s;
(ii) damped frequency=0.707 cycles/s;
(iii) number of cycles = 4, approximately.

20.4. (a) Let $x = 3^{\frac{1}{2}} \cos \omega t + \sin(\omega t + \frac{1}{4}\pi)$. Then

$$\begin{aligned} x &= 3^{\frac{1}{2}} \cos \omega t + \cos(\omega t - \frac{1}{4}\pi) = (3^{\frac{1}{2}} + 2^{-\frac{1}{2}}) \cos \omega t + 2^{-\frac{1}{2}} \sin \omega t \\ &= \sqrt{(\sqrt{6}+4)} \cos \omega t + 2^{-\frac{1}{2}} \sin \omega t \\ &= (\sqrt{6}+1) 2^{-\frac{1}{2}} \cos(\omega t + \phi) \end{aligned}$$

where $\cos\phi = (\sqrt{6}+1)/(\sqrt{(8+2\sqrt{6})})$, $\sin\phi = -1/(\sqrt{(8+2\sqrt{6})}$. Hence, $\phi = -0.282$ radians.
(b) Let $x = 3^{-\frac{1}{2}} \cos \omega t - \sin(\omega t + \frac{1}{4}\pi)$. Then, as in (a),

$$\begin{aligned} x &= 3^{\frac{1}{2}} \cos \omega t - \cos(\omega t - \frac{1}{4}\pi) = (3^{\frac{1}{2}} - 2^{-\frac{1}{2}}) \cos \omega t - 2^{-\frac{1}{2}} \sin \omega t \\ &= (\sqrt{6}-1) 2^{-\frac{1}{2}} \cos \omega t - 2^{-\frac{1}{2}} \sin \omega t \\ &= \sqrt{(4-\sqrt{6})} \cos(\omega t + \phi) \end{aligned}$$

where $\cos\phi = (\sqrt{6}-1)/(\sqrt{(8-2\sqrt{6})})$, $\sin\phi = 1/(\sqrt{(8-2\sqrt{6})})$. Hence, $\phi = 0.604$ radians.
(c) Let $x = -3^{\frac{1}{2}} \cos \omega t + \sin(\omega t + \frac{1}{4}\pi)$. Then

$$\begin{aligned} x &= -3^{\frac{1}{2}} \cos \omega t + 2^{-\frac{1}{2}} \sin \omega t + 2^{-\frac{1}{2}} \cos \omega t = -(\sqrt{6}-1) 2^{-\frac{1}{2}} \cos \omega t + 2^{-\frac{1}{2}} \sin \omega t \\ &= \sqrt{(4-\sqrt{6})} \cos(\omega t + \phi), \end{aligned}$$

where $\cos\phi = -(\sqrt{6}-1)/\sqrt{(8-2\sqrt{6})}$, $\sin\phi = -1/\sqrt{(8-2\sqrt{6})}$. Hence, $\phi = -2.538$ radians.
(d) Let $x = -3^{\frac{1}{2}} \cos \omega t - \sin(\omega t + \frac{1}{4}\pi)$. Then, as in (a)

$$x = -(\sqrt{6}+1) 2^{-\frac{1}{2}} \cos \omega t - 2^{-\frac{1}{2}} \sin \omega t = \sqrt{(4+\sqrt{6})} \cos(\omega t + \phi)$$

where $\cos\phi = -(\sqrt{6}+1)/(\sqrt{(8+2\sqrt{6})})$, $\sin\phi = 1/\sqrt{(8+2\sqrt{6})}$. Hence, $\phi = 2.860$ radians.

20.5. (a) For $x(t) = C\mathrm{e}^{-kt} \cos(\omega t + \phi)$, the first derivative is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -C\mathrm{e}^{-kt}[k \cos(\omega t + \phi) + \omega \sin(\omega t + \phi)].$$

The maxima and minima occur where $\mathrm{d}x/\mathrm{d}t = 0$, that is, where t satisfies

$$\tan(\omega t + \phi) = -\frac{k}{\omega}.$$

Denoting these times by T_N, they are given by

$$\omega T_N + \phi = -\arctan\left[\frac{k}{\omega}\right] + N\pi,$$

where N is any integer.
(b) If

$$\tan(\omega t + \phi) = -\frac{k}{\omega},$$

then

$$\cos(\omega T_N + \phi) = \frac{\omega}{\sqrt{(\omega^2 + k^2)}}, \ \sin(\omega T_N + \phi) = -\frac{k}{\sqrt{(\omega^2 + k^2)}},$$

if N is even, and

$$\cos(\omega T_N + \phi) = -\frac{\omega}{\sqrt{(\omega^2 + k^2)}}, \ \sin(\omega T_N + \phi) = \frac{k}{\sqrt{(\omega^2 + k^2)}},$$

if N is odd. Hnece

$$x(T_N) = C\mathrm{e}^{-kT_N}\cos(\omega T_N + \phi) = \frac{(-1)^N \omega C \mathrm{e}^{-kT_N}}{(\omega^2 + k^2)^{\frac{1}{2}}}.$$

20.6. From Example 20.4, $Q_c = B\mathrm{e}^{-4t}\cos(\omega t + \phi)$, where $B = 0.9851$, $\omega = 99.92$ and $\phi = 2.777$. The time constant of Q_c is $\frac{1}{4}$.
(b) With $g(x)$ containing non-exponential terms only, the decay of x, which tends to zero with t, is controlled by $h(t) = \mathrm{e}^{-t/T}$. Let $t = \tau + T\ln 2$. Then

$$h(\tau + T\ln 2) = \exp[-(\tau + T\ln 2)/T] = \exp[-\tau - \ln 2] = \tfrac{1}{2}h(\tau).$$

Hence the exponential factor halves in every interval of duration $T\ln 2$.

20.7. Heavy damping: $x'' + 2kx' + \omega^2 x = 0$. The characteristic equation is

$$m^2 + 2km + \omega^2 x = 0,$$

which has the roots

$$m_1 = -k + \sqrt{(k^2 - \omega^2)}, \quad m_2 = -k - \sqrt{(k^2 - \omega^2)}.$$

For an overdamped oscillation (heavy damping), the friction term is large and satisfies the inequality $k^2 > \omega^2$, in which case both roots are real and negative. The general solution is

$$A\exp[\{-k + \sqrt{(k^2 - \omega^2)}\}t] + B\exp[\{-k - \sqrt{(k^2 - \omega^2)}\}t].$$

The solution contains no oscillatory terms, and is the sum of two exponentially decaying terms.

20.8. $x'' + 10x' + 24x = 0$. The characteristic equation is

$$m^2 + 10m + 24 = 0, \text{ which has the solutions } m_1 = -6, \ m_2 = -4.$$

This is an overdamped case (Problem 20.7) with general solution

$$x = A\mathrm{e}^{-6t} + B\mathrm{e}^{-4t}.$$

The initial conditions are $x(0) = -3$, $x'(0) = 20$ lead to the equations

$$A + B = -3, \quad -6A - 4B = 20,$$

which have the solution $A = -4$, $B = 1$. Hence the required solution is

$$x = -4\mathrm{e}^{-6t} + \mathrm{e}^{-4t}.$$

The solution crosses the t axis where

$$-4\mathrm{e}^{-6t} + \mathrm{e}^{-4t} = 0, \text{ or } \mathrm{e}^{-2t} = \tfrac{1}{4}.$$

By taking the logarithm of this equation we obtain the uinque solution $t = \ln 2$.

20.9. Critical damping: $x'' + 2kx' + \omega^2 x = 0$ ($k^2 = \omega^2$). The characteristic equation is

$$m^2 + 2km + k^2 x = 0,$$

which has the repeated root $m = -k$. In this special case the general solution is

$$x = (A + Bt)\mathrm{e}^{-kt}.$$

20.10. $x'' + 2kx' + \omega_0^2 x = K\cos\omega t$, $k = 0.5$, $\omega_0 = 6$, $K = 10$.
(a) The period of the free oscillation is

$$\frac{2\pi}{\sqrt{[\omega_0^2 - k^2]}} = \frac{2\pi}{\sqrt{[36 - 0.25]}} = 1.051.$$

(b) From (20.14), the amplitude of the forced oscillation is

$$A = \frac{K}{\sqrt{[(\omega_0^2 - \omega^2)^2 + 4k^2\omega^2]}} = \frac{10}{\sqrt{[1296 - 71\omega^2 + \omega^4]}},$$

and the phase Φ is the polar angle of the point

$$(\omega_0^2 - \omega^2, -2k\omega) = (36 - \omega^2, -\omega).$$

We can write

$$\Phi = -\arctan\left[\frac{\omega}{36 - \omega^2}\right].$$

(c) From (20.16) the resonant frequency is

$$\omega = \sqrt{[\omega_0^2 - 2k^2]} = \sqrt{[36 - 0.5]} = 5.958.$$

(d) The curves of amplitude and phase against ω in the range $4 \le \omega \le 8$ are shown in Figure 14. The peak in the first curve occurs at the resonant frequency.

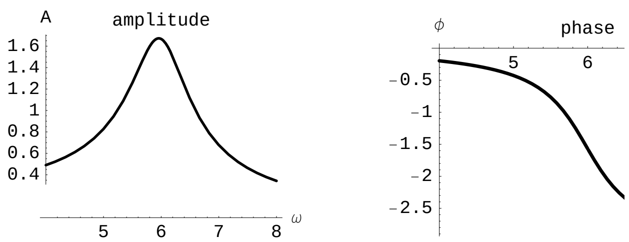

Figure 14: Problem 20.10

20.11. The equation of motion is

$$x'' + \frac{2ax(g + 2ax'^2)}{1 + 4a^2x^2} = 0.$$

The particle is in equilibrium at $x = 0$. For small x, x'^2 and x^2 can be ignored in the equation to leave the linearized equation

$$x'' + 2agx = 0.$$

The period of this simple harmonic motion is $2\pi/\sqrt{[2ag]}$.

20.12. The equation of motion is

$$x'' + \frac{2ax(2ax'^2 - g)}{1 + 4a^2x^2} = 0.$$

The particle is in equilibrium at $x = 0$. For small x, x'^2 and x^2 can be neglected in the equation to leave the linearized equation

$$x'' - 2agx = 0.$$

The general solution is

$$x = Ae^{-\sqrt{2ag}t} + Be^{\sqrt{2ag}t}.$$

Unless $B = 0$ (which implies a severe restriction on the initial conditions) the particle will move away from its initial position.

20.13. Displacement $x(t)$ satisfies

$$x'' + 4\left[x - \frac{2}{3 - x}\right] = 0.$$

(a) The system is in equilibrium is $x'' = 0$ which occurs where

$$x(3 - x) - 2 = 0 \text{ or } (x - 2)(x - 1) = 0.$$

There are two positions of equilibrium: $x = 1$ and $= 2$.

(b) Let $x = 1 + u$. Then the differential equation in terms of u becomes

$$u'' + 4\left[(1 + u) - \frac{2}{2 - u}\right] = u'' - 4u\left[\frac{u - 1}{2 - u}\right] = 0.$$

Let $x = 2 + v$. Then the equation becomes

$$v'' + \left[(2 + v) - \frac{2}{1 - v}\right] = v'' - 4v\left[\frac{v + 1}{1 - v}\right] = 0.$$

(c) For small $|u|$, u satisfies the linearized equation $u'' + 2u = 0$, and for small $|v|$, v satisfies $v'' - 4v = 0$.

(d) Near $x = 1$, u satisfies $u'' + 2u = 0$, which has the general oscillatory solution

$$u = A\cos\sqrt{2}t + B\sin\sqrt{2}t.$$

Near $x = 2$, v satisfies $v'' - 4v = 0$ which has the exponential unstable solution

$$v = Ae^{2t} + Be^{-2t}.$$

20.14. The particle has the equation of motion

$$\frac{d^2u}{d\theta^2} + u - \frac{\gamma}{H^2}u^{\alpha-2} = 0,$$

where $u = r^{-1}$. Let $u = u_0$, where u_0 is a constant. Then

$$0 + u_0 - \frac{\gamma}{H^2}u_0^{\alpha-2} = 0.$$

Hence, provided $\alpha \neq 3$,

$$u_0 = \left(\frac{\gamma}{H^2}\right)^{1/(3-\alpha)}, \qquad \text{(i)}$$

which generates a circular orbit of radius $r_0 = 1/u_0$.

Let $u = u_0 + x$. Then x satisfies

$$x'' + u_0 + x - \frac{\gamma}{H^2}(u_0 + x)^{\alpha-2} = 0.$$

Using the binomial expansion given:

$$x'' + u_0 + x - \frac{\gamma}{H^2}u_0^{\alpha-2}\left[1 + (\alpha - 2)\right] \approx 0.$$

Hence using (i) above to eliminate u_0, the linearized equation for x is

$$x'' + (3 - \alpha)x = 0.$$

The solutions are oscillatory (and therefore bounded) if $\alpha < 3$, which means that a disturbed orbit remains close to the circle. If $\alpha \geq 3$ then the disturbed orbit will diverge from the circle. In the gravitational case, $\alpha = 2$, the inverse square law.

20.15. The forced amplitude for the forced linear oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = K\cos\omega t.$$

is (see (20.16))

$$A = \frac{K}{[(\omega_0^2 - \omega^2)^2 + 4k^2\omega^2]^{\frac{1}{2}}}.$$

Assuming that k and ω_0 are given, resonance will occur when A as a function of ω takes its maximum value, or where

$$g(\omega) = (\omega_0^2 - \omega^2)^2 + 4k^2$$

is a minimum. Thus

$$\frac{\mathrm{d}g(\omega)}{\mathrm{d}\omega} = -4\omega(\omega_0^2 - \omega^2) + 8k^2\omega = 0$$

where $\omega^2 = \omega_0^2 - 2k^2$, which gives the resonant frequency. Substitute this frequency back into A to give the resonant amplitude

$$\frac{K}{[4k^4 + 4k^2(\omega_0^2 - 2k^2)]^{\frac{1}{2}}} = \frac{K}{2k(\omega_0^2 - k^2)^{\frac{1}{2}}}.$$

20.16. (a) Given the wavelength $\lambda = 1.2$m and the period $T = 1/250$ s, the speed of sound is

$$v = \lambda f = \frac{\lambda}{T} = 1.2 \times 250 = 30\text{ms}^{-1}.$$

(b) Given the tuning frequency $f = 100\text{MHz} = 100 \times 10^6\text{Hz} = 10^8\text{s}^{-1}$ and electromagnetic wave speed $v = 3 \times 10^8\text{s}^{-1}$. Hence the wavelength

$$\lambda = \frac{v}{f} = \frac{3 \times 10^8}{10^8} = 3\text{m}.$$

20.17. Given

$$u(t, x) = A\cos\left[\left(\frac{2\pi t}{T}\right) + \phi\right]\cos\left[\left(\frac{2\pi z}{\lambda}\right) + \alpha\right],$$

use the identity $\cos A\cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B)$ so that

$$u(t, x) = \frac{1}{2}A\cos\left[2\pi\left(\frac{t}{T} - \frac{z}{\lambda}\right) + (\phi - \alpha)\right] + \frac{1}{2}A\cos\left[2\pi\left(\frac{t}{T} + \frac{z}{\lambda}\right) + (\phi + \alpha)\right].$$

The terms

$$\left(\frac{t}{T} \pm \frac{z}{\lambda}\right)$$

indicate waves travelling in opposite directions.

20.18. $u = A\cos(\omega t - kz + \phi)$.
(i) By (20.25a), $u = \cos\left[2\pi\left(\frac{t}{T} - \frac{z}{\lambda}\right) + \phi\right]$.
(ii) By (20.28), $u = \cos\left[\omega\left(t - \frac{z}{v}\right) + \phi\right]$.

20.19. $u = \cos(\omega t - kz + \phi)$. Apply the identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

with, for example, $A = \omega t$, $B = kz - \phi$. Then

$$\begin{aligned} u &= \cos\omega t\cos(kz - \phi) + \sin\omega t\sin(kz - \phi) \\ &= \cos\omega t\cos(kz - \phi) + \sin\omega t\cos(kz - \phi - \frac{1}{2}\pi), \end{aligned}$$

which matches the form of (20.24), since $\omega = 2\pi/T$ and $k = 2\pi/\lambda$.

20.20. Given $u(t, z) = A\cos(4500t - 3z)$, $(\omega 4500,\ k = 3)$,
phase velocity, $v = \omega/k = 1500$ from (20.28);
frequency, $f = \omega/(2\pi) = 4500/(2\pi) \approx 716.2$ from (20.25b);
wavelength, $\lambda = 2\pi/k = 2\pi/3 \approx 2.09$.

20.21. From (20.31a), the plane wave in direction $\hat{\mathbf{s}}$ is

$$u(t, \mathbf{r}) = A\cos(\omega t - k\hat{\mathbf{s}}\cdot\mathbf{r} + \phi).$$

$\hat{\mathbf{s}}$ is a unit vector in the direction of $\mathbf{s} = (1, 1, 1)$, that is, $\hat{\mathbf{s}} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Hence

$$\hat{\mathbf{s}}\cdot\mathbf{r} = \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)\cdot(x, y, z) = \tfrac{1}{\sqrt{3}}(x + y + z),$$

where (x, y, z) are the original coordinates. Therefore

$$u(t, \mathbf{r}) = A\cos[\omega t - \tfrac{1}{\sqrt{3}}(x + y + z) + \phi].$$

20.22. (a) $\cos\omega_1 t + \cos\omega_2 t$ is periodic if, and only if, there exists a number T (the period) such that for all values of t

$$\cos\omega_1 t + \cos\omega_2 t = \cos\omega_1(t + T) + \cos\omega_2(t + T),$$

which is equivalent to the requirement that $\omega_1 T$ and $\omega_2 T$ are multiples of 2π. In that case $\omega_1 T = p2\pi$ and $\omega_2 T = q2\pi$ for some integer values of p and q, or

$$\frac{\omega_1}{\omega_2} = \frac{p}{q},$$

where p and q are integers. When ω_1/ω_2 is rational, the smallest period $T = 2\pi p/\omega_1$ or $2\pi q/\omega_2$ is got by reducing p/q to its lowest terms.
(b) Let

$$u = u_1 + u_2 = \cos 10t + \cos 13.1t.$$

Here, $\omega_1 = 10$ and $\omega_2 = 13.1$;

$$\frac{\omega_1}{\omega_2} = \frac{10}{13.1} = \frac{p}{q},$$

where $p = 100$ and $q = 131$. The exact period T of u is given by

$$T = \frac{2\pi p}{\omega_1} = 2\pi \frac{100}{10} = 2\pi \approx 62.8 \left(\text{ or } \frac{2\pi q}{\omega_2} = 2\pi \frac{131}{13.1} \right).$$

The periods of u_1 and u_2 are $2\pi/10 \approx 0.62$ and $2\pi/13.1 \approx 0.48$ respectively. The period of the beats is obtained from (20.36): it is

$$T_B = \frac{1}{2} \times \frac{2\pi}{\frac{1}{2}\Delta\omega} = \frac{2\pi}{\omega_2 - \omega_1} = \frac{2\pi}{3.1} \approx 2.02.$$

So $u_1 + u_2$ has a period of about 31 beats long.

20.23. (a) By expanding the cosines by Appendix B(b), and collecting the coefficients of $\cos\omega t$ and $\sin\omega t$ we obtain

$$u = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) = \alpha_1 \cos\omega t - \alpha_2 \sin\omega t, \qquad \text{(i)}$$

where

$$\alpha_1 = A_1 \cos\phi_1 + A_2 \cos\phi_2, \quad \alpha_2 = A_1 \sin\phi_1 + A_2 \sin\phi_2.$$

Following a similar procedure to that giving (1.18), put $R = \sqrt{(\alpha_1^2 + \alpha_2^2)}$, and choose a constant ϕ such that

$$\cos\phi = \frac{\alpha_1}{R}, \qquad \sin\phi = \frac{\alpha_2}{R}.$$

Then (i) becomes $u = R\cos(\omega t + \phi)$, where the constants R and ϕ are determined as above.

(b) By the same procedure, with $\omega t - kz$ in place of ωt, we obtain

$$u = R\cos(\omega t - kz + \phi).$$

20.24. (Reflection, phase and amplitude unchanged) Put

$$\begin{aligned} u &= A\cos(\omega t - kz + \phi) + A\cos(\omega t + kz + \phi) \\ &= 2A\cos(\omega t + \phi)\cos kz, \text{ (by using Appendix B(d)).} \end{aligned}$$

(This represents a stationary wave.)

(b) *Phase change only:*

$$\begin{aligned} u &= A\cos(\omega t - kz + \phi_1) + A\cos(\omega t + kz + \phi_2) \\ &= 2A\cos(\omega t + \frac{1}{2}\{\phi_1 + \phi_2\})\cos(-kz + \frac{1}{2}\{\phi_1 - \phi_2\}) \end{aligned}$$

(This represents a standing wave.)

Amplitude change only:

$$\begin{aligned} u &= A_1[\cos(\omega t + \phi)\cos kz + \sin(\omega t + \phi)\sin kz] \\ &\qquad + A_2[\cos(\omega t + \phi)\cos kz - \sin(\omega t + \phi)\sin kz] \\ &= [(A_1 + A_2)\cos kz + (A_1 - A_2)\sin kz]\cos(\omega t + \phi) \end{aligned}$$

(This is a standing wave: the coefficient of $\cos(\omega t + \phi)$ takes the form $A\cos(kz + \alpha)$ where A and α are constants by the result eqn (1.18).)

20.25. Let

$$\begin{aligned} u &= A\cos(\omega_1 t - k_1 z) + A\cos(\omega_2 t - k_2 z) \\ &= 2A\cos[\frac{1}{2}(\omega_2 - \omega_1)t - \frac{1}{2}(k_2 - k_1)z]\cos[\frac{1}{2}(\omega_2 + \omega_1)t - \frac{1}{2}(k_2 + k_1)z] \qquad \text{(i)} \end{aligned}$$

(compare eqn (20.42). Here, the phase velocity v is a function of λ. Also $\omega = kv$ (eqn (20.28)) and $k = 2\pi/\lambda$ (eqn (20.25a)). The beat profile is

$$\pm 2A\cos[\tfrac{1}{2}\Delta\omega\, t - \tfrac{1}{2}\Delta k\, z],$$

where $\Delta\omega = \omega_2 - \omega_1$ and $\Delta k = k_2 - k_1$. The group velocity v_0 is defined as the limit of the beat velocity as the parameters $\Delta\omega$, Δk, etc approach zero.

The beat velocity is

$$\frac{\Delta\omega}{\Delta k} = \frac{\omega_2 - \omega_1}{k_2 - k_1}, \text{ (eqn (20.48)).}$$

In terms of λ and v we have

$$\begin{aligned} v_g &= \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{\mathrm{d}(kv)}{\mathrm{d}k} = \frac{\mathrm{d}(v/\lambda)}{\mathrm{d}(1/\lambda} \\ &= \frac{\mathrm{d}(v/\lambda)}{\mathrm{d}\lambda}\Big/\frac{\mathrm{d}(1/\lambda)}{\mathrm{d}\lambda} = \left(-\frac{v}{\lambda^2} + \frac{1}{\lambda}\frac{\mathrm{d}v}{\mathrm{d}\lambda}\right)\Big/\left(-\frac{1}{\lambda^2}\right) \\ &= v - \lambda\frac{\mathrm{d}v}{\mathrm{d}\lambda}. \end{aligned}$$

20.26. Put f in place of f_1, Δf in place of $(f_2 - f_1)$, and similarly with the other variables. Since $f_1\lambda_1 = v_1$ and $f_2\lambda_2 = v_2$, we have

$$(f + \Delta f)(\lambda + \Delta\lambda) = v + \Delta v.$$

Divide the left-hand side by $f\lambda$ and the right side by $v = f\lambda$. This gives the first identity

$$\left(1 + \frac{\Delta}{f}\right)\left(1 + \frac{\Delta\lambda}{\lambda}\right) = 1 + \frac{\Delta v}{v}.$$

From the identity

$$kv = \frac{2\pi}{\lambda}(\lambda f) = \frac{2\pi}{\lambda}\frac{\lambda\omega}{2\pi} = \omega$$

we obtain the second identity.

20.27. Taking the sound speed in air as approximately 300 ms^{-1}, we have $v = 300\mathrm{ms}^{-1}$ and $u = 10^5/3600 = 2.777\mathrm{ms}^{-1}$. The frequency heard on the approach is

$$\frac{350}{(1 - 2.777/300)} = 352.2\mathrm{H}$$

and the frequency on recession is

$$\frac{350}{(1 + 2.777/300)} = 346.8\mathrm{H}.$$

The frequency drop is 5.4H.

Chapter 21: Steady forced oscillations: phasors, impedance, transfer functions

21.1. (a) $\mathbf{X} = 2\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}$ or 2i.

(b) $\mathbf{X} = 2\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}$ in polar form, or $\mathbf{X} = -2\mathrm{i}$.

(c) $x = 3\sin\omega t = 3\cos(\omega t - \frac{1}{2}\pi)$. Therefore $\mathbf{X} = 3\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}$ or $-3\mathrm{i}$.

(d) $x = 4\cos(3t - \frac{1}{4}\pi + \frac{1}{2}\pi) = 4\cos(3t + \frac{1}{4}\pi)$. Therefore $\mathbf{X} = 4\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$ or $2\sqrt{2}(1 + \mathrm{i})$.

21.2. (a) $\mathbf{X} = 1 - \mathrm{i} = \sqrt{2}\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}$. Therefore, $x = \sqrt{2}\cos(\omega t - \frac{1}{4}\pi)$.

(b) $\mathbf{X} = 2\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}$, $x = 2\cos(\omega t + \frac{1}{2}\pi)$.

(c) $\mathbf{X} = 3e^{-\frac{1}{2}\pi i}$, $x = 3\cos(\omega t - \frac{1}{2}\pi)$.

(d) $\mathbf{X} = 2e^{-\frac{3}{4}\pi i}$, $x = 2\cos(\omega t - \frac{3}{4}\pi)$.

(e) $\mathbf{X} = 4e^{-\frac{5}{6}\pi i}$, $x = 4\cos(\omega t - \frac{5}{6}\pi)$.

(f) $\mathbf{X} = 2e^{-\frac{2}{3}\pi i}$, $x = 2\cos(\omega t - \frac{2}{3}\pi)$.

(g) $\mathbf{X} = \sqrt{5}e^{i \arctan 2}$, $x = \sqrt{5}\cos(\omega t + \arctan 2)$.

(h) $\mathbf{X} = \sqrt{5}e^{i(\arctan 2 + \frac{1}{2}\pi)}$, $x = \sqrt{5}\cos(\omega t + \arctan 2 + \frac{1}{2}\pi)$.

(i) $|\mathbf{X}| = 1$, $\arg \mathbf{X} = \arg(2+3i) - \arg(2-3i) = 2\arg(2+3i) = 2\arctan\frac{3}{2}$ $(= 1.97)$.

(j) $\mathbf{X} = -\frac{1}{3}i + 2i = \frac{5}{3}i = \frac{5}{3}e^{\frac{1}{2}\pi i}$, $x = \cos(\omega t + \frac{1}{2}\pi)$.

21.3. Use the addition principle (21.3).

(a) $x = -\cos 2t + \cos(2t - \frac{1}{4}\pi)$. Then

$$\mathbf{X} = 1 + e^{-\frac{1}{4}\pi i} = 1 + \cos\tfrac{1}{4}\pi - i\sin\tfrac{1}{4}\pi = 1 + \tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}.$$

Hence $|\mathbf{X}| = \sqrt{(2+\sqrt{2})}$, $\arg \mathbf{X} = \arctan\left(\frac{1}{1+\sqrt{2}}\right)$.

(b) $x = \cos 3t - \sin 3t = \cos 3t - \cos(3t - \frac{1}{2}\pi)$. Therefore

$$\mathbf{X} = 1 - e^{-\frac{1}{2}\pi i} = 1 + i; \quad |\mathbf{X}| = \sqrt{2}, \quad \arg \mathbf{X} = \tfrac{1}{4}\pi.$$

(c) $x = \sin 3t + 2\cos 3t = \cos(3t - \frac{1}{2}\pi) + 2\cos 3t$. Therefore

$$\mathbf{X} = e^{-\frac{1}{2}\pi i} + 2 = 2 - i,$$

and

$$|\mathbf{X}| = \sqrt{5}, \quad \arg \mathbf{X} = \arctan(-\tfrac{1}{2}) = \arctan\tfrac{1}{2}.$$

21.4. Use the addition principle (21.3).

(a) $x = -\cos 2t + \cos(2t + \frac{1}{4}\pi) + \cos(2t - \frac{1}{2}\pi)$. Therefore

$$\mathbf{X} = -1 + e^{\frac{1}{4}\pi i} + e^{-\frac{1}{2}\pi i} = -1 + \tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}} + i = (\tfrac{1}{\sqrt{2}} - 1) + (\tfrac{1}{\sqrt{2}} + 1)i.$$

(b) $x = \cos 1760t - 3\cos(1760t - \frac{1}{2}\pi) + \cos(1760t + \frac{1}{2}\pi)$. Therefore

$$\mathbf{X} = 1 - 3e^{-\frac{1}{2}\pi i} + e^{\frac{1}{2}\pi i} = 1 + 4i.$$

21.5. Let $\mathbf{X}$ be represented by the vector $\overline{OP}$; the coordinates of P are

$$c\cos(\omega t + \phi), c\sin(\omega t + \phi).$$

P therefore lies on a circle, centre O and radius c. If θ is the angular coordinate then $\theta = \omega t + \phi$, so that the angular velocity $d\theta/dt = \omega$. The projection of OP on the x axis is simply the x coordinate of P.

21.6. (a) $Z = R + 1/(i\omega C)$ (from (21.7)).

(b) $Z = R + i\omega L$.

(c) $Z = i\omega L + 1/(i\omega C) = i(\omega^2 L - 1)/(\omega C)$.

(d) $1/Z = (1/R) + [1/(1/i\omega C)]$, from (21.7). Therefore, $Z = R/(1 + i\omega RC)$.

(e) From (21.7),

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{i\omega L}.$$

Therefore, $Z = i\omega LR/(R + i\omega L)$.

(f)
$$\frac{1}{Z} = \frac{1}{\mathrm{i}\omega L} + \frac{1}{1/(\mathrm{i}\omega C)}.$$

Therefore, $Z = \mathrm{i}\omega L/(1-\omega^2 LC)$.

(g)
$$\frac{1}{Z} = \frac{1}{R+\mathrm{i}\omega L} + \frac{1}{1/\mathrm{i}\omega C}.$$

Therefore
$$Z = \frac{R+\mathrm{i}\omega L}{(1-\omega^2 LC)+\mathrm{i}\omega RC}.$$

(h)
$$\frac{1}{Z} = \frac{1}{R+1/(\mathrm{i}\omega C)} + \frac{1}{\mathrm{i}\omega L}.$$

Therefore
$$Z = \frac{\omega L(1+\mathrm{i}\omega RC)}{\omega RC + \mathrm{i}(\omega^2 LC - 1)}.$$

(i) $Z = R +$ (impedance of L and C in parallel). From (f) we obtain
$$Z = R + \frac{\mathrm{i}\omega L}{1-\omega^2 LC}.$$

(j)
$$\frac{1}{Z} = \frac{1}{R+\mathrm{i}\omega L} + \frac{1}{R}.$$

Therefore
$$Z = \frac{R(R+\mathrm{i}\omega L)}{2R+\mathrm{i}\omega L}.$$

(k) Z is given by
$$\frac{1}{Z} = \frac{1}{\mathrm{i}\omega L} + \frac{1}{1/(\mathrm{i}\omega C)} + \frac{1}{R}.$$

(l) (NB: the problem has been simplified in the 2003 reprint by deleting Z_5.) The circuit is equivalent to two parallel circuits connected in series. The impedance Z_L of the left-hand circuit is given by
$$\frac{1}{Z_L} = \frac{1}{Z_1} + \frac{1}{Z_3},$$
and the impedance Z_R of the right-hand circuit is given by
$$\frac{1}{Z_R} = \frac{1}{Z_2} + \frac{1}{Z_4}.$$
The impedance of the whole circuit is given by
$$Z = Z_L + Z_R = \frac{Z_1 Z_3}{Z_1+Z_3} + \frac{Z_2 Z_4}{Z_2+Z_4}.$$

21.7. The solutions for (21.6a,b,c,d) are given. The voltage $v(t)$ has phasor $\mathbf{V} = 2$. The corresponding current phasors $\mathbf{I}$ are given by $\mathbf{I} = \mathbf{V}/Z$, where Z is the complex impedance obtained in (21.6a,b,c,d).

(a)
$$\mathbf{I} = \frac{2}{R-\mathrm{i}/(\omega C)} = \frac{2\omega C(\omega CR+\mathrm{i})}{\omega^2 C^2 R^2+1}.$$

Therefore $|\mathbf{I}| = 2\omega C/(\omega^2 C^2 R^2 + 1)^{\frac{1}{2}}$, and the phase of $\mathbf{I}$ is $\arctan[1/(\omega RC)]$.

(b) $$\mathbf{I} = \frac{2}{R + \mathrm{i}\omega L} = \frac{2(R - \mathrm{i}\omega L)}{R^2 + \omega^2 L^2}.$$

Therefore $|\mathbf{I}| = 2/(R^2 + \omega^2 L^2)^{\frac{1}{2}}$, and phase is $\arctan(\omega L/R)$.

(c) $$\mathbf{I} = \frac{2}{\mathrm{i}\omega L + 1/(\mathrm{i}\omega C)} = \frac{2\mathrm{i}\omega C}{\omega^2 LC - 1}.$$

Therefore, $|\mathbf{I}| = 2\omega C/(\omega^4 C^2 L^2 + 1)^{\frac{1}{2}}$; phase is $\frac{1}{2}\pi$.

(d) $$\mathbf{I} = \frac{2}{R/(1 + \mathrm{i}\omega RC)} = \frac{2}{R}(1 + \mathrm{i}\omega RC).$$

Therefore $|\mathbf{I}| = 2(1 + \omega^2 R^2 C^R)^{\frac{1}{2}}/R$; phase of $\mathbf{I}$ is $\arctan(\omega RC)$.

21.8. (a) The complex impedance $Z = 3 + 3\mathrm{i} - \mathrm{i} = 3 + 2\mathrm{i}$, and the current phasor is therefore

$$\mathbf{I}_1 = \frac{\mathbf{V}_0}{Z} = \frac{V_0}{3 + 2\mathrm{i}}.$$

(The transfer impedance is $\mathbf{V}_0/\mathbf{I}_1 = Z$ in this case.) The voltage phasor

$$\mathbf{V}_1 = 3\mathrm{i}\mathbf{I}_1 = 3\frac{V_0 \mathrm{i}}{3 + 2\mathrm{i}},$$

so the voltage gain

$$\frac{\mathbf{V}_1}{\mathbf{V}_0} = \frac{3\mathrm{i}}{3 + 2\mathrm{i}} = \frac{1}{13}(2 + 3\mathrm{i}).$$

(b) Let I represent the current input at the terminals. The impedance Z_R of the right-hand, parallel, circuit is given by

$$\frac{1}{Z_R} = \frac{1}{-2\mathrm{i}} + \frac{1}{1 - \mathrm{i}} = \frac{1 - 3\mathrm{i}}{-2 - 2\mathrm{i}}, \qquad \text{(i)}$$

and the impedance Z of the whole circuit is given by

$$Z = 1 + \frac{-2 - 2\mathrm{i}}{1 - 3\mathrm{i}} = -\frac{1 + 5\mathrm{i}}{1 - 3\mathrm{i}} = \frac{2}{13}(3 - 2\mathrm{i}). \qquad \text{(ii)}$$

The current $\mathbf{I}_0$ delivered by $\mathbf{V}_0$ to the whole circuit is

$$\mathbf{I}_0 = \frac{\mathbf{V}_0}{Z} = \mathbf{V}_0 \frac{1 - 3\mathrm{i}}{1 + 5\mathrm{i}}.$$

The voltage drop over the unit resistance in series is therefore equal to

$$\mathbf{I}_0 = -\mathbf{V}_0 \frac{1 - 3\mathrm{i}}{1 + 5\mathrm{i}},$$

and the voltage over the right-hand circuit is

$$\mathbf{V}_1 = \mathbf{V}_0 + \mathbf{V}_0 \frac{1 - 3\mathrm{i}}{1 + 5\mathrm{i}} = \mathbf{V}_0 \frac{2 + 2\mathrm{i}}{1 + 5\mathrm{i}}.$$

Therefore the voltage gain

$$\frac{\mathbf{V}_1}{\mathbf{V}_0} = \frac{2 + 2\mathrm{i}}{1 + 5\mathrm{i}} = \frac{2}{13}(3 - 2\mathrm{i}).$$

The transfer impedance $\mathbf{V}_0/\mathbf{I}_1$ is given by

$$\frac{\mathbf{V}_0}{\mathbf{I}_1} = \frac{\mathbf{V}_0}{\mathbf{V}_1}\frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1+5\mathrm{i}}{2+2\mathrm{i}}(1-\mathrm{i})$$

(since $1/\mathbf{I}_1$ is the impedance $1-\mathrm{i}$ of the $\mathbf{I}_1$, $\mathbf{V}_1$ branch). Therefore

$$\frac{\mathbf{V}_0}{\mathbf{I}_1} = \frac{1}{2}(5-\mathrm{i}).$$

(c) Problem (c) has been deleted from the 2003 reprint.

21.9. The components of the phasors are given to one decimal place.

(a) $\cos 10t + 2\cos(10t + 0..3)$. The phasors are $(1, 0)$ and $(2\cos 0.3, 2\sin 0.3) = (1.9, 0.6)$. Resultant phasor has components $(2.9, 0.6)$.

(b) $\cos 10t + 2\sin(10t + 10.2) = \cos 10t - 2\cos(10t + 10.2 - \frac{1}{2}\pi)$. The phasors are $(1, 0)$ and $(1.4, -1.4)$, and the resultant has components $(3.9, -0.6)$

(c) $\cos 10t + 3\cos(10t - 0.2)$. the phasors are $(1, 0)$ and $(3\cos(-0.2), 3\sin(-0.2))$. The resultant has components $(3.9, -0.6)$.

(d) $\sin 20t - 3\cos(20t + 0.75) = -\cos(20t - \frac{1}{2}\pi) - 3\cos(20t + 0.75)$. The phasors are $(0, 1)$ and $(-3\cos 0.75, -3\sin 0.75) = (-2.2, -2.0)$. The resultant is $(-2.2, -1.0)$.

(e) $2\cos(50t + 0.4) + \sin(50t + 0.3) - 3\cos(50t - 0.5)$. The phasors are $(1.8, 0.8)$, $(-0.3, 1.0)$ and $(-2.6, 1.4)$. The resultant is $(-1.1, 3.2)$.

21.10. (a) Without loss of generality take $\phi = 0$ and the amplitude $A = 1$. Then

$$u(x, y, z, t) = \cos[\omega t - 3^{-\frac{1}{2}}k(x + y + z)].$$

In the x, y plane the field is

$$u(x, y, z, t) = \cos[\omega t - 3^{-\frac{1}{2}}k(x + y)].$$

(b) The combined wave takes the form

$$u(x, y, z, t) = \cos[\omega t - 3^{-\frac{1}{2}}k(x + y + z)] + \cos(\omega t - kz). \qquad (i)$$

On $z = 0$

$$u = \cos[\omega t - 3^{-\frac{1}{2}}k(x + y)] + \cos\omega t.$$

In terms of the corresponding phasors this becomes

$$\mathbf{U} = \mathrm{e}^{-\mathrm{i}k(x+y)/\sqrt{3}} + 1 = (1 + \cos[k(x + y)/\sqrt{3}]) + \mathrm{i}\sin[k(x + y)/\sqrt{3}].$$

The intensity of the wave on $z = 0$ is proportional to $|\mathbf{U}|^2$, where

$$|\mathbf{U}|^2 = 2 + 2\cos[k(x + y/\sqrt{3}].$$

The maxima of $|\mathbf{U}|^2$ (the fringes) occur where $k(x + y)/\sqrt{3} = n\pi$ and n is any integer; that is, along the 45° straight lines $x + y = \sqrt{3}n\pi/k$. They are spaced at equal distances $\sqrt{(3/2)}\pi/k$.

Chapter 22: Graphical, numerical, and other aspects of first-order equations

22.1. The lineal-element diagrams were obtained using *Mathematica*: the x and y ranges chosen are indicated by the graphs.

(a) $y' = -y$:
(b) $y' = x - y$.
(c) $y' = x/y$.
(d) $y' = xy$.

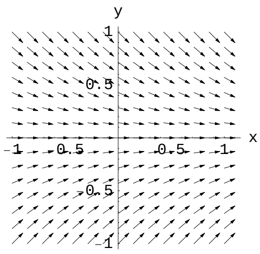

Figure 15: Problem 22.1(a)

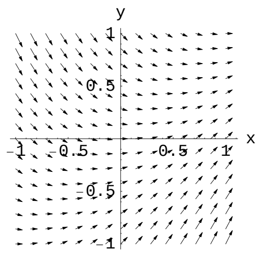

Figure 16: Problem 22.1(b)

(e) $y' = -y/x$
(f) $y' = y/x$.
(g) $y' = (x-1)y$.
(h) $y' = 1/(x^2 + y^2)$.
(i) $y' = 1/(x^2 + y^2 - 1)$.
(j) $y' = (1 - y^2)^{\frac{1}{2}}$.
(k) $y' = (y/x)^{\frac{1}{2}}$.

22.2. (a) $y' = -\frac{1}{2}y$, $y[0] = 1$, $0 \le x \le 2$. Euler's method for initial-value problems is given in (22.2). A *Mathematica* program for Euler's method is given below for this equation: the step-length is $h = 0.2$, and the number of steps $n = 10$.

Euler program

```
Clear[f, x, y, h]
f[x_, y_] = -y/2;
h = 0.2;
y[0] = 1;
x[n_] = n*h;
y[n_] := y[n] = y[n - 1] + h*f[x[n - 1], y[n - 1]];
euler = Table[x[i], y[i], {i, 0, 10}]
```

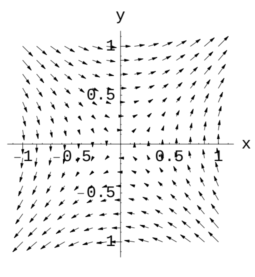

Figure 17: Problem 22.1(c)

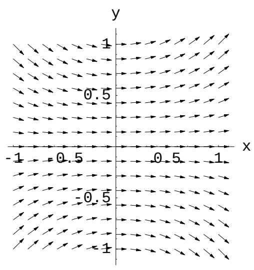

Figure 18: Problem 22.1(d)

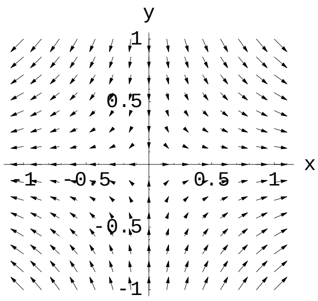

Figure 19: Problem 22.1(e)

To three decimal places the numerical solution gives

n	0	1	2	3	4	5	6	7	8	9	10
$x(n)$	0	0.2	0.4	0.6	0.8	01.0	1.2	1.4	1.6	1.8	2.0
$y(n)$	1	0.9	0.81	0.729	0.656	0.590	0.531	0.478	0.430	0.387	0.349

The exact solution is $y = \mathrm{e}^{-\frac{1}{2}x}$ which gives $y = 0.367\ldots$ at $x = 2$. Smaller values for h improve the accuracy. The figure shows the exact solution (the continuous curve) and the euler approximation (the dots).

(b) $y' = -x/y$, $y(-1) = -1$, $-1 \leq x \leq 1$. The Euler program is:

```
Clear[f, x, y, h]
f[x_, y_] = -x/y;
h = 0.1;
y[-1] = -1;
x[n_] = -1 + n*h;
y[n_] := y[n] = y[n - 1] + h*f[x[n - 1], y[n - 1]];
```

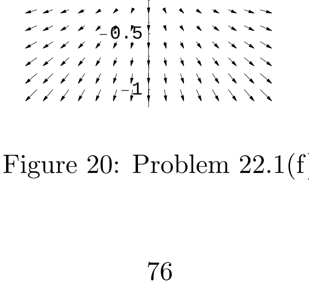

Figure 20: Problem 22.1(f)

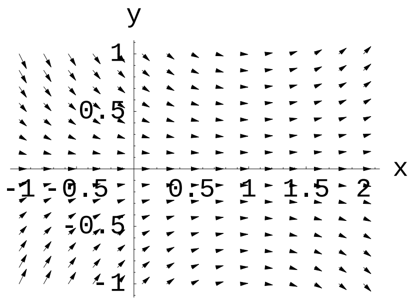

Figure 21: Problem 22.1(g)

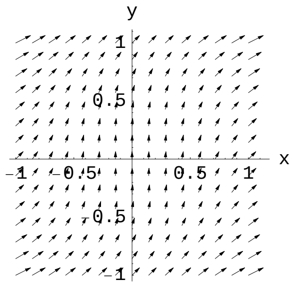

Figure 22: Problem 22.1(h)

euler = Table[x[i], y[-1 + i], i, 0, 20]

The figure shows the numerical approximation with the step-length $h = 0.1$ and the number of steps $n = 20$ compared with the exact solution $y = -(2 - x^2)^{\frac{1}{2}}$.

There is considerable divergence between the approximation and the exact solution. We need step-lengths of the order of $h = 0.01$ to achieve a close approximation. The Euler method can be applied to $-2 < x \leq -1$ by choosing negative values of h.
(c) $y' = (1 - y^2)^{\frac{1}{2}}$, $y(0) = 0$, $0 \leq x \leq \frac{1}{2}\pi$. The program in (a) can be adapted to this equation. A comparison between the exact solution $y = \sin x$ and the Euler approximation is shown for a step-length $h = \pi/20$.
22.3. A *Mathematica* program for Euler's method is given in Problem 22.2. Some typical solutions are shoen in each of thes cases.
(a) $y' = y(x+1)/[x(y+1)]$ for $-1 < x < 1$. The general solution is $ye^y = Axe^x$ (see Example 22.6). The solutions shown are for $A = -2, -1, -0.5, 0, 0.5, 1, 2$.
(b) $y' = 2y^{\frac{1}{2}}$ (see Example 22.7). Figure 30 shows a solution computed using Euler's method with initial condition $y(1) = 1$, $h = -0.01$: the computed solution ends near to $(0, 0)$. Other initial conditions are translated copies of this solution as shown in Figure 22.10(b).

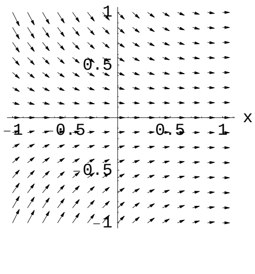

Figure 23: Problem 22.1(i): $y' = 1/(x^2 + y^2 - 1)$.

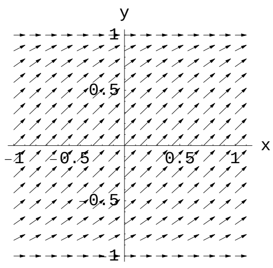

Figure 24: Problem 22.1(j)

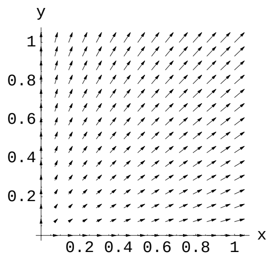

Figure 25: Problem 22.1(k)

(c) $y' = (y/x)^{\frac{1}{2}}$. A lineal-element diagram in $x, y > 0$ is shown in the figure for Problem 22.2(c). Two solution curves computed using Euler's method are shown in Figure 31 for initial values $y(0.1) = 0.2$ and $y(0.2) = 0.1$.

22.4. Separation of variables: these are equations of the form $\mathrm{d}y/\mathrm{d}x = g(x)h(y)$. The general solution is given by

$$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\mathrm{d} + C.$$

Remember that there often several ways of representing the solutions.
(a) $y' = x/y$. In this case $g(x) = x$ and $h(y) = 1/y$. Hence the solution is given by

$$\int y\mathrm{d}y = \int x\mathrm{d}x + C, \text{ or } \frac{1}{2}y^2 = \frac{1}{2}x^2 + C \text{ or } y^2 - x^2 = A,$$

a family of hyperbolas.
(b) $y' = 2x/y$. The solution is given by

$$\int y\mathrm{d}y = \int 2x\mathrm{d}x + C, \text{ or } \frac{1}{2}y^2 = x^2 + C.$$

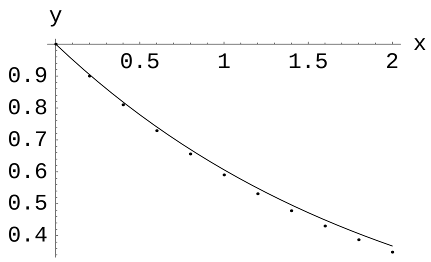

Figure 26: Problem 22.2(a)

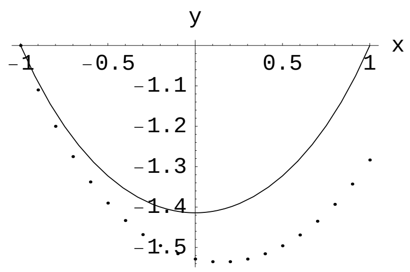

Figure 27: Problem 22.2(b)

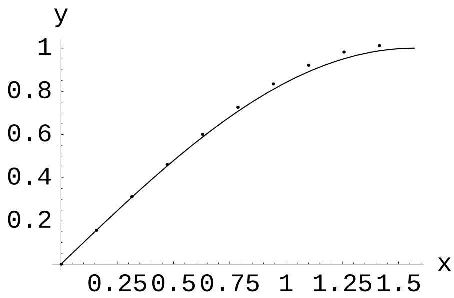

Figure 28: Problem 22.2(c)

(c) $y' = x/(y+2)$. The solution is given by

$$(y+2)^2 = x^2 + C.$$

(d) $y' = (x+3)/(y+2)$. The solution is

$$(y+2)^2 = (x+3)^2 + C.$$

(There are alternative answers such as $\frac{1}{2}y^2 + 2y = \frac{1}{2}x^2 + 3x + C$.)
(e) $y' = x^2/y^2$. The solution is given by

$$\int y^2 \mathrm{d}y = \int x^2 \mathrm{d}x + C, \text{ or } \frac{1}{3}y^3 = \frac{1}{3}x^3 + C.$$

(f) $y' = -x^2/y^2$. the general solution is

$$y^3 = -x^3 + C.$$

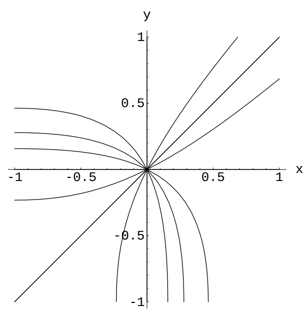

Figure 29: Problem 22.3(a)

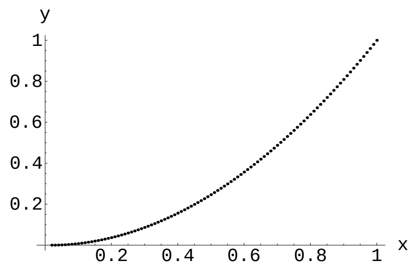

Figure 30: Problem 22.3(b)

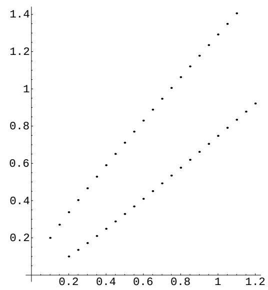

Figure 31: Problem 22.3(c)

(g) $y' = y^2/x^2$. The general solution is given by

$$\int \frac{\mathrm{d}y}{y^2} = \int \frac{\mathrm{d}x}{x^2} + C \text{ or } -\frac{1}{y} = -\frac{1}{x} + C.$$

This can be also expressed in the form

$$y = \frac{x}{1+Cx}.$$

(h) $y' = -y^2/x^2$. The general solution is

$$y = \frac{x}{Cx-1}.$$

(i) $2xy' = y^2$. Then

$$2\int \frac{\mathrm{d}y}{y^2} = \int \frac{\mathrm{d}x}{x}, \text{ or } \frac{-2}{y} = \ln|x| + C.$$

This can also be expressed in the form

$$y = \frac{-2}{C + \ln|x|}.$$

(j) $yy' + x = 1$. Then the general solution is given by

$$y^2 = 2x - x^2 + C.$$

(k) $\mathrm{d}x/\mathrm{d}t = 3t^2x^3$. The general solution is given by

$$x^2 = -\frac{1}{2t^3 + C}.$$

(l) $(\sin x)(\mathrm{d}x/\mathrm{d}t) = t$. Then separating the variables and integrating

$$\int \sin x \mathrm{d}x = \int t \mathrm{d}t + C$$

so that $-\cos x = \frac{1}{2}t^2 + C$.
(m) $\mathrm{e}^{x+y}(\mathrm{d}y/\mathrm{d}x) = 1$. The general solution is

$$\mathrm{e}^y = -\mathrm{e}^x + C.$$

(n) $(1 + x^2)(\mathrm{d}y/\mathrm{d}x) + (1 + y^2) = 0$, $y(0) = -1$. This is an initial-value problem. Separating the variables

$$\int \frac{\mathrm{d}y}{1+y^2} = -\frac{\mathrm{d}x}{1+x^2},$$

so that

$$\arctan y = -\arctan x + C.$$

Using the initial condition, $(-\arctan 1) = C$, or $C = -\frac{1}{4}\pi$. Hence

$$y = \tan(-\frac{1}{4}\pi - \arctan x) = \frac{1+x}{x-1}$$

(see Appendix B(b)).

22.5. $\mathrm{d}y/\mathrm{d}x = -x/y$. Consider

$$\int_2^x u\mathrm{d}u = -\int_1^y v\mathrm{d}v.$$

Differentiate this equation with respect to x:

$$x = -y\frac{\mathrm{d}y}{\mathrm{d}x},$$

by (15.20), which is the same as the differential equation. Also the integrals are both zero when $y = 1$ and $x == 2$, which agrees with the given condition.

The solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y), \quad y(a) = b$$

is

$$\int_a^x g(u)\mathrm{d}u = \int_b^y h(v)\mathrm{d}v.$$

22.6. The following comments and figures provide some help in plotting solutions, with warnings about spurious solutions.
(a) $x(\mathrm{d}y/\mathrm{d}x) = 2y^{\frac{1}{2}}$. Note that y must be positive. Separation of variables gives

$$y = (\ln|x| + C)^2.$$

The figure shows three curves in $x > 0$ for the stated values of C: note the special solution $y = 0$. Since $y^{\frac{1}{2}} > 0$, then $\mathrm{d}y/\mathrm{d}x > 0$ for $x > 0$, and $\mathrm{d}y/\mathrm{d}x < 0$ for $x < 0$ on solutions. Only portions of the curves will be solutions.

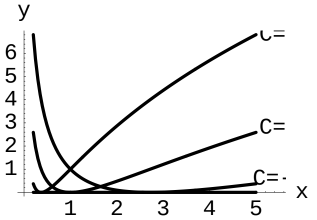

Figure 32: Problem 22.6(a)

(b) $\mathrm{d}y/\mathrm{d}x = xy^{\frac{1}{2}}$. Note that y must be positive. By separation of variables the general solution is

$$y = \frac{1}{16}(x^2 + C)^2,$$

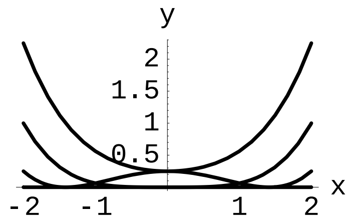

Figure 33: Problem 22.6(b)

and these are plotted for three values of C. However, for true solution curves, $\mathrm{d}y/\mathrm{d}x > 0$ for $x > 0$, and $\mathrm{d}y/\mathrm{d}x < 0$ for $x < 0$, so certain parts of the illustrated curves are spurious. Note also that that $y = 0$ is a special solution.
(c) $\mathrm{d}y/\mathrm{d}x = (1-y^2)^{\frac{1}{2}}$. Real solutions will be restricted to $-1 \le y \le 1$. Separating the variables

$$\int \frac{\mathrm{d}y}{(1-y^2)^{\frac{1}{2}}} = \int \mathrm{d}x + C, \text{ or } \arcsin y = x + C, \text{ or } y = \sin(x+C).$$

The figure shows 3 curves for $C = 0$ and $C = \pm 2$: the set of valid solutions is limited to the segments on which $\mathrm{d}y/\mathrm{d}x > 0$.

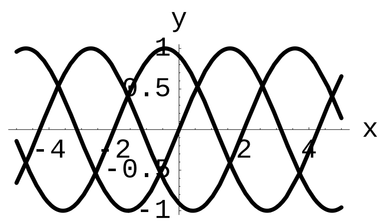

Figure 34: Problem 22.6(c)

(d) $x(\mathrm{d}y/\mathrm{d}x) = (1-y^2)^{\frac{1}{2}}$, $-1 \le y \le 1$. The general solution is

$$y = \sin(C + \ln|x|).$$

The figure shows three of these curves in $x > 0$, for $C = 0$ and $C = \pm 1$. Since $x > 0$ and $(1-y^2)^{\frac{1}{2}} > 0$, only those parts of the solutions which have positive slope are solution curves: the other sections of curves are spurious. Similar remarks apply for $x < 0$.

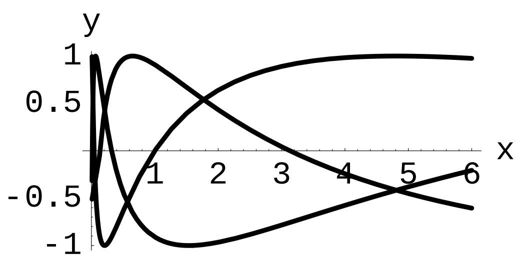

Figure 35: Problem 22.6(d)

22.7. Differential forms are given in Section 22.4.
(a) $\mathrm{d}y/\mathrm{d}x = (2x-y)/(x+2y)$. In differential form this can be written as

$$0 = (2x-y)\mathrm{d}x - (x+2y)\mathrm{d}y = \mathrm{d}(x^2) - \mathrm{d}(xy) - \mathrm{d}(y^2) = \mathrm{d}(x^2 - xy - y^2).$$

Therefore the general solution is given by

$$x^2 - xy - y^2 = C.$$

Check for solutions $y = mx$ in the differential equation: we obtain

$$m = \frac{2-m}{1+2m}, \text{ or } m^2 + m - 1 = 0,$$

which has roots $m = \frac{1}{2}(-1 \pm \sqrt{5})$. There are two straight-line solutions. They can also be obtained by putting $C = 0$ above.
(b) $\mathrm{d}y/\mathrm{d}x = y/(y^2 - x)$. In differential form the equation becomes

$$0 = y\mathrm{d}x - (y^2 - x)\mathrm{d}y = \mathrm{d}(xy) - \mathrm{d}(\tfrac{1}{3}y^3) = \mathrm{d}(xy - \tfrac{1}{3}y^3).$$

Therefore the general solution is given by $xy - \frac{1}{3}y^3 = C$.
(c) $\mathrm{d}y/\mathrm{d}x = (x^2 - y)/(x + y)$. In differential form this becomes

$$0 = (x^2 - y)\mathrm{d}x - (x + y)\mathrm{d}y = \mathrm{d}(\tfrac{1}{3}x^3) - \mathrm{d}(xy) - \mathrm{d}(\tfrac{1}{2}y^2) = \mathrm{d}(\tfrac{1}{3}x^3 - xy - \tfrac{1}{2}y^2).$$

Therefore the general solution is given by $\frac{1}{3}x^3 - xy - \frac{1}{2}y^2 = C$.
(d) $\mathrm{d}y/\mathrm{d}x = (2x - y)/(x - 2y)$. The general solution is given by

$$x^2 - xy + y^2 = C.$$

(e) $\mathrm{d}y\mathrm{d}x = (x - 2xy)/(x^2 - y)$. In differential form the equation becomes

$$0 = (x - 2xy)\mathrm{d}x - (x^2 - y)\mathrm{d}y = \mathrm{d}(\tfrac{1}{2}x^2 - x^2y + \tfrac{1}{2}y^2).$$

Therefore the general solution is given by

$$x^2 - 2x^2y + y^2 = C.$$

(f) $\mathrm{d}y/\mathrm{d}x = 3x^2/(3y^2 + 1)$. In differential form

$$0 = 3x^2\mathrm{d}x - (3y^2 + 1)\mathrm{d}y = \mathrm{d}(x^3 - y^3 - y).$$

Therefore the general solution is given by $x^3 - y^3 - y = C$. (Note that the equation is also separable.)
(g) $\mathrm{d}y/\mathrm{d} + [2xy/(x^2 - 1)] = 0$. In differential form

$$0 = 2xy\mathrm{d}x + (x^2 - 1)\mathrm{d}y = \mathrm{d}(x^2y - y).$$

Hence the general solution is $y(x^2 - 1) = C$.
(h) $(1 - \sin y)(\mathrm{d}y/\mathrm{d}x) + \cos x = 0$. In differential form x and y satisfy

$$0 = \cos x\mathrm{d}x + (1 - \sin y)\mathrm{d}y = \mathrm{d}(\sin x + y + \cos y).$$

Therefore the general solution is $\sin x + y + \cos x = C$. (This is another separable equation.)
(i) $(1 + 3\mathrm{e}^{3y})(\mathrm{d}y/\mathrm{d}x) = 2\mathrm{e}^{2x} - 1$. In differential form

$$0 = (2\mathrm{e}^{2x} - 1)\mathrm{d}x - (1 + 3\mathrm{e}^{3y})\mathrm{d}y = \mathrm{d}(\mathrm{e}^{2x} - x - y - \mathrm{e}^{3y}).$$

Hence the general solution is

$$\mathrm{e}^{2x} - x - y - \mathrm{e}^{3y} = C.$$

(j) $(\mathrm{e}^{x+y} + 1)(\mathrm{d}y/\mathrm{d}x) + (\mathrm{e}^{x+y} - 1) = 0$. In differential form

$$0 = (\mathrm{e}^{x+y} - 1)\mathrm{d}x + (\mathrm{e}^{x+y} + 1)\mathrm{d}y = \mathrm{d}(\mathrm{e}^{x+y} - x + y).$$

Therefore the general solution is given by

$$e^{x+y} - x + y = C.$$

(k) $\mathrm{d}y/\mathrm{d}x = (1 + \cos x \sin y)/(1 - \sin x \cos y)$. In differential form

$$0 = (1 + \cos x \sin y)\mathrm{d}x - (1 - \sin x \cos y)\mathrm{d}y = \mathrm{d}(x + \sin x \sin y - y).$$

Hence the general solution is given by

$$x + \sin x \sin y - y = C.$$

22.8. (a) $\mathrm{d}y/\mathrm{d}x = [y(y - 2x)/x(x - 2y)]$. In differential form the equation can be written as

$$0 = (y^2 - 2xy)\mathrm{d}x - (x^2 - 2xy)\mathrm{d}y = \mathrm{d}(y^2x - x^2y).$$

Therefore the general solution is given by

$$y^2x - x^2y = C.$$

(b) $\mathrm{d}y/\mathrm{d}x = [y(1 - x^2)]/[x(1 + x^2)]$. In differential form

$$0 = (y - yx^2)\mathrm{d}x - (x + x^3)\mathrm{d}y.$$

This is not a perfect differential, so an integrating factor is needed; say $1/x^2$. Multiply through by $1/x^2$:

$$0 = \left(\frac{y}{x^2} - y\right)\mathrm{d}x - \left(\frac{1}{x} + x\right)\mathrm{d}y = -\mathrm{d}\left(\frac{y}{x}\right) - \mathrm{d}(xy).$$

Therefore the general solution is given by

$$\frac{y}{x} + xy = C.$$

(c) $\mathrm{d}y/\mathrm{d}x = y^2/(y^2 - 1)$. In differential form

$$0 = \mathrm{d}x - (1 - y^{-2})\mathrm{d}y = \mathrm{d}(x - y - y^{-1}).$$

Therefore the general solution is given by

$$x - y - \frac{1}{y} = C.$$

(d) $\mathrm{d}y/\mathrm{d}x = [y(y - 1)]/[y^2 - x]$. In differential form

$$0 = y(y - 1)\mathrm{d}x - (y^2 - x)\mathrm{d}y.$$

An integrating factor is required. Multiply through by $1/y^2$:

$$0 = \left(1 - \frac{1}{y}\right)\mathrm{d}x - \left(1 - \frac{x}{y^2}\right)\mathrm{d}y = \mathrm{d}\left(x - y - \frac{x}{y}\right).$$

(e) $\mathrm{d}y/\mathrm{d}x = [y(x^2 + y^2 - y)]/[x(x^2 + y^2)]$. In differential form

$$0 = y(x^2 + y^2 - y)\mathrm{d}x - x(x^2 + y^2)\mathrm{d}y,$$

which requires an integrating factor. Multiply through by $1/(x^2y^2)$:

$$0 = \left(\frac{1}{y} + \frac{y}{x^2} - \frac{1}{x^2}\right)\mathrm{d}x - \left(\frac{x}{y^2} + \frac{1}{x}\right)\mathrm{d}y = \mathrm{d}\left(\frac{x}{y} - \frac{y}{x} + \frac{1}{x}\right).$$

Therefore the general solution is given by

$$\frac{x}{y} - \frac{y}{x} + \frac{1}{x} = C.$$

(f) $\mathrm{d}y/\mathrm{d}x = [y(x^2 - y)]/[x(x^3 + y)]$. In differential form

$$\begin{aligned} 0 &= y(x^3 - y)\mathrm{d}x - x(x^3 + y)\mathrm{d}y = x^3y^2\left(\frac{1}{y}\mathrm{d}x - \frac{x}{y^2}\mathrm{d}y\right) - y(y\mathrm{d}x + x\mathrm{d}y) \\ &= x^3y^2\mathrm{d}\left(\frac{x}{y}\right) - y\mathrm{d}(xy). \end{aligned}$$

We now perform a change of variable: $u = xy$ and $v = x/y$, so that the differential form becomes

$$v\mathrm{d}v = \frac{1}{u^2}\mathrm{d}u,$$

which has the general solution

$$\frac{1}{2}v^2 = -\frac{1}{u} + C, \text{ or } \frac{1}{2}\left(\frac{x}{y}\right)^2 = -\frac{1}{xy} + C.$$

22.9. The logistic equation is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP - bP^2.$$

This is a separable equation: therefore

$$\int \frac{\mathrm{d}P}{P(a - bP)} = \int \mathrm{d}t + C, \text{ or } \frac{1}{a}\int\left(\frac{1}{P} + \frac{b}{a - bP}\right)\mathrm{d}P = t + C.$$

Hence

$$\ln P - \ln|a - bP| = at + aC.$$

Solve this equation for P. If $a - bP > 0$, then, taking exponentials of both sides of the equation:

$$\frac{P}{a - bP} = B\mathrm{e}^{at}, \text{ or } P = \frac{Ba}{Bb + \mathrm{e}^{-at}}.$$

If $P(0) = P_0$, then the solution can be written as

$$P = \frac{aP_0}{bP_0 + (a - bP_0)\mathrm{e}^{-at}}.$$

It can be checked that the same solution also holds for $a - bP < 0$. Note also that the equation has special solutions $P = a/b$ and $P = 0$.

To reduce the number of parameters, write the equation in the form

$$P = \frac{\kappa P_0}{P_0 + (\kappa - P_0)\mathrm{e}^{-\tau}},$$

where $\kappa = a/b$ and $\tau = at$. In Figure 36 $\kappa = 1$ and $P_0 = 0.5$, 1, 2: the solutions $P = 0$ and $P = 1$ are also shown.

22.10. For small P, $\mathrm{d}P/\mathrm{d}t \approx aP$. With $P(0) = 10$, it follows that

$$a \approx \frac{P'(0)}{P(0)} = \frac{150}{100} = 1.5$$

Ultimately for large t, $P \to 25000$. Hence

$$b = a/25000 = 1.5/25000 = 0.00006.$$

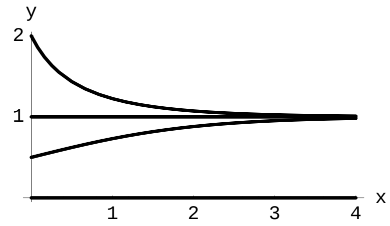

Figure 36: Problem 22.9

With t measured in days,

$$\begin{aligned} P(t) &= \frac{aP_0}{bP_0 + (a - bP_0)\mathrm{e}^{-at}} \\ &= \frac{15}{0.0006 + (1.5 - 0.0006)\mathrm{e}^{-1.5t}} = \frac{15}{0.0006 + 1.5\mathrm{e}^{-1.5t}}. \end{aligned}$$

If we conjecture the law

$$\frac{\mathrm{d}P}{\mathrm{d}t} = aP - bP^4,$$

then $b = 1.5/(25000)^4 = 9.6 \times 10^{-14}$. A comparison of the two solutions computed numerically is shown in the figure both using the initial value $P(0) = 10$.

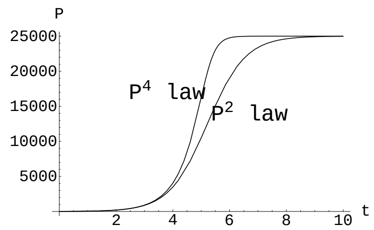

Figure 37: Problem 22.10

22.11. (a) The equation of motion of the falling body is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = g - \frac{K}{m}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{\alpha}.$$

When the body reaches its limiting speed it will be moving with constant speed, which means that its acceleration $\mathrm{d}^2x/\mathrm{d}t^2$ is zero. Therefore from the equation of motion its limiting speed v_s is given by

$$v_s = \left(\frac{mg}{K}\right)^{1/\alpha}.$$

(b) Let $v = \mathrm{d}x/\mathrm{d}t$. Then

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}(v^2)}{\mathrm{d}x}.$$

Therefore the equation of motion can be expressed as

$$\frac{\mathrm{d}(v^2)}{\mathrm{d}x} = 2\left(g - \frac{K}{m}(v^2)^{\frac{1}{2}\alpha}\right).$$

(c) The given data are $K = 4$, $m = 80$, $\alpha = 1.2$, $g = 10$. A numerical solution for v against t is shown. The graph shows that the speed reaches its limiting value of approximately $83\mathrm{ms}^{-1}$ at around 30m.

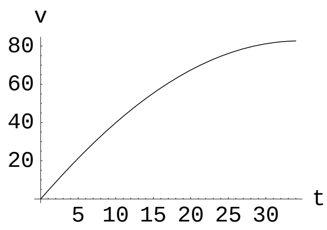

Figure 38: Problem 22.11

22.12. If $y = xw$, then
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(xw) = w + x\frac{\mathrm{d}w}{\mathrm{d}x}.$$
Most solutions are expressed in implicit form.
(a) $\mathrm{d}y/\mathrm{d}x = (x^2 - xy + y^2)/(xy)$. Substitute $y = xw$: then the equation in the variables x and w becomes
$$w + x\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{x^2 - -x^2w + x^2w^2}{x^2w} = \frac{1 - w + w^2}{w}.$$
Therefore
$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{1-w}{wx}.$$
Separating the variables
$$\int \frac{w\mathrm{d}w}{1-w} = \int \frac{\mathrm{d}x}{x}, \text{ or } \int \left[-1 + \frac{1}{1-w}\right]\mathrm{d}w = \ln|x| + C.$$
Integrating
$$w - \ln|1-w| = \ln|x| + C, \text{ or } y + x\ln|x-y| + Cx = 0$$
in terms of y and x.
(b) $\mathrm{d}y/\mathrm{d}x + (x^2 + y^2)/(xy) = 0$. Let $y = xw$: then
$$w + x\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{x^2 + x^2w^2}{x^2w} = -\frac{1+w^2}{w}.$$
Therefore
$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{1+2w^2}{wx}.$$
Hence
$$\int \frac{w\mathrm{d}w}{1+2w^2} = -\int \frac{\mathrm{d}x}{x} + C.$$
Integrating
$$\frac{1}{4}\ln(1+2w^2) = -\ln|x| + C, \text{ or } x^2(x^2+2y^2) = B.$$
(c) $\mathrm{d}y/\mathrm{d}x + (x-y)/(3x+y) = 0$. Let $y = xw$: then
$$w + x\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{1-w}{3+w}.$$
Therefore
$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{(1+w)^2}{x(3+w)}.$$
Separating and integrating
$$\int \frac{(3+w)\mathrm{d}w}{(1+w)^2} = -\int \frac{\mathrm{d}x}{x}, \text{ so that } -\frac{2}{1+w} + \ln|1+w| = -\ln|x| + C.$$

Hence the implicit solution is

$$-\frac{2x}{x+y} + \ln|x+y| = C.$$

(d) $\mathrm{d}y/\mathrm{d}x = 2xy/(3x^2 - 4y^2)$. Let $y = xw$: then

$$w + x\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{2w}{3-4w^2}.$$

Therefore

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{w(4w^2-1)}{3-4w^2}.$$

Using partial fractions

$$\int \frac{3-4w^2}{w(4w^2-1)}\mathrm{d}w = \int \left[-\frac{3}{w} + \frac{2}{2w-1} + \frac{2w}{2w+1}\right]\mathrm{d}w = \int \frac{\mathrm{d}x}{x} + C = \ln|x| + C.$$

Integrating

$$-3\ln|w| + \ln|4w^2-1| = \ln|x| + C.$$

Substitute $w = y/x$ and simplifying:

$$4y^2 - x^2 = By^3.$$

(The moduli can be accommodated by the sign of the constant B.)
(e) $\mathrm{d}y/\mathrm{d}x + 2(2x^2+y^2)/(xy) = 0$. Let $y = xw$: then

$$w + x\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{4+2w^2}{w}.$$

Separating the variables and integrating

$$\int \frac{w\mathrm{d}w}{3w^2+4} = \frac{1}{6}\ln(4+3w^2) = -\frac{\mathrm{d}x}{x} + C = -\ln|x| + C.$$

Hence y satisfies

$$(4x^2+3y^2)x^4 = B.$$

22.13. If $w = y^{1-n}$ $(n \neq 1)$, then $w' = (1-n)y^{-n}y'$. Eliminate y from the Bernoulli equation

$$y' + g(x)y = h(x)y^n,$$

to give

$$w' + (1-n)g(x)w = (1-n)h(x).$$

This is now a linear equation in w, to which the integrating-factor method (Section 19.5) can be applied.
(a) $y' + y = y^4$. In this example of the Bernoulli equation $g(x) = h(x) = 1$ and $n = -3$. Hence $w = y^{-3}$ which satisfies

$$w' - 3w = -3.$$

Using the integrating-factor method the general solution of this equation is

$$w = 1 + C\mathrm{e}^{3x}, \text{ so that } y = (1 + C\mathrm{e}^{3x})^{-\frac{1}{3}}.$$

(b) $y' + y = y^{-\frac{1}{3}}$. In this example $g(x) = f(x) = 1$ and $n = -\frac{1}{2}$. Hence $w = y^{\frac{3}{2}}$ which satisfies

$$w' + \tfrac{3}{2}w = \tfrac{3}{2}.$$

Using the integrating-factor method the general solution is

$$w = 1 + C\mathrm{e}^{-\frac{3}{2}x}, \text{ so that } y = w^{\frac{2}{3}} = (1 + C\mathrm{e}^{-\frac{3}{2}x})^{\frac{2}{3}}.$$

22.14. (a) The equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{b}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{c}{x^2}\right)y = 0.$$

is equidimensional.
(a) Let $y = x^M$. Then

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{b}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{c}{x^2}\right)y &= M(M-1)x^{M-2} + bMx^{M-2} + cx^{M-2} \\ &= [M^2 + (b-1)M + c]x^{M-2} \\ &= 0\end{aligned}$$

for all x if M satisfies $M^2 + (b-1)M + c = 0$. If the roots are M_1 and M_2, then the general solution is

$$x = Ax^{M_1} + Bx^{M_2}, \quad (M_1 \neq M_2)$$

If the roots are equal, $M_1 = M_2 = M$, say, then it can be verified that the general solution is

$$x = Ax^M + Bx^M \ln x.$$

(b) Let $x = \mathrm{e}^t$. Then, applying the change of variable,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \mathrm{e}^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{e}^{-t}\right) = \mathrm{e}^{-2t}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right).$$

Hence the differential equation becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (b-1)\frac{\mathrm{d}y}{\mathrm{d}t} + cy = 0,$$

which is a constant-coefficient second order differential equation (see Chapter 18).
(c) (i) $\mathrm{d}^2y/\mathrm{d}x^2 - (2/x)\mathrm{d}y/\mathrm{d}x + (2/x^2)y = 0$. Use method (a) by putting $y = x^M$. Then M satisfies

$$M^2 - 3M + 2 = 0, \qquad \text{or } (M-1)(M-2) = 0.$$

Hence the roots are $M_1 = 1$ and $M_2 = 2$. The general solution is therefore

$$y = Ax + Bx^2.$$

(ii) $\mathrm{d}^2y/\mathrm{d}x^2 - (1/x)\mathrm{d}y/\mathrm{d}x + 1/x^2 = 0$. Note that this is a forced equation. In this case use method (b) by making the change of variable $x = \mathrm{e}^t$. Then in terms of the differential equation is transformed into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} = -1.$$

The characteristic equation is $m^2 - 2m = 0$, which has roots $m_1 = 0$ and $m_2 = 2$. It is easy to confirm that $y = \frac{1}{2}t$ is a particular solution. Hence the general solution is

$$x = A + B\mathrm{e}^{2t} + \tfrac{1}{2}t = A + Bx^2 + \tfrac{1}{2}\ln x.$$

(iii) $\mathrm{d}^2y/\mathrm{d}x^2 + (3/x)\mathrm{d}y/\mathrm{d}x + (2/x^2)y = 0$. Using method (a), let $y = x^M$: the characteristic equation is

$$M^2 + 2M + 2 = 0, \text{ which has the roots } M = -1 \pm \mathrm{i}.$$

The general solution is therefore

$$y = Ax^{(-1+\mathrm{i})t} + B\mathrm{e}^{(-1-\mathrm{i})t},$$

which can be expressed in real form as indicated in (a).

22.15. In the location shown in the figure the velocity component of the boat in the x direction is the component of V, and in the y direction the velocity component is $v - V\sin\theta$. Therefore

$$\frac{\mathrm{d}x}{\mathrm{d}t} = V\cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v - V\sin\theta$$

Also from the figure

$$\cos\theta = \frac{H-x}{\sqrt{[(H-x)^2+y^2]}}, \quad \sin\theta = \frac{y}{\sqrt{[(H-x)^2+y^2]}}.$$

Finally the differential equation for the path is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{v\sqrt{[(H-x)^2+y^2]} - Vy}{V(H-x)}.$$

Let $z = H - x$. Then in terms of z the equation can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}z} = -\frac{\gamma\sqrt{[z^2+y^2]} - y}{z}, \quad \gamma = \frac{v}{V}.$$

This equation can be solved numerically but an analytic sokution is possible. Use the method of Section 22.5: let $y = zw$ so that

$$w + z\frac{\mathrm{d}w}{\mathrm{d}z} = -\gamma\sqrt{[1+w^2]} + w, \text{ or } z\frac{\mathrm{d}w}{\mathrm{d}z} = -\gamma\sqrt{[1+w^2]}.$$

This is a separable equation with solution given by

$$\int \frac{\mathrm{d}w}{\sqrt{[1+w^2]}} = -\int \frac{\mathrm{d}z}{z} + C.$$

Hence

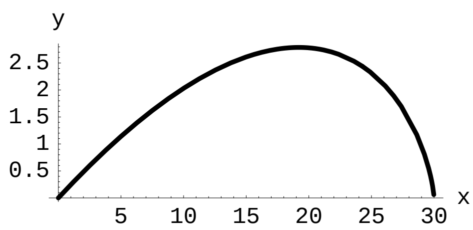

Figure 39: Problem 22.15

$$\sinh^{-1} w = -\ln z + C,$$

or

$$y = (H-x)\sinh[C - \gamma\ln(H-x)],$$

in terms of the original variables. The constant C is given by the condition that $y = 0$ when $x = 0$. Hence $C = \gamma\ln H$. Finally the solution is given by

$$y = (H-x)\sinh\left[\frac{v}{V}\ln\left(\frac{H}{H-x}\right)\right].$$

The graph of the path is shown for the given data.

22.16. From the previous problem we can quote the differential equation for the path with v replaced by $v(x)$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v(x)\surd[(H-x)^2+y^2]-Vy}{V(H-x)} = \frac{ax(H-x)\surd[(H-x)^2+y^2]-Vy}{V(H-x)}.$$

It is unlikely that there is a simple solution of this differential equation. Before fixing any values, it is convenient to nondimensionalize the equation with respect to the length H and boat speed V. Let $X = x/H$ and $Y = y/H$. Then

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\beta X(1-X)\surd[(1-X)^2+Y^2]-Y}{1-X},$$

where $\beta = aH^2/V$: there is just one parameter β. The maximum river speed $v_m = \frac{1}{4}aH^2$ occurs mid-stream. Hence $\beta = 4v_m/V$. We need only specify this parameter in the numerical solution. The figure is computed with $\beta = 2$, that is, $V = 2v_m$.

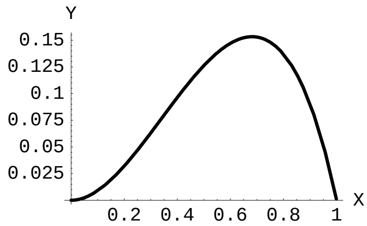

Figure 40: Problem 22.16

22.17. Impose a speed v in the direction HO on both the mouse and the cat so that effectively the mouse is stationary. The radial and transverse components of the velocity of the cat are (see p.193)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -V + v\cos\theta, \quad r\frac{\mathrm{d}\theta}{\mathrm{d}t} = -v\sin\theta$$

relative to the stationary mouse.

The equation of the pursuit path of the cat relative to the mouse is

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r/\mathrm{d}t}{\mathrm{d}\theta/\mathrm{d}t} = -\frac{r(v\cos\theta - V)}{v\sin\theta}.$$

This is a separable equation with solution:

$$\begin{aligned} \int\frac{\mathrm{d}r}{r} = \ln r &= -\int\frac{\cos\theta - V}{\sin\theta} + C \\ &= -\ln\sin\theta + \frac{V}{2v}\ln\left[\frac{1-\cos\theta}{1+\cos\theta}\right] + C, \end{aligned}$$

(see Appendix E). This can be rewritten as

$$r = B\operatorname{cosec}\theta\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{\frac{V}{2v}},$$

where B is a constant. To find the constant B, put $OB = a$, say, when $\theta = \frac{1}{2}\pi$, so that $B = a$. Therefore the path of the cat is

$$r = a\operatorname{cosec}\theta\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{\frac{V}{2v}},$$

22.18. The equation of motion of the satellite is

$$m\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{\gamma Mm}{r^2}.$$

Let

$$v = \frac{\mathrm{d}r}{\mathrm{d}t}, \quad \text{so that} \quad \frac{\mathrm{d}^2r}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}r}.$$

Hence the equation above becomes the separable equation

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{\gamma M}{r^2}.$$

Separating the variables and integrating:

$$\int v\mathrm{d}v = -\gamma M\int\frac{\mathrm{d}r}{r^2}, \text{ or } \frac{1}{2}v^2 = \gamma M\frac{1}{r} + C.$$

The constant C is given by the condition that $v = V$ when $r = a$. Therefore the required solution the velocity is given by

$$\frac{1}{2}(v^2 - V^2) = \gamma\left(\frac{1}{r} - p\frac{1}{a}\right).$$

The minimum value of V in order that the satellite should escape the satellite's gravitation is given byh the condition that $V = V_e$ in the limit $r \to \infty$. Hence, the escape velocity is $V_e = \sqrt{(2\gamma M/a)}$.

Chapter 23: Nonlinear differential equations and the phase plane

23.1. These are all linear systems (see Sections 23.2 and 23.3).
(a) $\dot{x} = y$, $\dot{y} = -4x$. The phase paths are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4x}{y},$$

which generates the phase paths $4x^2 + y^2 = C$. This is a centre (see Figure 41).

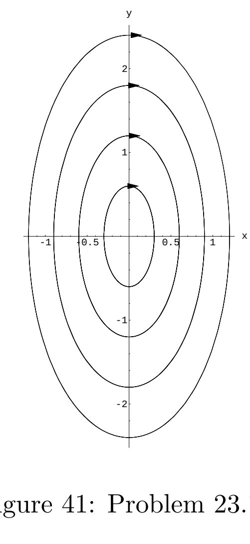

Figure 41: Problem 23.1(a)

(b) $\dot{x} = y$, $\dot{y} = x$. The phase paths are given by $x^2 - y^2 = C$ which is a saddle (see Figure 42).
(c) $\dot{x} = y$, $\dot{y} = -2x - 3y$. This phase diagram is that of a stable node (see Figure 43).
(d) $\dot{x} = y$, $\dot{y} = -3x - y$. This is a stable spiral (see Figure 44).
(e) $\dot{x} = y$, $\dot{y} = -2x + y$. This is an unstable spiral (see Figure 45).

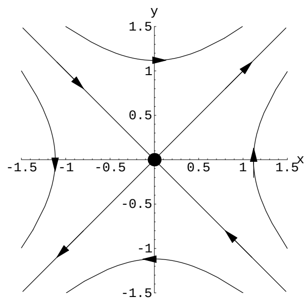

Figure 42: Problem 23.1(b)

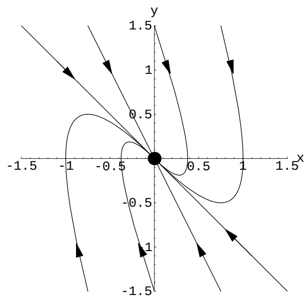

Figure 43: Problem 23.1(c)

(f) The phase diagram Figure 46) from (a) shows time-steps of interval 0.3 for paths starting on the positive x axis.
(g) Time-steps of interval 0.3 are shown on a separatrix and a phase path see Figure 47).
(h) $\dot{x} = y$, $\dot{y} = -2y$. Elimination of y leads to the second-order equation $\ddot{x} + 2\dot{x} + 0$. The phase paths (Figure 48) are given by $y = -2x + C$. which is a family of parallel straight lines. Note that all points on the x axis are equilibrium points.

23.2. These are all linear systems.
(a) $\dot{x} = y$, $\dot{y} = x$. The general solution for the phase paths is $x^2 - y^2 + C$. The phase diagram is given in Problem 23.1(b).
(b) $\dot{x} = x$, $\dot{y} = y$. The equation of the phase paths is given by $\mathrm{d}y/\mathrm{d}x = y/x$, which has the general solution $y = Cx$. The system has an unstable equilibrium point at the origin with radial phase paths.
(c) $\dot{x} = -y$, $\dot{y} = x$. The system has a centre at the origin. The phase diagram ia a family of concentric circles with the sense of the phase paths being described in the counter-clockwise sense.

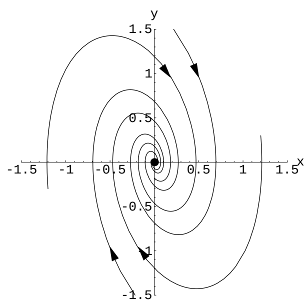

Figure 44: Problem 23.1(d)

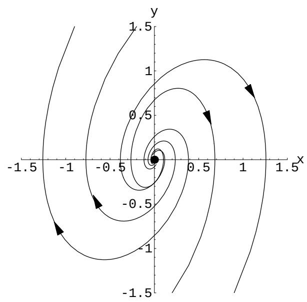

Figure 45: Problem 23.1(e)

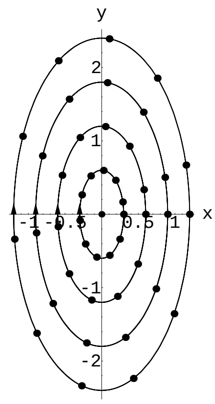

Figure 46: Problem 23.1(f)

(d) $\dot{x} = -x$, $\dot{y} = y$. The system has an equilibrium point at the origin. The phase paths are given by $xy = C$. The phase diagram is that of a saddle: it can be obtained from Fig. 23.4 by rotating the figure through 45° counter-clockwise.

(e) $\dot{x} = 2y$, $\dot{y} = x$. The system is a saddle point with phase paths $y^2 - \frac{1}{2}x^2 = C$.

(f) $\dot{x} = -2y$, $\dot{y} = x$. The phase diagram is a centre with phase paths $x^2 + 2y^2 + C$, but with the phase paths described in a counter-clockwise sense. This can be seen since $\dot{x} < 0$, $\dot{y} > 0$ in the first quadrant: by continuity the sense of the paths can be deduced.

23.3. In all cases put $y = \dot{x}$ for the phase plane.
(a) $\ddot{x} = \mathrm{e}^x$. Use the energy transformation:

$$\frac{1}{2}\frac{\mathrm{d}\dot{x}^2}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}y^2}{\mathrm{d}x} = \mathrm{e}^x,$$

which can be integrated to give the phase paths $\frac{1}{2}y^2 = \mathrm{e}^x + C$. The phase diagram has no equilibrium points.
(b) $\ddot{x} + \dot{x}^2 + x = 0$. The equation can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) + 2y^2 = -2x,$$

which is of first-order integrating-factor type. Its general solution is

$$y^2 = -\frac{1}{4} + \frac{1}{2}x + C\mathrm{e}^{-2x}.$$

The equation has one equilibrium point at the origin which is a centre.

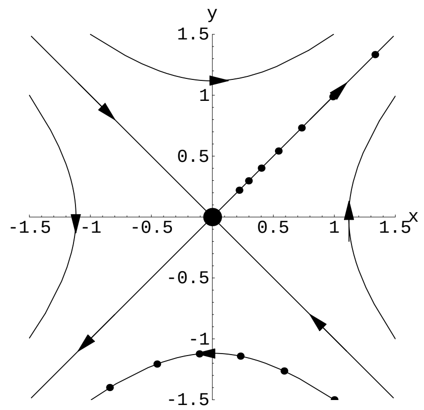

Figure 47: Problem 23.1(g)

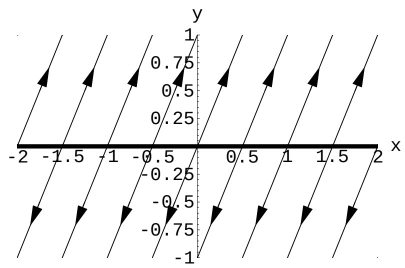

Figure 48: Problem 23.1(h)

(c) $\ddot{x} = 8x\dot{x}$. The phase paths are given by $y = 4x^2 + C$, which is a family of parabolas. All points on the y axis are equilibrium points.
(d) $\ddot{x} = \mathrm{e}^x - \mathrm{e}^{-x}$. The phase paths are given by $\frac{1}{2}y^2 = \mathrm{e}^x + \mathrm{e}^{-x} + C$. The system has one equilibrium point at the origin. Near the origin

$$\ddot{x} = \mathrm{e}^x - \mathrm{e}^{-x} \approx 1 + x - 1 + x = 2x,$$

for small $|x|$. This indicates a saddle point. Qualitatively the phase diagram will be similar to Fig. 23.4.

23.4. The only equilibrium point is at the origin in each case. The classification of linear systems is given by (23.22) for the linear equations

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy.$$

(a) $\dot{x} = x - 5y$, $\dot{y} = x - y$. The coefficients are $a = 1$, $b = -5$, $c = 1$, $d = -1$, so that

$$p = a + d = 0, \quad q = ad - bc = 4, \quad \Delta = p^2 - 4q = -16.$$

Since $p = 0$ and $q > 0$, the origin is a centre. See Fig. 23.2.
(b) $\dot{x} = x + y$, $\dot{y} = x - 2y$. The coefficients are $a = 1$, $b = 1$, $c = 1$, $d = -2$, so that

$$p = -1, \quad q = -3, \quad \Delta = 13.$$

Since $q < 0$, the equilibrium point is a saddle and therefore unstable. The separatrices are given by $y = mx$ where

$$m = \frac{1 - 2m}{1 + m}, \text{ so that } m^2 + 3m - 1 = 0.$$

Hence $m = \frac{1}{2}[-3 \pm \sqrt{13}]$. Qualitatively the phase diagram will be similar to Fig. 23.4.

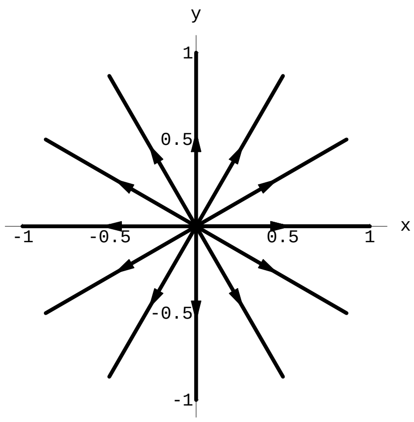

Figure 49: Problem 23.2(b)

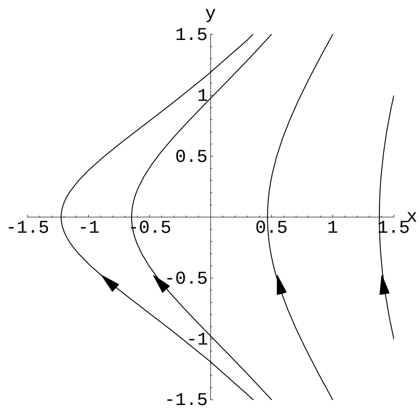

Figure 50: Problem 23.3(a)

(c) $\dot{x} = -4x + 2y$, $\dot{y} = 3x - 2y$. The coefficients are $a = -4$, $b = 2$, $c = 3$, $d = -2$, so that

$$p = -6, \quad q = 2, \quad \Delta = 28.$$

Since $q > 0$, $\Delta > 0$ and $p < 0$, the equilibrium point is a stable node. There are straight line solutions $y = mx$ with $m = \frac{1}{2}[1 \pm \sqrt{7}]$. A typical stable node is shown in Fig. 23.6.
(d) $\dot{x} = x + 2y$, $\dot{y} = 2x + 2y$. The coefficients are $a = 1$, $b = 2$, $c = 2$, $d = 2$, so that

$$p = 3, \quad q = -2, \quad \Delta = 1.$$

Since $q < 0$, the phase diagram is a saddle. The separatrices are given by $y = 2x$ and $y = -x$. A typical saddle point is shown in Fig. 23.4.
(e) $\dot{x} = 4x - 2y$, $\dot{y} = 3x - y$. The coefficients are $a = 4$, $b = -2$, $c = 3$, $d = -1$, so that

$$p = 3, \quad q = 2, \quad \Delta = 1.$$

Since $q > 0$, $\Delta > 0$ and $p > 0$, the origin is an unstable node. The node has the straight line solutions $y = x$ and $y = \frac{3}{2}x$.
(f) $\dot{x} = 2x + 3y$, $\dot{y} = -3x - 3y$. The coefficients are $a = 2$, $b = 3$, $c = -3$, $d = -3$, so that

$$p = -1, \quad q = 3, \quad \Delta = -11.$$

The phase diagram is a stable spiral, which is qualitatively similar to Fig. 23.5.

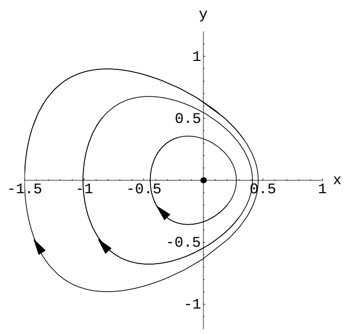

Figure 51: Problem 23.3(b)

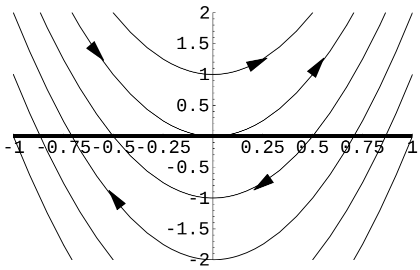

Figure 52: Problem 23.3(c)

23.5. & 23.6. For the general system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, equilibrium points are given by any simultaneous solutions of $P(x, y) = 0$, $Q(x, y) = 0$. Phase paths have zero slope where they cross the curve $Q(x, y) = 0$, and infinite slope (that is, parallel to the y axis) where they cross the x axis.

(a) $\dot{x} = x - y$, $\dot{y} = x + y - 2xy$. Equilibrium points are given by

$$x - y = 0, \quad x + y - 2xy = 0.$$

Elimination of y leads to $x(1-x) = 0$: hence equilibrium points occur at $(0, 0)$ and $(1, 1)$. Linearize the equations near the equilibrium points.

Near $(0, 0)$. $\dot{x} = x - y$, $\dot{y} \approx x + y$. Since $p = 2$, $q = 2$ and $\Delta = -4$, the origin is locally an unstable spiral according to (23.22).

Near $(1, 1)$. Let $x = 1 + X$, $y = 1 + Y$. Then the linear approximation is given by

$$\dot{X} = X - Y, \quad \dot{Y} = 2 + X + Y - 2(1 + X)(1 + Y) \approx -X - Y.$$

Hence the coefficients are $a = 1$, $b = 1$, $c = -1$, $d = -1$, so that $q = -2 < 0$, which, according to (23.22), means that $(1, 1)$ is locally a saddle. A computed phase diagram is shown in Figure 54.

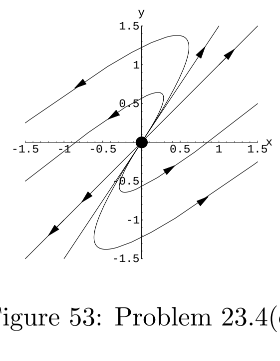

Figure 53: Problem 23.4(e)

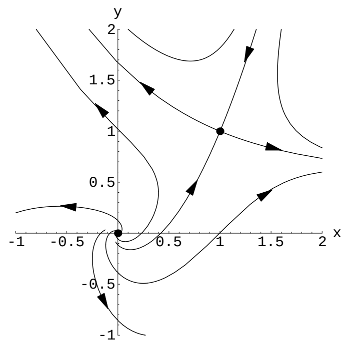

Figure 54: Problem 23.5(a)

(b) $\dot{x} = 1 - xy$, $\dot{y} = (x-1)(y+1)$. Equilibrium points are given by solutions of

$$1 - xy = 0, \quad (x-1)(y+1) = 0,$$

and occur at $(1,1)$, $(-1,-1)$.
Near $(1,1)$. Let $x = 1 + X$, $y = 1 + Y$. Then

$$\dot{X} \approx -X - Y, \quad \dot{Y} = X.$$

Hence the coefficients are $a = -1$, $b = -1$, $c = 1$, $d = 0$. Therefore $p = -1$, $q = 1$, $\Delta = -3$. By (23.22), $(1,1)$ is a stable spiral.
Near $(-1,-1)$. Let $x = -1 + X$, $y = -1 + Y$. Then

$$\dot{X} \approx X + Y, \quad \dot{Y} \approx -2Y.$$

Here $a = 1$, $b = 1$, $c = 0$, $d = -2$. Since $q = -2 < 0$, $(-1,-1)$ is a saddle. A computed phase diagram is shown in Figure 55.

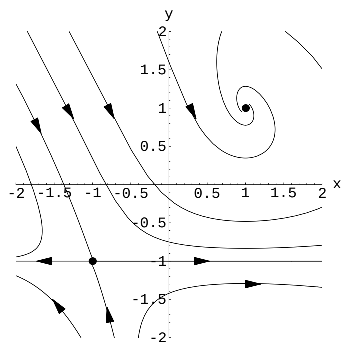

Figure 55: Problem 23.5(b)

(c) $\dot{x} = x - y$, $\dot{y} = x^2 - 1$. Equilibrium points are given by the solutions of

$$x - y = 0, \quad x^2 - 1 = 0.$$

Hence $(1,1)$ and $(-1,-1)$ are equilibrium points.
Near $(1,1)$. Let $x = 1 + X$, $y = 1 + Y$. Then

$$\dot{X} = X - Y, \quad \dot{Y} \approx 2X.$$

The coefficients of this linear approximation are $a = 1$, $b = -1$, $c = 2$, $d = 0$. Therefore $p = 1$, $q = 2$, $\Delta = -7$. According to (23.22), $(1,1)$ is an unstable spiral.

Near $(-1,-1)$. Let $x = -1 + X$, $y = -1 + Y$. Then

$$\dot{X} = X - Y, \quad \dot{Y} \approx -2X.$$

The coefficients are $a = 1$, $b = -1$, $c = -2$, $d = 0$. Therefore $p = 1$ and $q = -2$. Hence $(-1,-1)$ is a saddle. A computed phase diagram is shown in Figure 56

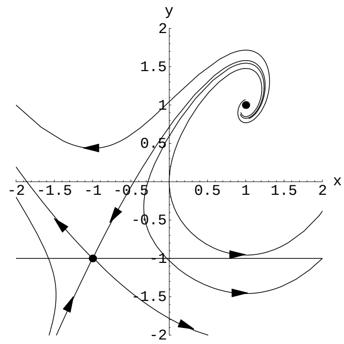

Figure 56: Problem 23.5(c)

(d) $\ddot{x} + x - x^3$, $\dot{x} = y$. The corresponding first-order system is $\dot{x} = y$, $\dot{y} = -x + x^3$. There are equilibrium points at $(0,0)$, $(1,0)$ and $(-1,0)$.

Near $(0,0)$. The linear approximation is $\dot{x} = y$, $\dot{y} = -x$, which is locally a centre.

Near $(1,0)$ *and* $(-1,0)$. These equilibrium points are saddles. A computed phase diagram is shown in Figure 57

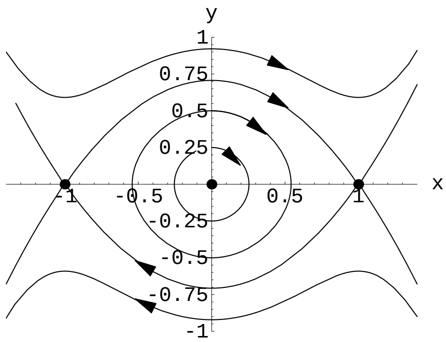

Figure 57: Problem 23.5(d)

(e) $\dot{x} = 4x - 2xy$, $\dot{y} = -2y + xy$. The equilibrium points are the simultaneous solutions of

$$x(2 - y) = 0, \quad y(-2 + x) = 0,$$

which are $(0,0)$ and $(2,2)$.

Near $(0,0)$. The linearized equations are $\dot{x} = 4x$, $\dot{y} = -2y$, for which $a = 4$, $b = 0$, $c = 0$, $d = -2$. Hence $q = -8 < 0$, which means that the origin is a saddle. Note also that $x = 0$ and $y = 0$ are phase paths. The isoclines of zero slope are the lines $y = 0$ and $x = 2$, passing through the equilibrium point at $(2,2)$.

Near $(2,2)$. Let $x = 2 + X$ and $y = 2 + Y$. Then

$$\dot{X} = 4(2 + X) - 2(2 + X)(2 + Y) \approx -4Y, \quad \dot{Y} = -2(2 + Y) + (2 + X)(2 + Y) \approx 2X.$$

Hence $(2,2)$ is locally a centre.

The phase diagram is essentially similar to that given in Fig. 23.10.

23.7. The direction of the paths can be either way as long as there is continuity of direction between adjacent paths. Below are some possible phase diagrams.
(a) The phase diagram shown as a centre at $(0,0)$ and a saddle at $(1,1)$. It has been computed from the system $\dot{x}=y$, $\dot{y}=x^2-x$.

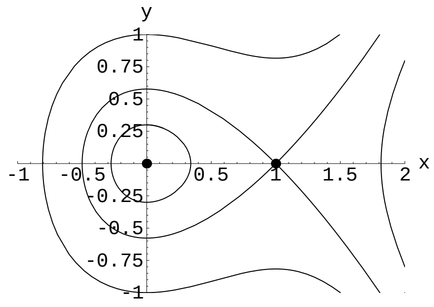

Figure 58: Problem 23.7(a)

(b) A possible phase diagram is given by the phase diagram of Problem 23.5(d).

(c) Need to draw an unstable node near the origin as in the figure for Problem 23.4(e) and a stable node near $x=1$ as in Fig. 23.6, and then join up the paths without further equiliubrium points.

(d) Let $\dot{x}=xy$, $\dot{y}=1-x^2$. This example has just two equilibrium points at $(\pm 1,0)$ which are both centres. Note that $x=0$ is a phase path.

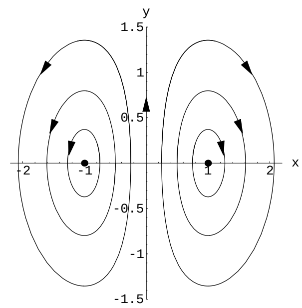

Figure 59: Problem 23.7(d)

23.8. In all problems except (d) and (e) let $y=\dot{x}$.
(a) $\ddot{x}+|\dot{x}|\dot{x}+x=0$. Then $\dot{y}=-|y|y-x$. The system has just one equilibrium point at the origin. The linear approximation near the origin is $\dot{x}=y$, $\dot{y}=-x$, which predicts a centre. However, the computed phase diagram clearly shows that the origin is a stable spiral. Note that the linear approximation can fail if the linear approximation turns out to be a centre.

(b) $\ddot{x}+|\dot{x}|\dot{x}+x^3=0$. The system has one equilibrium point at the origin, but a linear approximation near the origin is not helpful; the origin is a 'higher order' equilibrium point. The phase diagram shows that the equilibrium is a stable spiral.

(c) $\ddot{x}=x^4-x^2$. There are equilibrium points at $(0,0)$ and $(\pm 1,0)$, but the linear approximation is identically zero, so is of no help at the origin. The phase diagram is shown in the figure. Near the origin, if we neglect the term x^4, then $\dot{x}=y$ and $\dot{y}\approx -x^2$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x^3}{y},$$

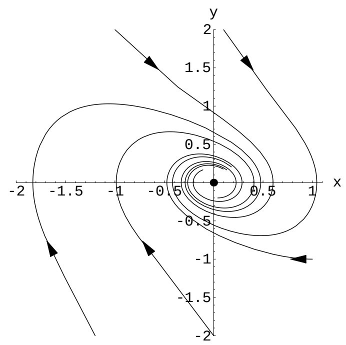

Figure 60: Problem 23.8(a)

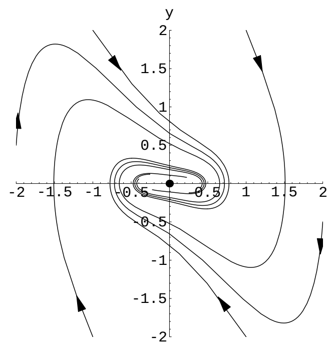

Figure 61: Problem 23.8(b)

which can be integrated to give the phase paths

$$3y^2 = -2x^3 + C.$$

The paths which meet at the origin are given by $3y^2 = -2x^3$, which can only be defined for $x < 0$. The linear approximations close to $x = -1$ indicates a centre (which is the case for this problem), and close to $x = 1$ indicates a saddle.

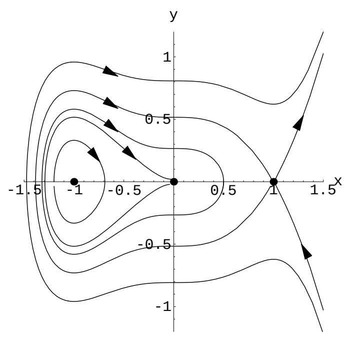

Figure 62: Problem 23.8(c)

(d) $\dot{x} = 2xy$, $\dot{y} = y^2 - x^2$. The system has one equilibrium point at the origin. Note that $x = 0$ is an equilibrium path since the first equation is satified identically.
(e) $\dot{x} = 2xy$, $\dot{y} = x^2 - y^2$. The system has one equilibrium point at the origin.
  Let $y = mx$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - y^2}{2xy} \text{ becomes } m = \frac{1 - m^2}{2m}.$$

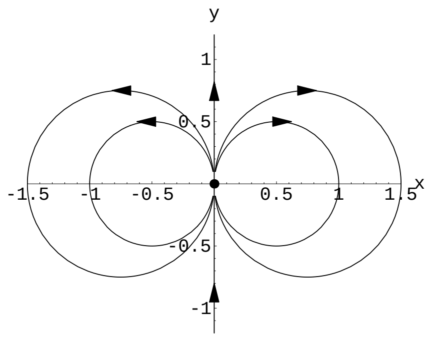

Figure 63: Problem 23.8(d)

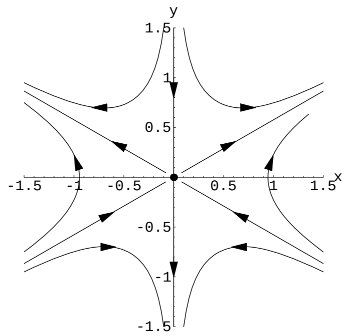

Figure 64: Problem 23.8(e)

Hence $3m^2 = 1$, so that $m = \pm 1/\sqrt{3}$. Hence $y = \pm x/\sqrt{3}$ are separatrices. Further $x = 0$ is also a solution which is a third separatrix as shown in the figure. The origin is an example of a higher order saddle point.A computed phase diagram is shown in Figure 64.

(f) $\ddot{x} + \dot{x}(x^2 + \dot{x}^2) + x = 0$. The equation has one equilibrium point at the origin. Note that the origin is clearly a stable spiral although the linear approximation is a centre. A computed phase diagram is shown in Figure 65.

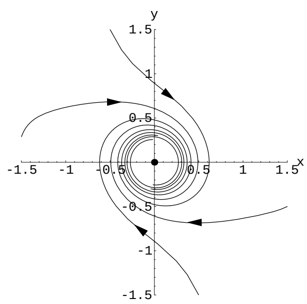

Figure 65: Problem 23.8(f)

23.9. The pendulum equation (Section 23.4) is $\ddot{x} + \omega^2 \sin x = 0$, which, for small x, can be approximated by $\ddot{x} + \omega^2(x - \frac{1}{6}x^3) = 0$ as stated. With $\dot{x} = y$, the approximation has equilibrium points at $(0, 0)$ and at $(\pm\sqrt{6}, 0)$. The exact equation has an infinite number of equilibrium points where $\sin x = 0$, the two closest to the origin being $x = \pm\pi$ which differ significantly from $x = \pm\sqrt{6}$. Near the origin $\ddot{x} + \omega^2(x - x^3) = 0$ has a centre whilst near $x = \pm\sqrt{6}$, the equation has saddle points. The phase diagram of Figure 66 has been computed for the equation $\ddot{x} + x - \frac{1}{6}x^3 = 0$, since the time t can always be rescaled by putting $\tau = \omega t$.

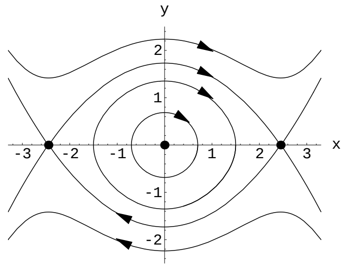

Figure 66: Problem 23.9

23.10. $\dot{x} = 4x - 2xy - x^2$, $\dot{y} = -2y + xy - 2y^2$. Equilibrium points occur where

$$x(4 - 2y - x) = 0, \quad y((-2 + x - 2y) = 0,$$

but remember that since this is a population problem, we require $x, y \geq 0$. There are three equilibrium points at $(0,0)$, $(4,0)$ and $(3, \frac{1}{2})$. The linear approximations near these points are as follows.

Near $(0,0)$. $\dot{x} \approx 4x$ and $\dot{y} \approx -2y$. This is a saddle point with separatrices $x = 0$ and $y = 0$: note that these are also solutions of the full equations.

Near $(4,0)$. Let $x = 4 + X$, $y = Y$. Then the linear approximations are

$$\dot{X} \approx -4X - 8Y, \quad \dot{Y} \approx 2Y.$$

In the usual notation, $a = -4$, $b = -8$, $c = 0$, $d = 2$. Hence $q = -8$, which means that $(4,0)$ is a saddle.

Near $(3, \frac{1}{2})$. Let $x = 3 + X$ and $y = \frac{1}{2} + Y$. Then

$$\dot{X} \approx -3X - 6Y, \quad \dot{Y} \approx \frac{1}{2}X - Y.$$

Then $a = -3$, $b = -6$, $c = \frac{1}{2}$, $d = -1$, so that $p = -4$, $q = 6$ and $\Delta = -8$. Hence $(3, \frac{1}{2})$ is a stable spiral.

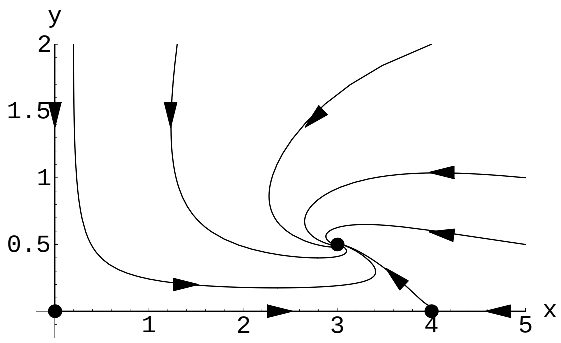

Figure 67: Problem 23.10

23.11. The equations can be expressed as $\dot{H} = (a - bP)H$, $H\dot{P} = (cH - dP)P$. The equations are satisfied if $H = 0$ and $P = 0$, and if $P = a/b$ and $H = ad/(bc)$. Hence $(0,0)$ and $(a/b, ad/(bc))$ are equilibrium points. The origin is a saddle, and the full equations have solutions $H = 0$ and $P = 0$. The linear approximations near the other equilibrium point are, if $P = (a/b) + p$ and $H = ad/(bc) + h$,

$$\dot{h} \approx -\frac{ad}{c}, \quad \dot{p} \approx -\frac{c^2}{d}h - cp.$$

Hence

$$p = -\frac{ad^2 + c^3}{cd} < 0, \quad q = ac > 0, \quad \Delta = \frac{(ad^2 - c^3)^2}{cd} > 0,$$

which, by (23.22), means that the equilibrium point is a stable node.

23.12. The equation of motion is

$$\ddot{x} + \frac{s}{m}\left(1 - \frac{l}{(h^2 + x^2)^{\frac{1}{2}}}\right) x = 0.$$

Equilibrium points occur where

$$x[(h^2 + x^2)^{\frac{1}{2}} - l] = 0.$$

$l < h$. There is only one equilibrium point, at $x = 0$, which has the linear approximation

$$\dot{x} = y, \qquad \dot{y} \approx -\frac{s}{m}\left(1 - \frac{l}{h}\right) x.$$

This predicts a centre.

$l = h$. There is still just one equilibrium point, at $x = 0$, but the linear approximation will not decide the type of equilibrium point.

$l > h$. This system has three equilibrium points at $x = 0$ and $x = \pm(l^2 - h^2)^{\frac{1}{2}}$. The point $(0, 0)$ is a saddle.

For the other equilibrium points, let $x = \pm(l^2 - h^2)^{\frac{1}{2}} + X$, $y = Y$. Then $\dot{X} = Y$ and

$$\begin{aligned} \dot{Y} &= -\frac{s}{m}\left(1 - \frac{l}{[h^2 + \{\pm(l^2 - h^2)^{\frac{1}{2}} + X\}^{\frac{1}{2}}}\right)[(l^2 - h^2)^{\frac{1}{2}} + X] \\ &\approx -\frac{s}{ml}(l^2 - h^2)X. \end{aligned}$$

Both equilibrium points are centres.

23.13. The equation of motion with friction included is

$$\ddot{x} + k\dot{x} + \frac{s}{m}\left(1 - \frac{l}{(h^2 + x^2)^{\frac{1}{2}}}\right) x = 0.$$

The equilibrium points are still at $x = 0$ if $l \leq h$ and at $x = 0$ and $x = \pm(l^2 - h^2)^{\frac{1}{2}}$ if $l > h$. The approximation near the origin is

$$\dot{x} = y, \quad \dot{y} = -\frac{s}{m}\left(1 - \frac{l}{h}\right) - ky.$$

This equilibrium point is a stable spiral if $l < h$ and a saddle if $l > h$.

If $l > h$ and k is small, then the other equilibrium points are both stable spirals.

23.14. $\ddot{x} + k\dot{x} - x + x^2 = 0$. With $\dot{x} = y$, the equation has equilibrium points at $(0, 0)$ and $(1, 0)$. Near the origin

$$\dot{x} = y, \quad \dot{y} \approx x - ky,$$

for which $q = -1$. Hence, for all k, the origin is a saddle with separatrices in the directions with slopes

$$m = -k \pm \sqrt{(k^2 + 4)}.$$

Near $(1, 0)$, let $x = 1 + X$ and $y = Y$. Then the linear approximation is given by

$$\dot{X} = Y, \quad \dot{Y} \approx -X - kY.$$

The parameters are $p = -k$, $q = 1$, $\Delta k^2 - 4$. If $k^2 < 4$, then $(1, 0)$ is a spiral (stable if $k > 0$; unstable if $k < 0$). If $k^2 > 4$, then $(1, 0)$ is a node (stable if $k > 0$; unstable if $k < 0$).

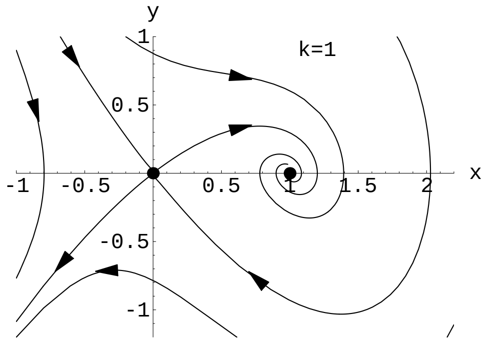

Figure 68: Problem 23.14

The figure shows the phase diagram for $k = 1$, which gives a stable spiral at $x = 1$. For $k > 2$, the spiral at $x = 1$ is replaced by a stable node (not shown).

23.15. Let $\dot{x} = y$ in each case. The first-order system $\dot{x} = y$, $\dot{y} = f(x, y)$ has been solved numerically.
(a) $\ddot{x} + \frac{1}{2}(x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$. The system has one equilibrium point, which is an unstable spiral, at the origin. Check that $x = \cos(t + \alpha)$ is a solution for all α, so that $x^2 + y^2 = 1$ is a limit cycle . A computed phase diagram showing the stable limit cycle is displayed in Figure 69.

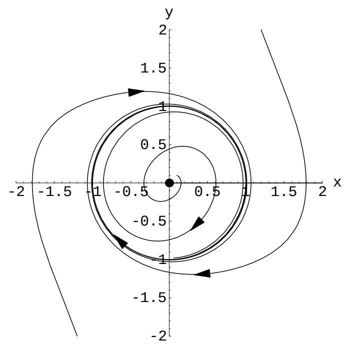

Figure 69: Problem 23.15(a)

(b) $\ddot{x} + \frac{1}{5}(x^2 - 1)\dot{x} + x = 0$. The system has a stable limit cycle. A computed phase diagram is shown in Figure 70

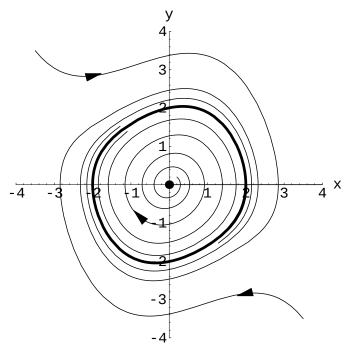

Figure 70: Problem 23.15(b)

(c) $\ddot{x} + \frac{1}{5}(\frac{1}{3}\dot{x}^2 - 1)\dot{x} + x = 0$. This equation has a stable limit cycle, and its phase diagram is very similar to that of the previous problem.

(d) $\ddot{x} + 5(x^2 - 1)\dot{x} + x = 0$. The equation has a stable limit cycle, shown in Figure 71, which is distorted compared with that of Problem 23.15(c). This is caused by the large parameter associated with the middle term. The equation represents the van der Pol oscillator.

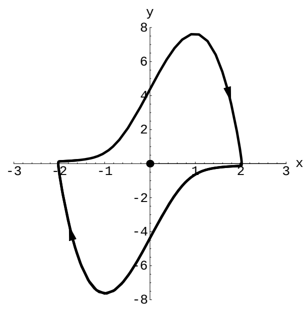

Figure 71: Problem 23.15(d)

23.16. (a) The phase diagram of Problem 23.15(a) is an example of this.

(b) The system

$$\ddot{x} - (1 - x^2 - \dot{x}^2)(4 - x^2 - \dot{x}^2) + x = 0$$

has the exact solutions $x = \cos t$ and $x = 2\cos t$ which generate the limit cycles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. The inner on is stable and the outer one is unstable.

(c) The phase diagram (Figure 72) shows a possible configuration of the equilibrium points, the limit cycle and the separatrices through the saddle points. The paths can be taken in either sense so long as they are consistent between adjacent paths.

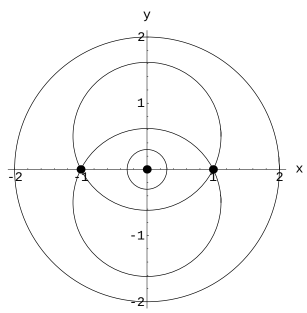

Figure 72: Problem 23.16(c)

(d) The phase diagram (Figure 73) shows a saddle at the origin with separatrices surrounding centres at $x = \pm 1$. The larger circle is a phase path which is a stable limit cycle if paths inside and outside it approach it as $t \to \infty$. The direction of the limit cycle can be in either direction.

(e) The phase diagram shows a centre at the origin, a circular limit cycle with radius 2 and two paths spiralling into the limit cycle from the inside and the outside to indicate its stability.

23.17. $\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = x + y(1 - x^2 - y^2)$. Differentiate $r^2 = x^2 + y^2$ with respect to t:

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} = 2\{x[-y + x(1 - x^2 - y^2)] + y[x + y(1 - x^2 - y^2)]\} = 2r^2(1 - r^2)$$

Hence $\dot{r} = r(1 - r^2)$. Differentiate $\tan\theta = y/x$ with respect to t:

$$\sec^2\theta\dot{\theta} = \frac{x\dot{y} - x\dot{y}}{x^2} = \frac{x^2 + y^2}{x^2} = \sec^2\theta.$$

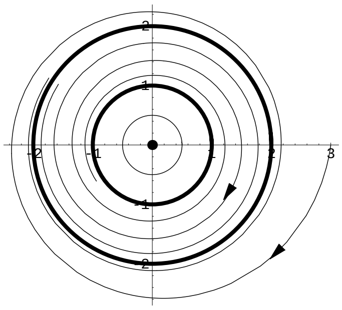

Figure 73: Problem 23.16(e)

Hence $\dot{\theta} = 1$.

Any periodic solutions are given by $\dot{r} = 0$ which means that there is one non-zero solution $r = 1$ (r is greater than 0). It is stable since $\dot{r} < 0$ for $r > 1$, that is r is decreasing with t, and $\dot{r} > 0$ for $r < 1$.

23.18. $\dot{x} = (x^2 + y^2 - 1)y$, $\dot{y} = -(x^2 + y^2 - 1)x$. Equilibrium points are given by

$$(x^2 + y^2 - 1)y = 0, \quad (x^2 + y^2 - 1)x = 0.$$

The solutions are $x = 0$, $y = 0$ and $x^2 + y^2 = 1$. Hence the equilibrium points are the origin and *all* points on the circle $x^2 + y^2 = 1$.

23.19. Let $x = \cos\omega(t - t_0)$. Then

$$\begin{aligned} &\ddot{x} + \left(1 - x^2 - \frac{\dot{x}^2}{\omega^2}\right)\dot{x} + \omega^2 x = \\ &\quad -\omega^2 \cos\omega(t - t_0) - (1 - \cos^2\omega(t - t_0) - \sin^2\omega(t - t_0))\omega\sin\omega(t - t_0) \\ &\quad\quad +\omega^2\cos\omega(t - t_0) = 0 \end{aligned}$$

The limit cycle is given by $x = \cos\omega(t - t_0)$, $\dot{x} = y = \omega\sin\omega(t - t_0)$. Elimination of t gives

$$x^2 + \frac{y^2}{\omega^2} = 1,$$

which is an ellipse.

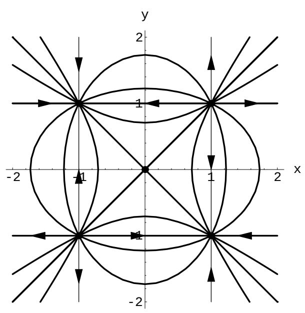

Figure 74: Problem 23.20: remaining phase path directions can be inserted by maintaining continuity of directions

Near the origin the linear approximation is

$$\ddot{x} + \dot{x} + \omega^2 x = 0,$$

which indicates a stable equilibrium point, a node if $0 < \omega < 2$, or a spiral if $2 < \omega$. The indicates that the limit cycle is unstable with phase paths spiralling away from it.

23.20. $\dot{x} = (x^2 - 1)y$, $\dot{y} = (y^2 - 1)x$. Equilibrium points are given by all simultaneous solutions of

$$(x^2 - 1)y = 0, \quad (y^2 - 1)x = 0.$$

Check that they are the points $(0, 0)$, $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. Note that the lines $x = \pm 1$ and $y = \pm 1$ are phase paths which each pass through two equilibrium points.

Near the origin $\dot{x} = -y$, $\dot{y} = -x$, which indicates a saddle. This information together with further linear approximations indicate that all the other equilibrium points are nodes with $\Delta = 0$ in which the paths are locally radial. The lines $y = \pm x$ are also phase paths: these are the separatrices of the saddle at the origin

The equations of the phase paths can be found since

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{(y^2 - 1)x}{(x^2 - 1)y}.$$

This first-order equation is of separable type with general solution

$$(x^2 - 1)(y^2 - 1) = C.$$

The phase diagram is shown in Figure 74.