PART II: Matrix and vector algebra

Chapter 7: Matrix algebra

7.1. The elements are $a_{13} = 3$ and $a_{31} = 2$.

7.2. Comparison of A and B shows that $A = B$ if $x = -2$ and $y - x = 3$. It follows that $y = 1$.

7.3. The answers are

$$A + B = \begin{bmatrix} 3 & 1 & 0 \\ 3 & 1 & 6 \end{bmatrix}, \qquad A - B = \begin{bmatrix} -1 & 3 & -6 \\ -5 & -1 & 2 \end{bmatrix},$$

$$2A - 3B = \begin{bmatrix} -4 & 7 & -15 \\ -14 & -3 & 2 \end{bmatrix}.$$

7.4. The distributive law is satisfied since, as follows:

$$B + C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}.$$

Hence

$$A(B + C) = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 6 & 4 \end{bmatrix}.$$

Also

$$\begin{aligned} AB + AC &= \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 3 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 6 & 4 \end{bmatrix} = A(B + C). \end{aligned}$$

7.5. Verify the left- and right-hand sides:

$$\begin{aligned} A(BC) &= \begin{bmatrix} -1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 5 \\ 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 10 \\ -1 & 14 \end{bmatrix} \end{aligned}$$

Also

$$
\begin{aligned}
(AB)C &= \left(\begin{bmatrix} -1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 10 \\ -1 & 14 \end{bmatrix} \\
&= A(BC),
\end{aligned}
$$

as required.

7.6. Multiplication gives

$$AB = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$BA = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -10 & -5 \\ 20 & 10 \end{bmatrix} \neq AB.$$

7.7. Since $A + C = \mathrm{I}_3$,

$$C = \mathrm{I}_3 - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & 0 \end{bmatrix}.$$

Hence

$$AC = A(\mathrm{I}_3 - A) = A - A^2 \text{ and } CA = (\mathrm{I}_3 - A)A = A - A^2 = AC.$$

Thus

$$AC = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 & -8 \\ 4 & -5 & -1 \\ 3 & 3 & 4 \end{bmatrix}.$$

Since $A + C = \mathrm{I}_3$, multiplication by A and C give

$$A^2 + AC = A,$$

and

$$CA + C^2 = C.$$

Adding:

$$A^2 + AC + CA + C^2 = A + C = \mathrm{I}_3.$$

Finally, since $AC = CA$,

$$A^2 + C^2 = \mathrm{I}_3 - 2AC = \begin{bmatrix} -5 & 6 & 16 \\ -8 & 11 & 2 \\ -6 & -6 & -7 \end{bmatrix}.$$

7.8. Using the transpose rules in Section 7.3 (following Example 7.7),

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T,$$

which means that $A + A^T$ is a symmetric matrix (see Section 7.3).

Similarly

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T),$$

which means that $A - A^T$ is skew-symmetric.

Use the result that

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

For the given A

$$A^T = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix},$$

so that

$$A = \begin{bmatrix} 2 & -\frac{1}{2} & 3 \\ -\frac{1}{2} & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & 0 \\ -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7.9. Given

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

Also

$$AA^T = \begin{bmatrix} 10 & 5 & 3 \\ 5 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \qquad A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 14 \end{bmatrix}.$$

7.10. The equation $A\mathbf{x} = \mathbf{d}$ becomes

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y + 2z \\ 3x + z \\ -x + 2y - 3z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

Hence

$$\begin{aligned} x - y + 2z &= 2 \\ 3x + z &= 0 \\ -x + 2y - 3z &= -1 \end{aligned}.$$

Also

$$\begin{aligned} \mathbf{x}^T A^T &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} x - y + 2z & 3x + z & -x + 2y - 3z \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}. \end{aligned}$$

The verification follows.

7.11.

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2b + ac & 0 & 1 \end{bmatrix}.$$

If $2b + ac = 0$, then $A^2 = \mathrm{I}_3$. Hence $A^{-1} = A$: in other words A is its own inverse. In this case, eliminating, say, b:

$$A^{-1} = A = \begin{bmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ -\frac{1}{2}ac & c & 0 \end{bmatrix},$$

for any a and c.

It follows that

$$A^{2n-1}A^{2n-1} = A^{4n-2} = (A^2)^{2n-1} = (\mathrm{I}_3)^{2n-1} = \mathrm{I}_3.$$

Hence the inverse of A^{2n-1} is A^{2n-1}.

7.12. The two matrix products are

$$AB = \begin{bmatrix} 2 & 0 & 1 \\ 2 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & -1 & -\frac{1}{4} \\ 1 & -1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_3.$$

Similarly

$$BA = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & -1 & -\frac{1}{4} \\ 1 & -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_3.$$

By Section 7.4, B must be the inverse of A.

7.13. The powers of A are

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 3 \\ 0 & 12 & 0 \\ 8 & -4 & 9 \end{bmatrix},$$

and

$$A^3 = AA^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 3 \\ 0 & 12 & 0 \\ 8 & -4 & 9 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 15 \\ 24 & -24 & 24 \\ 8 & 44 & 9 \end{bmatrix}.$$

Hence

$$\begin{aligned} A^3 - A^2 - 12A &= \begin{bmatrix} 16 & 4 & 15 \\ 24 & -24 & 24 \\ 8 & 44 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 4 & 3 \\ 0 & 12 & 0 \\ 8 & -4 & 9 \end{bmatrix} - 12 \begin{bmatrix} 2 & 0 & 1 \\ 2 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \\ &= -12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Multiplying both sides by A^{-1} (A is nonsingular):

$$A^{-1}A^3 - A^{-1}A^2 - 12A^{-1}A = -12A^{-1}\mathrm{I}_3,$$

that is,

$$A^2 - A - 12\mathrm{I}_3 = -12A^{-1}.$$

Hence

$$A^{-1} = \frac{1}{12}[A^2 - A - 12\mathrm{I}_3] = \frac{1}{6} \begin{bmatrix} 5 & -2 & -1 \\ 1 & -1 & 1 \\ -4 & 4 & 2 \end{bmatrix},$$

using A^2 previously found.

7.14. Let A be the matrix in each case, and use rule (7.8).

(a) Then

$$\det(A) = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3.$$

Hence

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

(b)

$$\det(A) = \begin{vmatrix} 2 & 3 \\ -7 & 11 \end{vmatrix} = 43, \text{ and } A^{-1} = \begin{bmatrix} \frac{11}{43} & -\frac{3}{43} \\ \frac{7}{43} & \frac{2}{43} \end{bmatrix}.$$

(c)

$$\det(A) = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2, \text{ and } A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

(d)

$$\det(A) = \begin{vmatrix} 10 & -7 \\ 8 & 0 \end{vmatrix} = 56, \text{ and } A^{-1} = \begin{bmatrix} 0 & \frac{1}{8} \\ -\frac{1}{7} & \frac{5}{28} \end{bmatrix}.$$

(e)

$$\det(A) = \begin{vmatrix} -99 & 100 \\ 97 & 98 \end{vmatrix} = -19402, \text{ and } A^{-1} = \begin{bmatrix} -\frac{49}{9701} & \frac{50}{9701} \\ \frac{97}{19402} & \frac{99}{19402} \end{bmatrix}.$$

7.15. The inverse of A must satisfy $AA^{-1} = \mathrm{I}_4$. Note that A has just one element in any row or column. Hence to achieve the zeros in the correct positions in I_4, we must have

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This suggests the form of inverse appropriate to the more general matrix:

$$A^{-1} = \begin{bmatrix} 0 & 0 & c^{-1} & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \\ 0 & 0 & 0 & d^{-1} \end{bmatrix},$$

assuming that a, b, c, d are all non-zero.

7.16. The equation $A\mathbf{x} = \mathbf{d}$ becomes

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix},$$

or

$$\begin{array}{ccccccc} & & y & + & z & = & 6 \\ x & - & 2y & + & 2z & = & 3 \\ x & & & + & z & = & -9 \end{array}.$$

To find the inverse first calculate

$$\det(A) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 3.$$

Using (7.10)

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

Multiplying $A\mathbf{x} = \mathbf{d}$ on the right by A^{-1}:

$$A^{-1}A\mathbf{x} = \mathrm{I}_3 A\mathbf{x} = \mathbf{x} = A^{-1}\mathbf{d}.$$

Hence

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -17 \\ -2 \\ 8 \end{bmatrix},$$

which is the solution of the set of equations.

7.17. Thus,

$$(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA) = A^{-1}BAA^{-1}BA = A^{-1}B\mathrm{I}_nBA = A^{-1}B^2A.$$

Using this result, observe that

$$A^{-1}B^4A = (A^{-1}B^2A)^2 = (A^{-1}BA)^4.$$

Using rule (7.8)

$$A^{-1}BA = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{8}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

Finally

$$A^{-1}B^4A = (A^{-1}BA)^4 = \begin{bmatrix} \frac{1}{3} & \frac{8}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}^4 = \begin{bmatrix} 1 & -8 \\ 2 & 0 \end{bmatrix}.$$

7.18. Since the parabola must pass through the points (x_1, y_1), (x_2, y_2) and (x_3, y_3),

$$y_1 = a + bx_1 + cx_1^2,$$

$$y_2 = a + bx_2 + cx_2^2,$$

$$y_3 = a + bx_3 + cx_3^2.$$

These can be viewed as three linear equations in a, b and c, which can be expressed in the matrix form

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Let A be the 3×3 matrix on the left. By (7.11)

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2x_3^2 - x_2^2x_3) - x_1(x_3^2 - x_2^2) + x_1^2(x_3 - x_2) \\ &= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1). \end{aligned}$$

The determinant will be non-zero if x_1, x_2 and x_3 are all different. By (7.10) the inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} \left| \begin{matrix} x_2x_3^2 - x_3x_2^2 & -(x_1x_3^2 - x_3x_1^2) & x_1x_2^2 - x_2x_1^2 \\ -(x_3^2 - x_2^2) & x_3^2 - x_1^2 & -(x_2^2 - x_3^2) \\ x_3 - x_2 & -(x_3 - x_1) & x_2 - x_1 \end{matrix} \right],$$

which equals the answer given in the question after some factorization. It follows that

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{x_2x_3y_1}{(x_2-x_3)(x_3-x_1)} & \frac{x_3x_1y_2}{(x_3-x_2)(x_1-x_2)} & \frac{x_1x_2y_3}{(x_1-x_3)(x_2-x_3)} \\ -\frac{(x_2+x_3)y_1}{(x_2-x_3)(x_3-x_1)} & -\frac{(x_3+x_1)y_2}{(x_3-x_2)(x_1-x_2)} & -\frac{(x_1+x_2)y_3}{(x_1-x_3)(x_2-x_3)} \\ \frac{y_1}{(x_2-x_3)(x_3-x_1)} & \frac{y_2}{(x_3-x_2)(x_1-x_2)} & \frac{y_3}{(x_1-x_3)(x_2-x_3)} \end{bmatrix}.$$

Note that if, for example, $x_1 = x_2$ then there will no parabola of the form given if $y_1 \neq y_2$, but if $y_1 = y_2$ (that is, two points coincide) there will be an infinite set of such parabolas.

For the three points given $x_1 = -2$, $x_2 = 1$ and $x_3 = 3$,

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} \frac{1}{5} & 1 & -\frac{1}{5} \\ -\frac{4}{15} & \frac{1}{6} & \frac{1}{10} \\ \frac{1}{15} & -\frac{1}{6} & \frac{1}{10} \end{bmatrix}.$$

Hence

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{14}{5} \\ \frac{1}{15} \\ \frac{11}{15} \end{bmatrix}.$$

The required parabola through the given points is

$$y = -\tfrac{14}{5} + \tfrac{1}{15}x + \tfrac{11}{15}x^2.$$

7.19. If $a_{ij} = (-j)^i - ij$, then

$$A = \begin{bmatrix} -2 & -4 & -6 \\ -1 & 0 & 3 \\ -4 & -14 & -36 \end{bmatrix}.$$

Also

$$\det A = \begin{vmatrix} -2 & -4 & -6 \\ -1 & 0 & 3 \\ -4 & -14 & -36 \end{vmatrix} = 24, \qquad A^{-1} = \begin{bmatrix} \frac{7}{4} & -\frac{5}{2} & -\frac{1}{2} \\ -2 & 2 & \frac{1}{2} \\ \frac{7}{12} & -\frac{1}{2} & -\frac{1}{6} \end{bmatrix}.$$

7.20. Calculate the powers of A:

$$A^2 = AA = \begin{bmatrix} 8 & 7 & 11 \\ 3 & 6 & 3 \\ 5 & 1 & 8 \end{bmatrix},$$

$$A^3 = AA^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 8 & 7 & 11 \\ 3 & 6 & 3 \\ 5 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 34 & 23 & 49 \\ 15 & 3 & 24 \\ 19 & 20 & 25 \end{bmatrix}.$$

Hence

$$\begin{aligned} A^3 - 2A^2 - 9A &= \begin{bmatrix} 34 & 23 & 49 \\ 15 & 3 & 24 \\ 19 & 20 & 25 \end{bmatrix} - 2\begin{bmatrix} 8 & 7 & 11 \\ 3 & 6 & 3 \\ 5 & 1 & 8 \end{bmatrix} - 9\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \\ &= 0. \end{aligned}$$

Also

$$\begin{aligned} A^2 - 2A - 9\mathrm{I}_3 &= \begin{bmatrix} 8 & 7 & 11 \\ 3 & 6 & 3 \\ 5 & 1 & 8 \end{bmatrix} - 2\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 5 & 5 \\ 1 & -1 & -1 \\ 3 & -3 & -3 \end{bmatrix} \neq 0. \end{aligned}$$

Suppose that A^{-1} exists. Then

$$A^{-1}(A^3 - 2A^2 - 9A) = A^2 - 2A - 9\mathrm{I}_3 = 0,$$

which contradicts the result above. We conclude that the inverse of A does not exist.

7.21. Given $A^2 = A$ and $A \neq \mathrm{I}_n$.
(a) Suppose that A^{-1} exists. Then

$$A^{-1}A^2 = A^{-1}A \text{ or } A = \mathrm{I}_n,$$

which contradicts the assumption that A is not the identity matrix. We conclude that the inverse of A does not exist.

(b) Verify the result:

$$(\mathrm{I}_n + A)(\mathrm{I}_n - \tfrac{1}{2}A) = \mathrm{I}_n^2 + A - \tfrac{1}{2}A - \tfrac{1}{2}A^2 = \mathrm{I}_n + A - \tfrac{1}{2}A - \tfrac{1}{2}A = \mathrm{I}_n.$$

Hence $\mathrm{I}_n - \frac{1}{2}A$ must be the inverse of $\mathrm{I}_n + A$.
(c) Use proof by induction. For $m = 2$,

$$(\mathrm{I}_n + A)^2 = (\mathrm{I}_n + A)(\mathrm{I}_n + A) = \mathrm{I}_n + 2A + A^2 = \mathrm{I}_n + 3A,$$

since $A^2 = A$. Hence the result is certainly true for $m = 2$. Suppose we know that the result is true for some value of m, say $m = r$; that is that

$$(\mathrm{I}_n + A)^r = \mathrm{I}_n + (2^r - 1)A.$$

Then

$$\begin{aligned}(\mathrm{I}_n + A)^{r+1} &= (\mathrm{I}_n + A)(\mathrm{I}_n + A)^r = (\mathrm{I}_n + A)(\mathrm{I}_n + (2^r - 1)A \\ &= \mathrm{I}_n^2 + A + (2^r - 1)A + (2^r - 1)A^2 \\ &= \mathrm{I}_n + A + (2^r - 1)A + (2^r - 1)A \\ &= \mathrm{I}_n + (2^{r+1} - 1)A.\end{aligned}$$

Hence if the conjecture is true for $m = r$, then it is true for $m = r + 1$. Hence by induction it is true for $m = 3, 4, \ldots$.

7.22.

$$A_1 + A_2 = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{bmatrix}, \qquad A_1A_2 = \begin{bmatrix} x_1x_2 - y_1y_2 & x_2y_1 + x_1y_2 \\ -x_2y_1 - x_1y_2 & x_1x_2 - y_1y_2 \end{bmatrix}.$$

Note that $A_2A_1 = A_1A_2$. The inverse

$$A_1^{-1} = \frac{1}{x_1^2 + y_1^2}\begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix}.$$

These results parallel the rules for complex numbers. For the complex numbers:

$$z_1 + z_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2), \quad z_1z_2 = x_1x_2 - y_1y_2 + \mathrm{i}(x_1y_2 + x_2y_1) = z_2z_1,$$

$$\frac{1}{z_1} = \frac{x_1 + \mathrm{i}y_1}{x_1^2 + y_1^2}.$$

The top row in the matrix operations gives the real and imaginary parts of the corresponding complex ones.

Also $|z_1|^2 = x_1^2 + y_1^2$, corresponding to $\det A_1 = x_1^2 + y_1^2$.

For the exponential function

$$\mathrm{e}^{z_1} = 1 + z_1 + \frac{1}{2!}z_1^2 + \cdots.$$

Interpret the **exponential of the matrix** as

$$\mathrm{e}^{A_1} = \mathrm{I}_n + A_1 + \frac{1}{2!}A_1^2 + \cdots.$$

The real and imaginary parts of e^{z_1} are given by the elements on the top rows of the terms of e^{A_1}, since z^n corresponds to A^n.

Chapter 8: Determinants

8.1. (a)

$$\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = (1)\times(3) - (2)\times(-1) = 5.$$

(b)

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (1)\times\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - (0)\times\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + (1)\times\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1.$$

(c)

$$\begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix} = 1.$$

(d)

$$\begin{vmatrix} 2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 3 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{vmatrix} = 20.$$

(e) Expanding by the top row at each step

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1.$$

(f) After repeated expansion by the top rows

$$\begin{vmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2\times 5 - 4 = 6.$$

8.2. (a) The elements of row 1 are twice the elements in row 2 (Rule 5).
(b) Add the elements in row 2 to row 1 to give (Rule 6)

$$\begin{vmatrix} -1 & 2 & 3 \\ 3 & 1 & -2 \\ -2 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ -2 & --3 & -1 \end{vmatrix}.$$

The determinant is zero since the elements in row 1 are (-1) times the elements in row 3.
(c) The determinant is unchanged if the elements in row 1 become the elements in row 1 minus the elements in row 2 (Rule 6). The determinant is therefore zero since the first and third rows have the same elements (Rule 5).
(d) The determinant is zero since the elements in rows 2 and 3 are in the same ratio (3/5) (Rule 6).

8.3. Since a is a factor of each element in row 1 and also a factor of each element in column 1, and c is factor of each element in column 3, by Rule 2,

$$\begin{vmatrix} a^3 & ab & ac^2 \\ a & c & ac \\ bac & a & bc \end{vmatrix} = a^2c\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a^2c\Delta.$$

8.4. There are many ways of simplifying determinants using the rules in Section 8.2. The usual aim is to introduce zero elements and to reduce the size of numbers. These methods are illustrated in the following solutions. Note that operations between columns (c_i) are employed as well as operations between rows (r_i).

(a)

$$\begin{aligned}
\begin{vmatrix} 99 & 100 & 200 \\ 98 & 102 & 199 \\ -1 & 2 & 3 \end{vmatrix} &= \begin{vmatrix} 1 & -2 & 1 \\ 98 & 102 & 199 \\ -1 & 2 & 3 \end{vmatrix} \quad (r_1' = r_1 - r_2) \\
&= \begin{vmatrix} 1 & -2 & 0 \\ 98 & 102 & 101 \\ -1 & 2 & 3 \end{vmatrix} \quad (c_3' = c_3 - c_1) \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 98 & 298 & 101 \\ -1 & 0 & 4 \end{vmatrix} \quad (c_2' = c_2 - 2c_1) \\
&= 298 \times 4 = 1192.
\end{aligned}$$

(b)

$$\begin{aligned}
\begin{vmatrix} 77 & 84 & 55 \\ 75 & 87 & 57 \\ 1 & -2 & 3 \end{vmatrix} &= \begin{vmatrix} 2 & -3 & -2 \\ 75 & 87 & 57 \\ 1 & -2 & 3 \end{vmatrix} \quad (r_1' = r_1 - r_2) \\
&= 3 \begin{vmatrix} 25 & 29 & 19 \\ 1 & -2 & 3 \end{vmatrix} \quad (r_1' = r_1 - 2r_2) \\
&= 3 \begin{vmatrix} 0 & 1 & -8 \\ 25 & 3 & -6 \\ 1 & -3 & 2 \end{vmatrix} \quad (c_2' = c_2 - c_1,\ c_3' = c_3 - c_1) \\
&= -3 \begin{vmatrix} 25 & -6 \\ 1 & 2 \end{vmatrix} - 24 \begin{vmatrix} 25 & 4 \\ 1 & -3 \end{vmatrix} \\
&= -3 \times 56 - 24 \times (-79) = 1728
\end{aligned}$$

(c)

$$\begin{aligned}
\begin{vmatrix} 2 & -1 & 1 \\ 99 & 98 & 55 \\ 200 & 197 & 111 \end{vmatrix} &= \begin{vmatrix} 3 & -1 & 1 \\ 1 & 98 & 55 \\ 3 & 197 & 111 \end{vmatrix} \quad (c_1' = c_1 - c_2) \\
&= \begin{vmatrix} 0 & -198 & -110 \\ 1 & 98 & 55 \\ 3 & 197 & 111 \end{vmatrix} \quad (r_1' = r_1 - r_3) \\
&= \begin{vmatrix} 0 & -198 & -110 \\ 1 & -1 & 0 \\ 3 & -1 & 1 \end{vmatrix} \quad (r_2' = r_2 - \tfrac{1}{2}r_1,\ r_3' = r_3 + r_1) \\
&= \begin{vmatrix} 0 & -198 & -110 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{vmatrix} \\
&= -22
\end{aligned}$$

(d)

$$\begin{vmatrix} 87 & 84 & 83 & 81 \\ 77 & 76 & 77 & 75 \\ 54 & 53 & 52 & 54 \\ -43 & -44 & -46 & -4 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 2 & 81 \\ 0 & -1 & 2 & 75 \\ 2 & 1 & -2 & 54 \\ 3 & 2 & -42 & -4 \end{vmatrix} \quad \begin{matrix} (c_1' = c_1 - c_3) \\ (c_2' = c_2 - c_3) \\ (c_3' = c_3 - c_4) \end{matrix}$$

$$
\begin{aligned}
&= \begin{vmatrix} 4 & 0 & 4 & 156 \\ 0 & -1 & 2 & 75 \\ 2 & 0 & 0 & 129 \\ 3 & 0 & -38 & 146 \end{vmatrix} \quad \begin{matrix} (r_1' = r_1 + r_2) \\ (r_3' = r_3 + r_2) \\ (r_4' = r_4 + 2r_2) \end{matrix} \\
&= -\begin{vmatrix} 4 & 4 & 156 \\ 2 & 0 & 129 \\ 3 & -38 & 146 \end{vmatrix} \quad \text{(expanding by column 2)} \\
&= -\begin{vmatrix} 0 & 4 & 156 \\ 2 & 0 & 129 \\ 41 & -38 & 146 \end{vmatrix} \quad (c_1' = c_1 - c_2) \\
&= 4\begin{vmatrix} 2 & 129 \\ 41 & 146 \end{vmatrix} - 156\begin{vmatrix} 2 & 0 \\ 41 & -38 \end{vmatrix} = -8132
\end{aligned}
$$

8.5. If $b = a$, then the first two columns have the same elements which means that the determinant will be zero (Rule 5). We conclude that the determinant has a factor $(a - b)$. Similarly the determinant has factors $(b - c)$ and $(c - a)$. The determinant is of degree three in a, b and c which implies that

$$\Delta = k(b - c)(c - a)(a - b),$$

where k is a number. Compare the leading diagonal term $1 \times b \times c^2 = bc^2$ in Δ with the corresponding term in the expansion which is kbc^2. Hence $k = 1$, and

$$\Delta = (b - c)(c - a)(a - b).$$

8.6. Operations between columns give

$$
\begin{aligned}
\Delta &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3 - a^3 & c^3 - a^3 \end{vmatrix} \quad \begin{matrix} (c_2' = c_2 - c_1) \\ (c_3' = c_3 - c_1) \end{matrix} \\
&= (b - a)(c^3 - a^3) - (c - a)(b^3 - a^3).
\end{aligned}
$$

Taking out factors

$$
\begin{aligned}
\Delta &= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2 + a^2 + ba & c^2 + a^2 + ac \end{vmatrix} \\
&= (b-a)(c-a)\begin{vmatrix} 1 & 0 \\ b^2 + a^2 + ba & (c^2 - b^2) + a(c - b) \end{vmatrix} \quad (c_2' = c_2 - c_1) \\
&= (b-a)(c-a)(c-b)(a+b+c) = (b-c)(c-a)(a-b)(a+b+c)
\end{aligned}
$$

8.7. The determinant equation is linear equation in x and y, and therefore represents the equation of a straight line. Also the determinant is zero if $x = a_1$, $y = b_1$ since two rows have the same elements. Hence the line passes through the point (a_1, b_1). Similarly the line also passes through the point (a_2, b_2).

The cofactors of x and y are $X_1 = b_1 - b_2$ and $X_2 = a_2 - a_1$, and the slope is $-X_1/X_2$ $(a_1 \neq a_2)$.
(a) The required line is $y = 4x - 5$.
(b) The required line is $5y = -x - 1$.

8.8.

$$
\begin{aligned}
\begin{vmatrix} 1 & 1 & -1 \\ 1 & a & 2 \\ -1 & 1 & 2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & a-1 & 3 \\ -1 & 2 & 1 \end{vmatrix} \quad \begin{matrix} (c_3' = c_3 - c_1) \\ (c_2' = c_2 - c_1) \end{matrix} \\
&= (a - 1) - 6 = a - 7.
\end{aligned}
$$

The determinant is zero if $a = 7$.

8.9. Each term in the expansion contains three elements each from a different row and column. Since there are x's are in different rows and columns, terms in x^3 will appear in the expansion, although it is possible that terms could cancel. In the second determinant there is no x in row 1, the expansion will be of degree 2, at most, in x.

$$\begin{aligned}\begin{vmatrix} x & 2 & -2 \\ 2 & x & 3 \\ x & -1 & x \end{vmatrix} &= \begin{vmatrix} 2x+2 & x+1 & x+1 \\ 2 & x & 3 \\ x & -1 & x \end{vmatrix} \quad (r_1' = r_1 + r_2 + r_3) \\ &= (x+1)\begin{vmatrix} 2 & 1 & 1 \\ 2 & x & 3 \\ x & -1 & x \end{vmatrix} \\ &= x^3 + x^2 + 5x + 4\end{aligned}$$

Hence the determinant is zero if $x = -1$ or $x^2 + x + 4 = 0$. The solutions are

$$x = -1, \quad x = \tfrac{1}{2}(-1 \pm \mathrm{i}\sqrt{15}).$$

The expansion of the second determinant is

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & x & 2 \\ x & 1 & x \end{vmatrix} = 4 - x - x^2.$$

Hence the determinant is zero where $x = \frac{1}{2}(-1 \pm \sqrt{17})$.

8.10. By the expansion (8.3)

$$\begin{aligned}&\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= (a_{11}+b_{11})(a_{22}a_{33} - a_{32}a_{23}) - (a_{12}+b_{12})(a_{21}a_{33} - a_{31}a_{23}) + \\ &\quad (a_{13}+b_{13})(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) + \\ &\quad b_{11}(a_{22}a_{33} - a_{32}a_{23}) - b_{12}(a_{21}a_{33} - a_{31}a_{23}) + b_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.\end{aligned}$$

8.11. Expansion by row 1, as in the previous problem, will lead to 2 determinants. Then expansion by row 2 for each of these 2 determinants will result in $2^2 = 4$ determinants: the number doubles for each row. Therefore there will be $2^3 = 8$ determinants.

For an $n \times n$ determinant with the sum of two terms in each element, the expansion will lead to 2^n determinants.

8.12.

$$\begin{aligned}\begin{vmatrix} 1 & a_1 - b_1 & a_1 + b_1 \\ 1 & a_2 - b_2 & a_2 + b_2 \\ 1 & a_3 - b_3 & a_3 + b_3 \end{vmatrix} &= \begin{vmatrix} 1 & 2a_1 & a_1 + b_1 \\ 1 & 2a_2 & a_2 + b_2 \\ 1 & 2a_3 & a_3 + b_3 \end{vmatrix} \quad (c_2' = c_2 - c_3) \\ &= \begin{vmatrix} 1 & 2a_1 & b_1 \\ 1 & 2a_2 & b_2 \\ 1 & 2a_3 & b_3 \end{vmatrix} \quad (c_3' = c_3 - \tfrac{1}{2}c_2) \\ &= 2\begin{vmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{vmatrix}\end{aligned}$$

8.13. Easy to verify that

$$D_1 = 2, \quad D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

Expanding D_n by the top row

$$D_n = 2D_{n-1} - 1 \times 1 \times D_{n-2} = 2D_{n-1} - D_{n-2},$$

or, equivalently,

$$D_n - D_{n-1} = D_{n-1} - D_{n-2}.$$

It follows that

$$Q_n = D_n - D_{n-1} = Q_{n-1} = Q_{n-2} = \ldots = Q_2 = D_2 - D_1 = 2 - 1 = 1.$$

Also

$$D_n = D_{n-1} + Q_n = D_{n-1} + 1 = D_{n-2} + 2 = \cdots = D_2 + (n-1) = n + 1.$$

8.14. First observe that the determinant is a quartic in x, that is, a polynomial of the fourth degree. If $x = a$, then rows 1 and 2 have the same elements, which mean that the determinant is zero. Similarly, if $x = b$ and $x = c$, two rows have identical elements. There will be one more solution. Then

$$\begin{aligned} & \begin{vmatrix} x & a & b & c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix} \\ = & \begin{vmatrix} x+a+b+c & x+a+b+c & x+a+b+c & x+a+b+c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix} \\ & \quad (r_1' = r_1 + r_2 + r_3 + r_4) \\ = & \quad (x+a+b+c) \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix}, \end{aligned}$$

showing a factor $(x + a + b + c)$. Hence, the solutions are

$$x = a, b, c, -(a + b + c).$$

8.15. The determinants of A and B are

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}, \quad \det B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}b_{21}.$$

(a) The product AB is given by

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

Then it can be verified that

$$\begin{aligned} \det(AB) &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - \\ &\qquad (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22}) \\ &= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) = \det A \det B. \end{aligned}$$

Put $B = A$ to obtain $\det(A^2) = (\det A)^2$.
(b)

$$\det(A^T) = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{21} - a_{12}a_{21} = \det A.$$

(c) By (7.8)

$$A^{-1} = \frac{1}{\det A}\begin{vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{vmatrix}.$$

Since $\det A$ effectively divides each element in the matrix,

$$\det(A^{-1}) = \frac{1}{(\det A)^2}(a_{22}a_{11} - a_{21}a_{12}) = \frac{1}{(\det A)^2}\det A = \frac{1}{\det A}.$$

(d) Result follows from (c) since $A^{-1} = \operatorname{adj} A/\det A$.

8.16. The numerical answers are:
$\det A = -2$; $\det B = 18$; $\det AB = -36$; $\det(A^T) = -2$;
$\det(\operatorname{adj} A) = -2$. Since

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 1 & -5 \\ -2 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix},$$

then $\det(A^{-1}) = -\frac{1}{2}$

8.17. The matrix A in full is

$$A = \begin{bmatrix} \alpha-2 & 2\alpha-4 & 3\alpha-8 \\ \alpha+2 & 2\alpha+4 & 3\alpha+8 \\ \alpha-2 & 2\alpha-4 & 3\alpha-8 \end{bmatrix} = 0,$$

and its determinant is zero since rows 1 and 3 have the same elements.

Chapter 9: Elementary operations with vectors

9.1. (a) $\overline{PQ} = -\overline{QP} = (5,-3)$; (b) $\overline{PQ} = -\overline{QP} = (-1,-3)$; (c) $\overline{PQ} = -\overline{QP} = (-1,-3)$; (d) $\overline{PQ} = -\overline{QP} = (1,1)$.

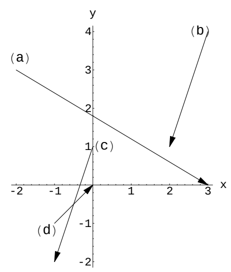

Figure 1: Problem 9.1

9.2. (a) $\sqrt{3}$, 0; (b) $\sqrt{2}$, 90°; (c) $\sqrt{2}$, 135°; (d) $\sqrt{2}$, 45°; (e) $\sqrt{2}$, −135°;
(f) 5, 127°; (g) 5, −126°; (h) $\sqrt{5}$, 153°.

9.3. (a) $(\sqrt{2}, \sqrt{2})$; (b) $(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$; (c) $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$; (d) $(-\frac{3\sqrt{2}}{2}, -\frac{3}{2})$.

9.4. $\overline{BE} = \overline{QE} - \overline{QB} = (3,3) + (1,1) + (2,3) - [(2,4) + (4,1)] = (6,1) - (6,5) = (0,-4)$. The bearing is due south.

9.5. (a) Distance between $(0,0,0)$ and $(1,2,3)$ is $\sqrt{(1^2+2^2+3^2)} = \sqrt{14}$.
(b) Distance between $(1,2,3)$ and $(3,2,1)$ is $\sqrt{[(1-3)^2 + (2-2)^2 + (3-1)^2]} = 2\sqrt{2}$.

(c) Distance between $(1, 0, -1)$ and $(-1, 1, 0)$ is

$$\sqrt{[(1-(-1))^2 + (0-1)^2 + (-1-0)^2]} = \sqrt{6}.$$

9.6. The vector $\overline{PQ} = (2, 3, 3) - (1, 2, 1) = (1, 1, 2)$. The projections respectively are the components of $\overline{PQ}$, namely $1, 1, 2$.

9.7. (a) $2\mathbf{a} = 2(1, 2, 1) = (2, 4, 2)$; $3\mathbf{b} = 3(2, 1, 2) = (6, 3, 6)$; $2\mathbf{a} - 3\mathbf{b} = (2, 4, 2) - (6, 3, 6) = (-4, 1, -4)$.
(b) $2\mathbf{a} = (6, 4, 6)$; $3\mathbf{b} = (3, 3, 6)$; $2\mathbf{a} - 3\mathbf{b} = (3, 1, 0)$.
(c) $2\mathbf{a} = (12, 6, 2)$; $3\mathbf{b} = (12, 6, 3)$; $2\mathbf{a} - 3\mathbf{b} = (0, 0, -1)$.

The vector $2\mathbf{a} - 3\mathbf{b}$ is parallel to the (x, y) plane in (b) because the z component is zero, and parallel to the z axis in (c) since both its x and y components are zero.

9.8. Figure 2 shows that $\overline{AB} + \overline{BC} + \overline{CA} = \mathbf{0}$ by the triangle law.

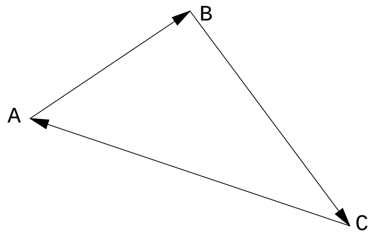

Figure 2: Problem 9.8

9.9. Figure 3 shows that $\overline{CD} = \overline{CB} + \overline{BA} + \overline{AD}$.

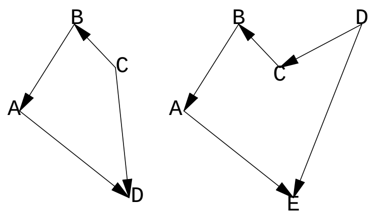

Figure 3: Problem 9.9

9.10. The vector $\overline{OP} = (5, 2, -3)$, and the vector giving the origin Q in terms of O is $\overline{OQ} = (2, -1, 3)$. Hence $\overline{QP} = \overline{OP} - \overline{OQ} = (3, 3, -6)$. In $QXYZ$, the point P has coordinates $(3, 3, -6)$.

The relation between coordinates is $x = X + 2$, $y = Y - 1$, $z = Z + 3$. Hence the equation of the sphere $x^2 + y^2 + z^2 = 1$ becomes the sphere

$$(X + 2)^2 + (Y - 1)^2 + (Z + 3)^2 = 1.$$

9.11. Let $\overline{AB} = \mathbf{a}$, $\overline{BC} = \mathbf{b}$, $\overline{CD} = \mathbf{c}$. By repeated application of the triangle rule, $\overline{AD} = \mathbf{a}+\mathbf{b}+\mathbf{c}$. Since P and Q are midpoints

$$\overline{PQ} = \overline{PB} + \overline{BQ} = \frac{1}{2}\overline{AB} + \frac{1}{2}\overline{BC} = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Simlarly

$$\overline{SR} = \overline{SD} + \overline{DR} = \frac{1}{2}\overline{AD} + \frac{1}{2}\overline{DC} = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \frac{1}{2}\mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \overline{PQ}.$$

The quadrilateral $PQRS$ has opposite sides which are parallel and equal in length, and is therefore a parallelogram.

9.12. $\overline{AP}$ is the median vector from A; F is a representative point, position vector $\mathbf{r}$, on AP. The parametric equation for AP is

$$\begin{aligned}\mathbf{r} &= \mathbf{r}_A + \lambda\overline{AP} = \mathbf{r}_A + \lambda(\mathbf{b} + \tfrac{1}{2}\mathbf{a}) \qquad (i)\\ &= \mathbf{r}_A + \lambda[\mathbf{r}_C - \mathbf{r}_A + \tfrac{1}{2}(\mathbf{r}_B - \mathbf{r}_C)]\\ &= (1-\lambda)_{\mathbf{r}A} + \tfrac{1}{2}\lambda\mathbf{r}_B + \tfrac{1}{2}\lambda\mathbf{r}_C, \qquad (ii)\end{aligned}$$

where $\mathbf{a} = \overline{CB}$, $\mathbf{b} = \overline{AC}$, $\mathbf{c} = \overline{AC}$ and λ is a parameter. By permuting the suffixes, $A \to B$, $B \to C$, $C \to A$, we obtain for the other medians:

$$\mathbf{r} = (1-\mu)\mathbf{r}_B + \tfrac{1}{2}\mu\mathbf{r}_C + \tfrac{1}{2}\mu\mathbf{r}_A, \qquad (iii)$$

and

$$\mathbf{r} = (1-\nu)\mathbf{r}_C + \tfrac{1}{2}\nu\mathbf{r}_A + \tfrac{1}{2}\nu\mathbf{r}_B, \qquad (iv)$$

where μ, ν are parameters. These lines meet at a single point C if values of λ, μ, ν can be found that make the right-hand sides (ii), (iii), (iv) equal. If we put $\lambda = \mu = \nu = \frac{2}{3}$ we obtain the common point G, with

$$\overline{OG} = \tfrac{1}{3}(\mathbf{r}_A + \mathbf{r}_B + \mathbf{r}_C).$$

Also the term $\lambda\overline{AP} = \frac{2}{3}\overline{AP}$ in (i) shows that G is two-thirds of the way along $\overline{AP}$.

9.13. Can one vector, say $\overline{OC}$, be expressed in terms of the other two? We require constants α and β such that, from eqn (9.12)

$$\overline{OC} = \alpha\overline{OA} + \beta\overline{OB}, \text{ or } (5,5,7) = \alpha(1,1,2) + \beta(1,1,1).$$

This will be possible if $\alpha + \beta = 5$ and $2\alpha + \beta = 7$. Hence $\alpha = 2$ and $\beta = 3$, and the three vectors drawn from the origin lie in the same plane.

Similarly, for $\overline{OA} = (a,a,p)$, $\overline{OB} = (b,b,q)$, $\overline{OC} = (c,c,r)$, we require α and β such that

$$c = \alpha a + \beta b, \qquad r = \alpha p + \beta q.$$

Hence, the lines lie in the same plane since solutions for α and β can be found:

$$\alpha = \frac{cq - rb}{aq - pb}, \qquad \beta = \frac{ra - cp}{aq - bp},$$

provided $aq - pb \neq 0$. If $aq - pb = 0$, the vectors $\overline{OA}$ and $\overline{OB}$ lie in the same direction, and the three vectors will still lie in a plane. The vectors all lie in a plane through the z axis, at 45° to the x and y axes.

9.14. If $\mathbf{v}_E$ is the velocity of the glider relative to the earth, then

$$\mathbf{v}_E = \mathbf{v} - \mathbf{w} = (40,30,10) - (5,-10,0) = (35,40,10).$$

9.15. Let $\mathbf{v}_{BS}$ be the velocity of the boat relative to the sea, $\mathbf{v}_B$ be the velocity of the boat and $\mathbf{v}_S$ the velocity of the sea. Then $\mathbf{v}_{BS} = \mathbf{v}_B - \mathbf{v}_S$, so that

$$\mathbf{v}_S = \mathbf{v}_B - \mathbf{v}_{BS} = (4,1) - (5,4) = (-1,-3).$$

9.16. Let $\mathbf{v}_{WC}$ be the velocity of the wind relative to the cyclist, $\mathbf{v}_W$ the velocity of the wind, and $\mathbf{v}_C$ the velocity of the cyclist. Suppose that $\mathbf{v}_W = (a,b)$. Then, in case (i):

$$\mathbf{v}_{WC} = (a,b) - (0,10) = (a,b-10).$$

Since $\mathbf{v}_{WC}$ has zero component in the northerly direction, it follows that $b = 10$. In case (ii)

$$\mathbf{v}_{WC} = (a, b) - (0, 20) = (a, b - 20) = (a, -10).$$

Since the wind appears to come from the north-west, $a = 10$. Its speed is

$$|\mathbf{v}_W| = \sqrt{[10^2 + 10^2]} = 10\sqrt{2}\text{km h}^{-1}.$$

9.17. Let $\mathbf{v}_S$ be the velocity of the ship, $\mathbf{v}_W$ the velocity of the wind and $\mathbf{v}_{WS}$ be the velocity of the wind relative to the ships. In both cases $\mathbf{v}_{WS} = \mathbf{v}_W - \mathbf{v}_S$. Let $\mathbf{v}_W = (a, b)$. Then

$$(a, b) - (0, -u) = (a, b - u) \text{ has direction } -\hat{\mathbf{i}}.$$

Hence $b = u$, and a is negative. Secondly

$$(a, b) - \left(-\frac{2u}{\sqrt{3}}, 0\right) = \left(a + \frac{2u}{\sqrt{3}}, b\right)$$

has direction

$$\tan\left(\frac{b}{a + \frac{2u}{\sqrt{3}}}\right) = -120°.$$

Therefore

$$\frac{u}{a + \frac{2u}{\sqrt{3}}} = -\sqrt{3},$$

so that

$$u = -\sqrt{3}a - 2u, \text{ or } a = -\sqrt{3}u.$$

The true velocity of the wind is $\mathbf{v}_W = (\sqrt{3}u, u)$.

9.18. The terminal point

$$\mathbf{R} = \mathbf{a} + 2\mathbf{r} = (2, 3, 1) + 2(1, 1, 2) = (4, 5, 5).$$

9.19. (a) 0°, 90°, 90°; (b) 45°, 45°, 90°;
(c) 90°, 90°, 180°; (d) $\arctan(1/\sqrt{3}) = 54.7°$, 54.7°, 54.7°;
(e) $\arctan(1/\sqrt{3}) = 54.7°$, 54.7°, 125.3°.

9.20. (a) The position vector of S is given by (a sketch of the points is helpful)

$$\overline{OS} = \overline{OP} + (\overline{PQ} + \overline{PR}) = (1, 1, 0) + (0, 0, 1) + (0, 1, 1) = (1, 2, 2).$$

(b) The diagonal $\overline{PS} = \overline{PQ} + \overline{PR} = (0, 1, 2)$. Hence the coordinates of the midpoint are

$$\overline{OP} + \tfrac{1}{2}\overline{PS} = (1, 1, 0) + \tfrac{1}{2}(0, 1, 2) = (1, \tfrac{3}{2}, 1).$$

(c) As in (b) the midpoint of QR has coordinates

$$\overline{OQ} + \tfrac{1}{2}\overline{QR} = \overline{OQ} + \tfrac{1}{2}(\overline{QS} + \overline{QP}) = (1, 1, 1) + \tfrac{1}{2}[(0, 1, 1) + (0, 0, -1)] = (1, \tfrac{3}{2}, 1),$$

which has the same coordinates as the midpoint of PS.

The coordinates of the midpoints are:
A has coordinates $\overline{OP} + \frac{1}{2}\overline{PR} = (1, 1, 0) + \frac{1}{2}(0, 1, 1) = (1, \frac{3}{2}, \frac{1}{2})$;
B has coordinates $\overline{OP} + \overline{PR} + \frac{1}{2}\overline{RS} = (1, 1, 0) + (0, 1, 1) + \frac{1}{2}(0, 0, 1) = (1, 2, \frac{3}{2})$;
C has coordinates $\overline{OP} + \overline{PQ} + \frac{1}{2}\overline{QS} = (1, 1, 0) + (0, 0, 1) + \frac{1}{2}(0, 1, 1) = (1, \frac{3}{2}, \frac{3}{2})$;
D has coordinates $\overline{OP} + \frac{1}{2}\overline{PQ} = (1, 1, 0) + \frac{1}{2}(0, 0, 1) = (1, 1, \frac{1}{2})$.
Hence $ABCD$ is a parallelogram since $\overline{AB} = (0, \frac{1}{2}, 1) = \overline{DC}$.

9.21. Since $\overline{AB} = (2, 1, -1)$ and $\overline{AC} = (-4, -2, 2)$, then $\overline{AC} = -2\overline{AB}$.
(a) Since $\overline{AC}$ is of the opposite sign to $\overline{AB}$, then A is between C and B.
(b) If P is a point on the line, then $\overline{OP} = \overline{OA} + \overline{AP} = \overline{OA} + \lambda\overline{AB}$, where λ is a parameter.
(c) As in (b)

$$(x, y, z) = (2\lambda + 1, \lambda + 2, -\lambda - 1) = \lambda(2, 1, -1) + (1, 2, -1) = \lambda\overline{AB} + \overline{OA},$$

which is the equation of the straight line.

9.22. (a) By the triangle law $\overline{AB} = \mathbf{b} - \mathbf{a}$. Then

$$\overline{OC} = \overline{OA} + \frac{1}{2}\overline{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

(b) As in (a)

$$\overline{OU} = \overline{OA} + \frac{1}{4}(\mathbf{b} - \mathbf{a}) = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}.$$

(c) As in (b)

$$\overline{OV} = \overline{OA} + \overline{AV} = \overline{OA} + \frac{1}{2}\overline{BA} = \mathbf{a} + \frac{1}{2}(\mathbf{a} - \mathbf{b}) = \frac{3}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}.$$

9.23. Let $\overline{OA} = \mathbf{a}$ and $\overline{OB} = \mathbf{b}$.
(a) We are given that $AU/UB = \lambda$ where $0 < \lambda < 1$. Hence $\overline{AU} = \lambda\overline{UB}$, and

$$\overline{OU} = \overline{OA} + \overline{AU} = \mathbf{a} + \lambda\overline{UB} = \mathbf{a} + \lambda(\mathbf{b} - \overline{OU}).$$

Hence

$$\overline{OU} = \frac{1}{1 - \lambda}(\mathbf{a} + \lambda\mathbf{b}).$$

(b) Since V does not lie between A and B, $\overline{AV} = \lambda\overline{BA}$, where $0 < \lambda < 1$. Thus

$$\overline{OV} = \overline{OA} + \overline{AV} = \mathbf{a} + \lambda(\overline{OV} - \mathbf{b}).$$

Therefore

$$\overline{OV} = \frac{1}{1 + \lambda}(\mathbf{a} - \lambda\mathbf{b}).$$

(c) If $\lambda > 1$, then B is outside AB, in the direction $\overline{AB}$.

9.24. (a) The vector $(2, 3, 4) - (1, 2, 3) = (1, 1, 1)$ is a vector in the direction of the line. If $\mathbf{r}$ is the position vector a general point on the line, then

$$\mathbf{r} = (x, y, z) = (1, 4, 2) + \lambda(1, 1, 1), \text{ or } x = 1 + \lambda, y = 4 + \lambda, z = 2 + \lambda.$$

(b) From (a) $\lambda = x - 1 = y - 4 = z - 2$. The equation of the line can be represented in the form of two simultaneous equations

$$x - 1 = y - 4 = z - 2.$$

(c) Using the representation in (b), the line intersects the (x, y) plane where

$$x - 1 = y - 4 = -2, \text{ that is, at } (-1, 2, 0).$$

Similarly, the line meets the (y, z) plane where

$$-1 = y - 4 = z - 2, \text{ that is, at } (0, 3, 1).$$

(d) Using these points the equation can be expressed in the form

$$\mathbf{r} = (-1, 2, 0) + \nu[(-1, 2, 0) - (0, 3, 1)] = (-1, 2, 0) + \nu(-1, -1, -1).$$

Two more equations for the line could be

$$x + 1 = y - 2 = z.$$

9.25. The position vector of P can be expressed as

$$\mathbf{r} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b}).$$

Hence P describes a straight line through A and B.

Looking at the line through AB, if $\lambda < 0$ then P lies on the extension of AB, if $0 < \lambda < 1$, P lies between A and B, and if $\lambda > 1$, P lies on the extension of BA.

9.26. The equation of a plane through the points $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ can be expressed in the from

$$\mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}).$$

(a) The plane is

$$\mathbf{r} = (1, 0, 1) + \lambda(-1, 1, -1) + \mu(-1, 0, 0).$$

Hence

$$x = 1 - \lambda - \mu, \quad y = \lambda, \quad z = 1 - \lambda.$$

Eliminate λ between the second two equations gives the plane $y + z = 1$: x can take any value.

(b) The plane is

$$\mathbf{r} = (0, 0, 0) + \lambda(1, 2, -1) + \mu(2, 2, 2).$$

Hence

$$x = \lambda + 2\mu, \quad y = 2\lambda + 2\mu, \quad z = -\lambda + 2\mu.$$

Eliminate λ giving

$$x + z = 4\mu, \quad y + 2z = 6\mu.$$

Finally eliminate μ:

$$3x - 2y - z = 0$$

which is the equation of the plane.

9.27. The coordinates of any point on the line are $(1 + t, 2 + t, 3 + t)$. The square of its distance from the origin is

$$r^2 = x^2 + y^2 + z^2 = (1 + t)^2 + (2 + t)^2 + (3 + t)^2 = 14 + 12t + 3t^2.$$

Then

$$\frac{\mathrm{d}(r^2)}{\mathrm{d}t} = 12 + 6t,$$

which is zero where $t = -2$. The distance is a stationary minimum since the second derivative is 6 which is positive. The minimum distance is $\sqrt{(14 - 24 + 12)} = \sqrt{2}$.

9.28. The unit vectors required are $\pm\mathbf{a}/|\mathbf{a}|$.

(a) $\pm(\frac{3}{\sqrt{34}}, \frac{4}{\sqrt{34}}, \frac{3}{\sqrt{34}})$; (b) $\pm(\frac{2}{7}, \frac{3}{7}, \frac{6}{7})$;

(c) $\pm(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$; (d) $\pm(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$;

(e) $\pm(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$.

9.29. (a) The vector is $(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ with length $\sqrt{29}$.

(b) $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, and length $\sqrt{14}$.

9.30. The vector equation of the line is

$$\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \lambda(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}).$$

If $x=0$, then $1+2\lambda=0$ or $\lambda=-\frac{1}{2}$, in which case $\mathbf{r}=\frac{7}{2}\hat{\mathbf{k}}$. The $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components are both zero which means that the line cuts the z axis.

9.31. A square bounded by the 4 lines $\pm x \pm y = 1$.

9.32. An octahedron bounded by the 8 triangular faces $\pm x \pm y \pm z = 1$.

9.33. The centre of mass is at

$$\tilde{\mathbf{r}} = \tfrac{1}{6}[1(1,1,2)+2(-2,3,5)+3(0,3,2)] = \tfrac{1}{6}(-3,16,-2) = -\tfrac{1}{2}\hat{\mathbf{i}}+\tfrac{8}{3}\hat{\mathbf{j}}-\tfrac{1}{3}\hat{\mathbf{k}}.$$

9.34. The vector equation of the line is $\mathbf{r}=(1,1,1)+\lambda(1,2,-1)$. The line meets the plane where

$$-2 = x-y+z = (1+\lambda)-(1+2\lambda)+(1-\lambda 0 = 1-2\lambda.$$

Hence $\lambda=\frac{3}{2}$. Therefore they intersect at $\frac{5}{2}\hat{\mathbf{i}}+4\hat{\mathbf{j}}-\frac{1}{2}\hat{\mathbf{k}}$.

9.35. The aircraft's path is given by the position vector

$$\mathbf{r} = \hat{\mathbf{i}}[P+R\cos(Vt/R)]+\hat{\mathbf{j}}\sin(Vt/R)+H\hat{\mathbf{k}}.$$

9.36. Let $\overline{OA}$ and $\overline{OB}$ be unit vectors in the directions of $\mathbf{a}$ and $\mathbf{b}$. Thus

$$\overline{OA} = \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}, \quad \overline{OB} = \hat{\mathbf{b}} = \frac{\mathbf{a}}{|\mathbf{b}|}.$$

If C is the midpoint of AB, then OC bisects the angle AOB. Hence

$$\overline{OC} = \overline{OA}+\frac{1}{2}\overline{AB} = \mathbf{a}+\frac{1}{2}(\hat{\mathbf{b}}-\hat{\mathbf{a}}) = \frac{1}{2}\left(\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}\right).$$

9.37. (a) For A: its path is given by

$$\mathbf{r}_A = 0.41\hat{\mathbf{i}}+148t\hat{\mathbf{j}}+0.99\hat{\mathbf{k}},$$

which describes a straight line since the components are linear in t. The velocity of A is

$$\mathbf{v}_A = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 148\hat{\mathbf{j}}.$$

Hence its speed is 148 km.hr^{-1}. For B: its path is given by

$$\mathbf{r}_B = 100t\hat{\mathbf{i}}+250t\hat{\mathbf{j}}+250t\hat{\mathbf{k}},$$

which also describes a straight line. The velocity of B is

$$\mathbf{v}_B = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 100\hat{\mathbf{i}}+250\hat{\mathbf{j}}+250\hat{\mathbf{k}}.$$

Its speed is $\sqrt{[100^2+250^2+250^2]} = 150\sqrt{6} \approx 367$ km.hr^{-1}.
(b) Let $S(t)$ be the distance between A and B at time t. Then

$$S(t) = \sqrt{[(100t-0.41)^2+(250t-148t)^2+(250t-0.99)^2]}.$$

We require the minimum value of $S(t)$, which is the same as that of $[S(t)]^2$. Differentiate $[S(t)]^2$:

$$\frac{\mathrm{d}[S(t)]^2}{\mathrm{d}t} = 200(100t-0.41)+2\times 102+500(250t-0.99),$$

which is zero when $t=0.0035$ hr=12.6 s. The minimum distance is $S(0.0035)=0.38$ km, which is a close encounter for aircraft.

9.38. (a) The position vector of B relative to A is $\mathbf{r}_B - \mathbf{r}_A$. The velocity of B relative A is

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}_B - \mathbf{r}_A] = \frac{\mathrm{d}\mathbf{r}_B}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{r}_A}{\mathrm{d}t}.$$

(b) For the given points $A : (t, -t^2, t)$ and $B : (t^3, 2t^2, 1 + 3t)$, the velocity of B relative to A is

$$\frac{\mathrm{d}}{\mathrm{d}t}[(t - t^3)\hat{\mathbf{i}} + (-t^2 - 2t^2)\hat{\mathbf{j}} + (t - 1 - 3t)\hat{\mathbf{k}}] = (1 - 3t^2)\hat{\mathbf{i}} - 6t\hat{\mathbf{j}} - 2\hat{\mathbf{k}}.$$

The velocity of A relative to B is the vector in the opposite direction, namely

$$\frac{\mathrm{d}\mathbf{r}_A}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{r}_B}{\mathrm{d}t} = -(1 - 3t^2)\hat{\mathbf{i}} + 6t\hat{\mathbf{j}} + 2\hat{\mathbf{k}}.$$

(c) The relative speed is $v(t) = \sqrt{[(1 - 3t^2)^2 + 36t^2 + 4]}$. Since $v(t)$ is never zero, any stationary point of $v(t)$ occurs at the same time as any stationary point of $[v(t)]^2$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}[v(t)]^2 = 12t(5 + 3t^2),$$

which is zero only when $t = 0$. The stationary value is a minimum since the second derivative is obviously positive.

9.39. Given $\mathbf{r} = \hat{\mathbf{i}}a\cos\omega t + \hat{\mathbf{j}}\sin\omega t$, the velocity and acceleration are given by

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -\hat{\mathbf{i}}a\omega\sin\omega t + \hat{\mathbf{j}}b\omega\cos\omega t,$$

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\hat{\mathbf{i}}a\omega^2\cos\omega t - \hat{\mathbf{j}}b\omega^2\sin\omega t = -\omega^2\mathbf{r}.$$

Hence the acceleration vector is in the opposite direction to $\mathbf{r}$ at the particle, and must, therefore, be directed towards the origin.

9.40. The first and second derivatives of $r = \sec t$ and $\theta = t$ are

$$\dot{r} = \sec t\tan t, \quad \ddot{r} = 2\sec^3 t - \sec t, \qquad \dot{\theta} = 1, \quad \ddot{\theta} = 0.$$

From Example 9.15, the radial and transverse components of acceleration are given by

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta = 2\sec t\tan^2 t\,\hat{\mathbf{e}}_r + 2\sec t\tan t\,\hat{\mathbf{e}}_\theta.$$

9.41. Given $\mathbf{r} = \hat{\mathbf{i}}a\cos\omega t\sin\nu t + \hat{\mathbf{j}}a\sin\omega t\sin\nu t + \hat{\mathbf{k}}\cos\nu t$,

$$\begin{aligned}|\mathbf{r}|^2 &= a^2\cos^2\omega t\sin^2\nu t + a^2\sin^2\omega t\sin^2\nu t + a^2\cos^2\nu t,\\ &= a^2\sin^2\nu t + a^2\cos^2\nu t = a^2\end{aligned}$$

The velocity is given by

$$\begin{aligned}\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \hat{\mathbf{i}}a(-\omega\sin\omega t\sin\nu t + \nu\cos\omega t\cos\nu t)\\ &\quad +\hat{\mathbf{j}}a(\omega\cos\omega t\sin\nu t + \nu\sin\omega t\cos\nu t) - \hat{\mathbf{k}}a\nu\sin\nu t\end{aligned}$$

Its magnitude is

$$\begin{aligned}|\mathbf{v}| &= a[(-\omega\sin\omega t\sin\nu t + \nu\cos\omega t\cos\nu t)^2\\ &\quad +(\omega\cos\omega t\sin\nu t + \nu\sin\omega t\cos\nu t)^2 + \nu^2\sin^2\nu t]^{\frac{1}{2}}\\ &= a(\nu^2 + \omega^2\sin^2\nu t)^{\frac{1}{2}} = v(t),\ \text{say}.\end{aligned}$$

To find the minimum and maximum speeds differentiate $v(t)$ with respect to t:

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{a\omega^2 \nu \sin \nu t \cos \nu t}{(\nu^2 + \omega^2 \sin^2 \nu t)^{\frac{1}{2}}}.$$

This zero when $\sin \nu t \cos \nu t = 0$, that is when $t = 0, \frac{\pi}{2\nu}, \frac{\pi}{\nu}, \frac{3\pi}{2\nu}$. At $t = 0$ and $t = \frac{\pi}{\nu}$, the sign of $\mathrm{d}v/\mathrm{d}t$ changes from negative to positive as t increases through these values of t. Hence they are both minima, and they occur at $\mathbf{r} = \pm\hat{\mathbf{k}}a$, the highest and lowest points of the sphere. Similarly, maxima occur at the times $t = \frac{\pi}{2\nu}$ and $t = \frac{3\pi}{2\nu}$. At these times the $\hat{\mathbf{k}}$ component of $\mathbf{r}$ is zero which means that the maximum speeds occur on the equator of the sphere.

Chapter 10: The scalar product

10.1. (a) $(2,2,1)\cdot(3,1,2) = 10$; (b) $(2,-3,2)\cdot(-2,3,-1) = -15$;
(c) $(2,2,-3)\cdot(-1,1,-2) = 6$; (d) $(2,3,4)\cdot(1,-2,1) = 0$;
(e) $p-q, p+q, p)\cdot(p+q, q, -p-q) = (p^2-q^2) + q(p+q) + p(-p-q) = 0$.

10.2. (a) $(2,3)\cdot(3,4) = 18$; (b) $(1,0)\cdot(0,1) = 0$; (c) $(5,6)\cdot(0,-4) = -24$; (d) $(2,3)\cdot(3,-2) = 0$.

10.3. By (10.1c)

$$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\ &= 2\mathbf{a}\cdot\mathbf{a} + 2\mathbf{b}\cdot\mathbf{b} = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2), \end{aligned}$$

using (10.1c) again.

10.4. (a) $(2,-3,4)\cdot(-1,-2,3) = 16$.
(b) $\mathbf{a}\cdot\mathbf{b} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\cdot(-1\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = -2 + 6 + 12 = 16$,
since $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = 1$, $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = 0$, and so on.

10.5. Given that $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$,
(a) $\mathbf{a}\cdot\mathbf{b} = (1,2,-1)\cdot(1,3,1) = 6$;
(b) $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (0,-1,-2)\cdot(2,5,0) = -5$;
(c) $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = (0,-1,-2)\cdot(0,-1,-2) = 5$;
(d)

$$\begin{aligned} \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} &= (1,2,-1)\cdot(1,2,-1) + 2(1,2,-1)\cdot(1,3,1) + (1,3,1)\cdot(1,3,1) \\ &= 6 + 12 + 11 = 29, \end{aligned}$$

or note that it is the same as $|\mathbf{a}+\mathbf{b}|^2$.
(e) $(\mathbf{a}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\mathbf{b} = 6(1,2,-1) - 11(1,3,1) = (6,12,-6) - (11,33,11) = (-5,-21,-17)$

10.6. Use (10.4) which states that the angle θ between $\mathbf{a}$ and $\mathbf{b}$, in the range $0 \le \theta \le 180°$, is given by

$$\theta = \arccos\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

(a) $\theta = \arccos[2/(\sqrt{3}\sqrt{2}) = 35.3°$;
(b) $\theta = \arccos 0 = 90°$;
(c) $\theta = \arccos[(2-3+6)/(\sqrt{14}\sqrt{14})] = \arccos[5/14] = 69.1°$.

10.7. (a) $\arccos 0 = 90°$;
(b) $\theta = \arccos[(2+2)/(\sqrt{5}\sqrt{5})] = \arccos[4/5] = 36.9°$;
(c) $\theta = \arccos 0 = 90°$.

10.8. Imagine a cube placed with the origin of coordinate axes at one corner, and with three edges coincident with the positive directions of the axes. Assume that these three edges are represented by the vectors $a\hat{\mathbf{i}}$, $a\hat{\mathbf{j}}$ and $a\hat{\mathbf{k}}$ (the cube has side-length a). The diagonal joins the origin $(0,0,0)$ to (a,a,a) which can be represented by the vector $\mathbf{r} = a\hat{\mathbf{i}} + a\hat{\mathbf{j}} + a\hat{\mathbf{k}}$. The angle α between this diagonal and, say, the x axis is given by

$$\alpha = \arccos[a\hat{\mathbf{i}} \cdot \mathbf{r}/(a.a\sqrt{3})] = \arccos[1/\sqrt{3}] = 54.7^\circ.$$

10.9. The position vector $\mathbf{r}$ of any point on the cone will be in the cone, and always make an angle α with the axial unit vector $\hat{\mathbf{a}}$. Hence, by (10.4a) the equation of the cone is

$$\hat{\mathbf{a}} \cdot \mathbf{r} = |\mathbf{r}| \cos \alpha.$$

Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, $\hat{\mathbf{a}} = \frac{2}{7}\hat{\mathbf{i}} - \frac{3}{7}\hat{\mathbf{j}} - \frac{6}{7}\hat{\mathbf{k}}$ and $\alpha = 60^\circ$. Then the cartesian equation of the cone is

$$(\tfrac{2}{7}, -\tfrac{3}{7}, -\tfrac{6}{7}) \cdot (x, y, z) = \tfrac{1}{2}\sqrt{(x^2 + y^2 + z^2)},$$

or, after squaring both sides,

$$4(2x - 3y - 6z)^2 = 49(x^2 + y^2 + z^2),$$

or

$$33x^2 + 13y^2 - 95z^2 + 48xy - 144yz + 96zx = 0.$$

10.10. Let $\overline{BC} = \mathbf{a}$, $\overline{CA} = \mathbf{b}$, and $\overline{AB} = \mathbf{c}$. By the triangle law,

$$\mathbf{a} = (-1, 1, -1), \quad \mathbf{b} = (3, 0, -1), \quad \mathbf{c} = (-2, -1, 2).$$

In each case the scalar product gives the *supplement* of the corresponding internal angle. Thus, $\alpha = \widehat{BAC}$, $\beta = \widehat{CBA}$, $\gamma = \widehat{ACB}$, then

$$\cos(\pi - \alpha) = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = \frac{-8}{\sqrt{10}\sqrt{9}} = -\frac{8}{3\sqrt{10}},$$

$$\cos(\pi - \beta) = \frac{\mathbf{c} \cdot \mathbf{a}}{|\mathbf{c}||\mathbf{a}|} = \frac{-1}{\sqrt{9}\sqrt{3}} = -\frac{1}{3\sqrt{3}},$$

$$\cos(\pi - \gamma) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-2}{\sqrt{3}\sqrt{10}} = -\frac{2}{\sqrt{30}}.$$

We can now find the internal angles;

$$\alpha = 32.5^\circ, \quad \beta = 78.9^\circ, \quad \gamma = 68.6^\circ.$$

10.11. Using (10.1c)

$$\begin{aligned} \frac{1}{4}(|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2) &= \frac{1}{4}[(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) - (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})] \\ &= \mathbf{a} \cdot \mathbf{b} \end{aligned}$$

10.12. From the definition of the scalar product

$$\frac{\mathbf{F} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{|\mathbf{F}||\mathbf{a}| \cos \theta}{|\mathbf{a}|} = |\mathbf{F}| \cos \theta,$$

which is the component of $\mathbf{F}$ in the direction of $\mathbf{a}$. Due regard to sign occurs since $0 \le \theta \le \pi$. For each of the vectors

$$\frac{\mathbf{F} \cdot \mathbf{a}}{|\mathbf{a}|} = \mathbf{F} \cdot \hat{\mathbf{a}} = (8, 15, 9) \cdot (2, 3, 6)/7 = (16 + 45 + 54)/7 = 115/7;$$

$$\mathbf{F}\cdot\hat{\mathbf{b}} = (8,15,9)\cdot(0,3,4)/7 = (45+36)/7 = 81/7,$$
$$\mathbf{F}\cdot\hat{\mathbf{c}} = (8,15,9)\cdot(2,2,1) = (16+30+9)/7 = 55/7.$$

Looking at the components in $\mathbf{F} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$, we have

$$8 = 2\lambda + 2\nu,$$
$$15 = 3\lambda + 3\mu + 2\nu,$$
$$9 = 6\lambda + 4\mu + \nu.$$

Now solve these equations by elimination: the solution is

$$\lambda = -13/11, \quad \mu = 310/11, \quad \nu = 57/11.$$

Hence
$$\mathbf{F} = -\tfrac{13}{11}\mathbf{a} + \tfrac{30}{11}\mathbf{b} + \tfrac{57}{11}\mathbf{c}.$$

10.13. Form the scalar product of **a** and **b**:

$$\mathbf{a}\cdot\mathbf{b} = (1,3,4)\cdot(-2,6,-4) = 0,$$

which means that the vectors are perpendicular.

The vector $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ will be perpendicular to both **a** and **b** if $\mathbf{c}\cdot\mathbf{a} = 0$ and $\mathbf{c}\cdot\mathbf{b} = 0$. This will be the case if
$$c_1 + 3c_2 + 4c_3 = 0,$$
and
$$-2c_1 + 6c_2 - 4c_3 = 0.$$

We have two equations in three unknowns, so specify one of them: put, say, $c_3 = 1$. Then

$$c_1 + 3c_2 = -4, \text{ and } -2c_1 + 6c_2 = 4.$$

Hence $c_1 = -3$ and $c_2 = -\frac{1}{3}$. The solution is any (nonzero) multiple of $(-9,-1,3)$. Unit vectors in the directions of **c** and $-\mathbf{C}$ are

$$\hat{\mathbf{c}} = \pm\frac{(-9,-1,3)}{\sqrt{(81+1+9)}} = \frac{1}{\sqrt{91}}.$$

10.14. By (10.4), the angle is given by

$$\arccos\frac{1,1,-1)\cdot(2,-1,2)}{3\sqrt{3}} = \arccos[1/\sqrt{3}] = 101.1^\circ.$$

Let $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ be a vector perpendicular to both **a** and **b**. Then we must have $\mathbf{c}\cdot\mathbf{a} = \mathbf{c}\cdot\mathbf{b} = 0$, which in component form become:

$$c_1 + c_2 - c_3 = 0, \text{ and } 2c_1 - c_2 + 2c_3 = 0.$$

These are two equations in three unknowns, so specify one component, $c_1 = 1$, and solve the two equations for c_2 and c_3. The result is $c_2 = -4$, $c_3 = -3$. Hence any vector which is perpendicular to both **a** and **b** is any multiple of $\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$.

10.15. The two vectors are perpendicular if

$$(\lambda,2,-1)\cdot(1,1,-3\lambda) = \lambda + 2 + 3\lambda = 2 + 4\lambda = 0.$$

Hence $\lambda = -\frac{1}{2}$.

10.16. The three vectors are mutually perpendicular if $\mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b} = 0$. Hence α, β, and γ must satisfy

$$-2 + 2\beta - 6\gamma = 0, \quad 2\alpha + 2 + 9\gamma = 0, \quad -\alpha + 4\beta - 6 = 0.$$

By elimination the solutions are

$$\alpha = -\tfrac{2}{5}, \quad \beta = \tfrac{7}{5}, \quad \gamma = \tfrac{2}{15}.$$

10.17. The vectors giving relevant edges of the tetrahedron are

$$\overline{AB} = (-1, 1, 0), \quad \overline{AD} = (-1, y, z), \quad \overline{BD} = (0, y-1, z),$$

$$\overline{BC} = (0, 0, 1), \quad \overline{DC} = (0, 1-y, 1-z).$$

Then $\widehat{BCD}$ is a right angle if $\overline{BC}\cdot\overline{DC} = (0,0,1)\cdot(0,1-y,1-z) = 1-z = 0$. Hence $z = 1$. The triangle ABD is equilateral if $AB = BD = DA$, that is, if

$$\sqrt{2} = \sqrt{[1 + (y-1)^2]} = \sqrt{(y^2+2)}.$$

Obviously, $y = 0$. Hence D is the point $(0, 0, 1)$.

10.18. The axes are shown in the figure. For a general point (x, y) whose coordinates are (X, Y) in axes rotated through an angle α anticlockwise

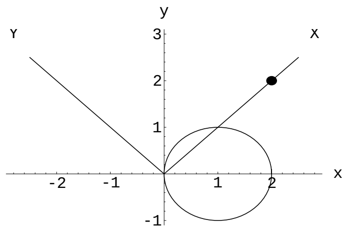

Figure 4: Problem 10.18

$$x = X\cos\alpha - Y\sin\alpha, \quad y = X\sin\alpha + Y\cos\alpha,$$

or solving for X and Y

$$X = x\cos\alpha + y\sin\alpha, \quad Y = -x\sin\alpha + y\cos\alpha.$$

(a) Given $\alpha = 45°$ and $P : (x, y) = (2, 2)$, $X = 2\frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{2}} = 2\sqrt{2}$, $Y = -2\frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{2}} = 0$.

(b) If $Q : (X, Y) = (1, -1)$, then $x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$, $y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$.

(c) Express x and y in terms of X and Y in the equation of the circle. It becomes

$$\left(\frac{1}{\sqrt{2}}X - \frac{1}{\sqrt{2}}Y - 1\right)^2 + \left(\frac{1}{\sqrt{2}}X + \frac{1}{\sqrt{2}}Y\right)^2 = 1,$$

which can be simplified to

$$\left(X - \frac{1}{\sqrt{2}}\right)^2 + \left(Y + \frac{1}{\sqrt{2}}\right)^2 = 1.$$

10.19. The lengths and direction cosines are (a) 1, $(0, 1, 0)$; (b) $\sqrt{3}$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$;
(c) 3, $(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3})$; (d) $\sqrt{3}$, $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$;
(e) $\sqrt{3}$, $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$; (f) 7, $(\frac{2}{7}, \frac{3}{7}, \frac{6}{7})$;
(g) 3, $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$; (h) 3, $(0, 0, 1)$; (i) 3, $(0, 0, -1)$.

10.20. (a) Check the scalar products:

$$19\mathbf{Y}\cdot 19\mathbf{Z} = (15,-10,6)\cdot(10,6,-15) = 150-60-90 = 0,$$

$$19\mathbf{Z}\cdot 19\mathbf{X} = (10,6,-15)\cdot(6,15,10) = 60+90-150 = 0,$$

$$19\mathbf{X}\cdot 19\mathbf{Y} = (6,15,10)\cdot(15,-10,6) = 90-150+60 = 0.$$

They are all zero: hence the vectors are mutually perpendicular. They are also all unit vectors since for each $6^2+15^2+10^2 = 361 = 19^2$.

(b) The base vectors for $OXYZ$ are $\hat{\mathbf{I}} = (6,15,10)/19$, $\hat{\mathbf{J}} = (15,-10,6)/19$, $\hat{\mathbf{K}} = (10,6,-15)/19$. Hence using (10.13a)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 6 & 15 & 10 \\ 15 & -10 & 6 \\ 10 & 6 & -15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Similarly

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 6 & 15 & 10 \\ 15 & -10 & 6 \\ 10 & 6 & -15 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

(c) From (b)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 6 & 15 & 10 \\ 15 & -10 & 6 \\ 10 & 6 & -15 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 56 \\ 7 \\ -8 \end{bmatrix}.$$

(d) The plane $x+y+z=0$ in the new coordinates is

$$[(6X+15Y+10Z)+(15X-10Y+6Z)+(10X+6Y-15Z)]/19 = 0,$$

or

$$31X+11Y+Z = 0.$$

10.21. The direction cosines are
(a) $\pm(\frac{3}{13},\frac{4}{13},\frac{12}{13})$; (b) $\pm(\frac{6}{19},-\frac{10}{19},\frac{15}{19})$.

10.22. We are given that for each particle with position vector $\mathbf{r(t)}$, its velocity is $\mathbf{v} = f(t)\mathbf{r}$. Let $\mathbf{s(t)}$ be the position vector of another particle. The velocity of the particle with position vector $\mathbf{r}$ relative to that with position vector $\mathbf{s}$ is

$$\mathbf{v}_s = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}-\mathbf{s}) = f(t)\mathbf{r} - f(t)\mathbf{s} = f(t)(\mathbf{r}-\mathbf{s}).$$

Hence the relative velocity obeys the same rule.

10.23. If α_1, α_2 and α_3 are the angles the vector makes with the axes then

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1.$$

In this problem $\cos\alpha_1 = \cos 45° = 1/\sqrt{2}$, $\cos\alpha_3 = \cos 60° = \frac{1}{2}$, so that

$$\tfrac{1}{2} + \cos^2\alpha_2 + \tfrac{1}{4} = 1, \text{ or } \cos^2\alpha_2 = \tfrac{1}{4}.$$

Therefore $\cos\alpha_2 = \pm\frac{1}{2}$. Hence $\mathbf{a}$ makes either 60° or 120° with y axis.

10.24. The direction cosines are
(a) $\pm(\frac{3}{13},\frac{4}{13},\frac{12}{13})$.
(b) $\pm(\frac{6}{19},\frac{-10}{19},\frac{15}{19})$.

10.25. (a) The coefficients of λ give a vector in the direction of the line, namely $(-1,3,1)$.
(b) The normal $\mathbf{n}$ to the plane is in the direction of the line. We can put $\mathbf{n} = (-1,3,1)$. Since the plane passes through the origin, its equation is $\mathbf{n}\cdot\mathbf{r} = 0$, that is, $-x+3y+z=0$.

(c) The plane passes through the origin. Therefore its equation takes the form $px + qy + rz = 0$. The plane must also pass through two points on the line. Choose obvious points, such as $\lambda = 1$ giving the point $(0, 5, 2)$ and $\lambda = -1$ giving $(2, -1, 0)$. Therefore p, q, r must satisfy

$$5q + 2r = 0, \qquad 2p - q = 0.$$

Put $q = 2$ (or any nonzero value). Then $p = 1$ and $r = -5$.The equation of the is

$$x + 2y - 5z = 0.$$

10.26. The angle between the planes is the same as the angle between the normals. For any plane $ax + by + cz = p$, the direction of its normal is the vector $\mathbf{n} = (p, q, r)$.
(a) From (10.22), the normals to the planes $2x - 3y + z = 2$ and $x - y = 0$ are, respectively, $\mathbf{n_1} = (2, -3, 1)$ and $\mathbf{n_2} = (1, -1, 0)$. If θ is the angle between the planes then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{5}{\sqrt{14}\sqrt{2}} = \frac{5}{2\sqrt{7}}.$$

Hence $\theta = 19.1°$.

(b) The normals to the planes $x + y + z = 0$ and $z = 0$ are, respectively, $\mathbf{n_1} = (1, 1, 1)$ and $\mathbf{n_2} = (0, 0, 1)$. If θ is the angle between the planes then $\cos\theta = 1/\sqrt{3}$. Hence $\theta = 54.7°$.

10.27. Since the directions of the normals to the planes are given by $\mathbf{a}$ and $\mathbf{b}$, the planes are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.

The direction of the normal to the plane $x+y+z = 0$ is $(1, 1, 1)$, from(10.22). Any vector (p, q, r) is perpendicular to $(1, 1, 1)$ if $p + q + r = 0$. The set of all vectors perpendicular to the normal is $(p, q, -p-q)$ for any p and q. The set of all perpendicular planes is given by $px+qy-(p+q)z = d$ for any p, q and d.

10.28. (a) The normal vector of the plane $ax+by+cz = d$ is (a, b, c) (see (10.22)) which is a fixed vector.
(b) The normal to the plane $2x + y - z = 2$ is $\mathbf{n} = (2, 1, -1)$. Since the line passes through the origin, the position vector must be in the direction of $\mathbf{n}$. Therefore $\mathbf{r} = \lambda(2, 1, -1)$.
(c) The point $\lambda(2, 1, -1)$ must lie on the plane. Hence $4\lambda + \lambda + \lambda = 2$, so that $\lambda = \frac{1}{3}$ and the point of intersection $(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3})$. The length is $\sqrt{[(4 + 1 + 1)/9]} = \sqrt{(2/3)}$.
(d) The planes are parallel. The line meets this plane where $4\lambda_1 + \lambda_1 + \lambda_1 = 1$ so that $\lambda_1 = \frac{1}{6}$. The point of intersection is $(\frac{1}{3}, \frac{1}{6}, -\frac{1}{6})$. The distance between the planes is the distance between the two points of intersection, namely

$$\sqrt{\left[\left(\frac{2}{3} - \frac{1}{3}\right)^2 + \left(\frac{1}{3} - \frac{1}{6}\right)^2 + \left(-\frac{1}{3} + \frac{1}{6}\right)^2\right]} = \sqrt{\left[\frac{1}{9} + \frac{1}{36} + \frac{1}{36}\right]} = \frac{1}{\sqrt{6}}.$$

10.29. The parametric equation for the straight line through $\mathbf{q}$ perpendicular to the plane is $\mathbf{r} - \mathbf{q} = \lambda\mathbf{p}$. The line intersects the plane $\mathbf{p} \cdot \mathbf{r} = d$ where

$$\mathbf{p} \cdot (\lambda\mathbf{p} + \mathbf{q}) = d, \text{ so that } \lambda = (d - \mathbf{p} \cdot \mathbf{q})/|\mathbf{p}|^2.$$

The line meets the plane at the point

$$\mathbf{r}_1 = \mathbf{q} + \frac{\mathbf{p}}{|\mathbf{p}|^2}(d - \mathbf{p} \cdot \mathbf{q}).$$

The distance of Q from the plane is

$$|\mathbf{r}_1 - \mathbf{q}| = |d - \mathbf{p} \cdot \mathbf{q}|/\mathbf{p}| = |\mathbf{p} \cdot \mathbf{q} - d|/|\mathbf{p}|.$$

For the given point and plane $\mathbf{q} = (1,1,2)$, $\mathbf{p} = (1,2,-4)$ and $d = -3$. Using the formula above the perpendicular distance is

$$\frac{|\mathbf{p}\cdot\mathbf{q} - d|}{|\mathbf{p}|} = \frac{|(1,2,-4)\cdot(1,1,2)+3|}{\sqrt{21}} = \frac{|-5+3|}{\sqrt{21}} = \frac{2}{\sqrt{21}}.$$

10.30. (a) For the plane P_1 through A, $-\hat{\mathbf{j}} + \hat{\mathbf{k}}$ will be the direction of its normal. Hence (see (10.22)) its equation is

$$0x - y + z = 0 + 1 + 3, \text{ or } -y + z = 4.$$

Let the equation of P_2 be $ax + by + cz = d$. It passes through $A : (0,-1,3)$, $B : (1,0,3)$ and $C : (0,0,5)$. Hence

$$-b + 3c = d, \quad a + 3c = d, \quad 5c = d.$$

Let $d = 5$. Then $c = 1$, $a = d - 3c = 2$, and $b = 3c - d = -2$. The equation of P_2 is

$$2x - 2y + z = 5.$$

(b) The normals of P_1 and P_2 are $\mathbf{n}_1 = (0,-1,1)$ and $\mathbf{n}_2 = (2,-2,1)$ so the angle θ between the planes is

$$\cos\theta = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2+1}{\sqrt{2}\sqrt{9}} = \frac{1}{\sqrt{2}}.$$

Hence $\theta = 45°$.
(c) The equation of the normal to P_1 which passes through the origin is $(x,y,z) = \lambda\mathbf{n}_1 = \lambda(0,-1,1)$. This meets P_1 where $\lambda + \lambda = 4$. Hence $\lambda = 2$, so that the line meets the plane at $(0,-2,2)$. The perpendicular distance is $\sqrt{[(-2)^2 + 2^2]} = 2\sqrt{2}$. (The result can also be found using the formula in Problem 10.29.)
(d) The line OD is given by

$$x = \lambda, \quad y = 4\lambda, \quad z = -4\lambda.$$

This meets P_1 where $-4\lambda - 4\lambda = 4$, that is where $\lambda = -\frac{1}{2}$. Similarly the line meets P_2 where $2\lambda - 8\lambda - 4\lambda = 5$, from which it follows that $\lambda = -\frac{1}{2}$ again. Since λ takes the same values on both planes the points of intersection coincide, and the point must therefore lie on the line of intersection of the planes.
(e) From (d), $\lambda = -\frac{1}{2}$ at the point of intersection of L with OD. Therefore the point of intersection has coordinates $(-\frac{1}{2}, -2, 2)$.

10.31. Since $\mathbf{F}\cdot\hat{\mathbf{n}} = |\mathbf{F}||\hat{\mathbf{n}}|\cos\theta = |\mathbf{F}|\cos\theta$ (see (10.4)), it follows that the right-hand side is the component in the direction $\hat{\mathbf{n}}$. Hence

$$\mathbf{F} = \mathbf{F}_s + (\mathbf{F}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}.$$

For the straight line $2x - 3y = 1$, we can choose $\hat{\mathbf{s}} = (3,2)/\sqrt{13}$ and $\hat{\mathbf{n}} = (2,-3)/\sqrt{13}$. Then

$$\mathbf{F}_n = (\mathbf{F}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} = (\tfrac{2}{\sqrt{13}} + \tfrac{9}{\sqrt{13}})(\tfrac{2}{\sqrt{13}}, -\tfrac{3}{\sqrt{13}}) = (\tfrac{22}{13}, -\tfrac{33}{13}).$$

The other component $\mathbf{F}_s$ is given by

$$\mathbf{F}_s = \mathbf{F} - \mathbf{F}_n = (1,-3) - (\tfrac{22}{13}, -\tfrac{33}{13}) = (-\tfrac{9}{13}, -\tfrac{6}{13}).$$

10.32. (a) The component of $\hat{\mathbf{u}}$ in the direction $\hat{\mathbf{s}}$ is

$$\hat{\mathbf{u}}_s = (\hat{\mathbf{u}}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}}$$

so that

$$\hat{\mathbf{u}}_n = \hat{\mathbf{u}} - (\hat{\mathbf{u}}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}}.$$

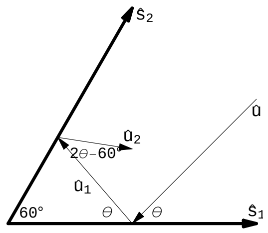

Figure 5: Problem 10.32

The vector $\hat{\mathbf{u}}_1$ is $\hat{\mathbf{u}}$ with the $\hat{\mathbf{n}}$ component reversed. Thus

$$\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_s - \hat{\mathbf{u}}_n = -\hat{\mathbf{u}} + 2(\hat{\mathbf{u}} \cdot \hat{\mathbf{s}})\hat{\mathbf{s}}.$$

(b) In this example, $\hat{\mathbf{s}} = \hat{\mathbf{i}}$, $\hat{\mathbf{n}} = \hat{\mathbf{j}}$, and $\hat{\mathbf{u}} = \frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}$. Therefore, by (a),

$$\hat{\mathbf{u}}_1 = \left(\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}\right) + 2\left(-\frac{1}{\sqrt{2}}\right)\hat{\mathbf{i}} = -\frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}.$$

(c) For the first reflection, $\hat{\mathbf{s}} = \hat{\mathbf{i}}$, $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$, and

$$\hat{\mathbf{u}} = -\hat{\mathbf{i}}\cos\theta - \hat{\mathbf{j}}\sin\theta.$$

Using (a)

$$\hat{\mathbf{u}}_1 = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta + 2(-\cos\theta)\hat{\mathbf{i}} = -\hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta.$$

For the second reflection, $\hat{\mathbf{s}}_2 = \frac{1}{2}\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\hat{\mathbf{j}}$, $\hat{\mathbf{n}}_2 = \frac{\sqrt{3}}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}$. Therefore

$$\begin{aligned}
\hat{\mathbf{u}} &= \hat{\mathbf{i}}\cos\theta - \hat{\mathbf{j}}\sin\theta + 2(-\tfrac{1}{2}\cos\theta + \tfrac{\sqrt{3}}{2}\sin\theta)(\tfrac{1}{2}\hat{\mathbf{i}} + \tfrac{\sqrt{3}}{2}\hat{\mathbf{j}}) \\
&= \hat{\mathbf{i}}(\tfrac{1}{2}\cos\theta + \tfrac{\sqrt{3}}{2}\sin\theta) + \hat{\mathbf{j}}(-\tfrac{\sqrt{3}}{2}\cos\theta + \tfrac{1}{2}\sin\theta) \\
&= \hat{\mathbf{i}}\cos(\theta - 60^\circ) + \hat{\mathbf{j}}\sin(\theta - 60^\circ).
\end{aligned}$$

10.33. For one point choose $x = 0$. Then $y + z = 2$ and $y - 2z = 1$. Solving these equations $y = \frac{5}{3}$ and $z = \frac{1}{3}$. For the other point choose $z = 0$. Then $x + y = 2$ and $2x + y = 1$. solving these equations $x = -1$ and $y = 3$. Two points on the line of intersection are $\mathbf{a} = (0, \frac{5}{3}, \frac{1}{3})$ and $\mathbf{b} = (-1, 3, 0)$.
(b) A vector equation of the line of intersection (see Example 9.9a) is

$$\mathbf{r} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b}) = (-1, 3, 0) + \lambda(1, -\tfrac{4}{3}, \tfrac{1}{3}).$$

(c) A cartesian equation can be written down from (b):

$$\frac{x+1}{1} = \frac{y-3}{-\frac{4}{3}} = \frac{z}{\frac{1}{3}}.$$

10.34. Select any two points on the line of intersection of $2x + 3y - z = 1$ and $x + y + z = 0$; say $\mathbf{a} = (0, \frac{1}{4}, -\frac{1}{4})$ and $\mathbf{b} = (\frac{1}{3}, 0, -\frac{1}{3})$. From(10.25) the line of intersection can be expressed as

$$\frac{x}{-\frac{1}{3}} = \frac{y - \frac{1}{4}}{\frac{1}{4}} - \frac{z + \frac{1}{4}}{\frac{1}{12}}.$$

(Note: an infinite number of alternative expressions may be obtained.)

10.35. Let (p, q, r) be a vector in the direction of the line of intersection. If the vector is parallel to both planes it must be parallel to the line of intersection, and it must also therefore be perpendicular to both normals of the planes. Hence

$$2p + 3q - 2r = 0, \qquad p - 3q + 2r = 0.$$

One obvious solution is $p = 0$, $q = 2$, $z = 3$. Hence direction ratios in the direction of the line of intersection are $0, 2, 3$ or any non-zero multiple of these numbers.

10.36. The plane P_1 with normal vector $\mathbf{n}_1 = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is $y + z = d$. This plane passes through $B : (-1, -2, 0)$ if $-2 = d$. Hence P_1 has the equation $y + z = -2$. The plane P_2 with normal vector $\mathbf{n}_2 = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is $2x - y + 3z = h$. This plane passes through $C : (-1, 0, 3)$ if $-2 + 9 = h$. Hence $h = 7$, and P_2 has the equation $2x - y + 3z = 7$. The vector $\overline{AC} = (-1, 0, 3) - (-1, -2, 1) = (0, 2, 2)$, which is a multiple of $\mathbf{n}_1$, the normal of P_1. Hence AC is perpendicular to P_1.

10.37. The vector (p, q, r) is perpendicular to both (a_1, b_1, c_1) an (a_2, b_2, c_2), the normals to the two planes, and therefore must be parallel to the line of intersection. The components p, q, r are possible direction ratios for this line.

10.38. Let the line EQ meet the screen at the point (X, Y, Z). Then, comparing directions,

$$\frac{X - 1}{x - 1} = \frac{Y - 1}{y - 1} = \frac{Z - 1}{z - 1} = \lambda.$$

The point (X, Y, Z) lies on the plane $1.1x + 1.1y + z = 1$ if

$$1.1[1 + \lambda(x - 1)] + 1.1[1 + \lambda(y - 1)] + 1 + \lambda[1 + \lambda(z - 1)] = 1,$$

Hence

$$\lambda = -\frac{1.2}{1.1x + 1.1y + z - 3.2}.$$

The apparent position is

$$(1 + (x - 1)\lambda, 1 + (y - 1)\lambda, 1 + z\lambda).$$

10.39. Pythagoras' theorem for an increment of arc-length gives

$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2.$$

Divide through by $(\delta t)^2$and let $\delta t \to 0$ (and take the square root):

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right]} = \sqrt{[a^2 \sin^2 t + b^2 \cos^2 t]}.$$

The unit tangent vector is

$$\hat{\mathbf{t}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{(-a \sin t, b \cos t)}{\sqrt{[a^2 \sin^2 t + b^2 \cos^2 t]}}.$$

A vector normal to $\hat{\mathbf{t}}$ is

$$\mathbf{n} = \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}t} = \frac{ab(-b \cos t, -a \sin t)}{(a^2 \sin^2 t + b^2 \cos^2 t)^{\frac{3}{2}}}.$$

A unit normal is

$$\hat{\mathbf{n}} = \frac{(-b \cos t, -a \sin t)}{\sqrt{(a^2 \sin^2 t + b^2 \cos^2 t)}}.$$

The curvature κ is given by

$$\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}s} = \kappa\hat{\mathbf{n}}.$$

Hence
$$\kappa = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}.$$
The radius of curvature
$$\rho = 1/|\kappa| = (a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}/(ab).$$
At $t = 0$,
$$\hat{\mathbf{t}} = (0,1); \quad \hat{\mathbf{n}} = (-1,0); \quad \kappa = a/b^2; \quad \rho = b^2/a.$$
At $t = \frac{1}{4}\pi$,
$$\hat{\mathbf{t}} = \frac{(-a,b)}{\sqrt{(a^2+b^2)}}; \quad \hat{\mathbf{n}} = \frac{(-b,-a)}{\sqrt{(a^2+b^2)}}; \quad \kappa = \frac{2^{\frac{3}{2}}ab}{(a^2+b^2)^{\frac{3}{2}}}; \quad \rho = \frac{(a^2+b^2)^{\frac{3}{2}}}{2^{\frac{3}{2}}ab}.$$
At $t = \frac{1}{2}\pi$,
$$\hat{\mathbf{t}} = (-1,0); \quad \hat{\mathbf{n}} = (0,-1); \quad \kappa = b/a^2; \quad \rho = a^2/b.$$

10.40. With x as the parameter, the arc-length satisfies
$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{[1+f'(x)^2]}$$
(since $\delta s^2 = \delta x^2 + \delta y^2$). The unit tangent vector is
$$\hat{\mathbf{t}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\hat{\mathbf{i}} + f'(x)\hat{\mathbf{j}}}{\sqrt{[1+f'(x)^2]}}.$$
A normal vector $\mathbf{n}$ is given by
$$\mathbf{n} = \frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}x} = \frac{-f'(x)f''(x)\hat{\mathbf{i}} + f''(x)\hat{\mathbf{j}}}{[1+f'(x)^2]^{\frac{3}{2}}}.$$
Further
$$\frac{\mathrm{d}\hat{\mathbf{t}}}{\mathrm{d}s} = \frac{-f'(x)f''(x)\hat{\mathbf{i}} + f''(x)\hat{\mathbf{j}}}{[1+f'(x)^2]^2} = -\frac{\hat{\mathbf{n}}}{[1+f'(x)^2]^2}.$$
Hence
$$\kappa = -f''(x)/[1+f'(x)]^{\frac{3}{2}},$$
(the sign of κ depends on the direction of the unit normal).
(a) For the parabola $y = x^2$, $f(x) = x^2$. Hence
$$\kappa = 2/(1+4x^2)^{\frac{3}{2}}.$$
(b) For the cosine curve $y = \cos x$, $f(x) = \cos x$. Hence
$$\kappa = -\cos x/(1+\sin^2 x)^{\frac{3}{2}}.$$

Chapter 11: Vector product

11.1. Given $\mathbf{a} = (1,-2,2)$, $\mathbf{b} = (3,-1,-1)$, and $\mathbf{c} = (-1,0,-1)$:
(a) $\mathbf{a}\times\mathbf{b} = (4,7,5)$; (b) $\mathbf{b}\times\mathbf{a} = (-4,-7,-5)$; (c) $\mathbf{a}\times\mathbf{a} = \mathbf{0}$;
(d) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -9$; (e) $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = -9$; (f) $\mathbf{b}\cdot\mathbf{a}\times\mathbf{c}) = 9$;
(g) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$; (h) $\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) = (-24,3,15)$;
(i) $(\mathbf{c}\times\mathbf{b})\times\mathbf{a} = (-6,3,6)$.

11.2. The vector $\mathbf{n}_1 \times \mathbf{n_2}$ is a vector perpendicular to both normals $\mathbf{n}_1$ and $\mathbf{n}_2$, and therefore must be in the direction of the line of intersection of the two planes. Hence the required plane must have this vector as its normal, and since it passes through the origin it can be represented by

$$\mathbf{r} \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0.$$

11.3. (a) $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} \times \mathbf{c})$, since $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = \lambda \mathbf{b} \cdot (\mathbf{b} \times \mathbf{c} = 0$, and similarly $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{c} = 0$.
(b) $\mathbf{r} = (1, 2, 1) + \lambda(1, -1, 0) \times (0, 1, 1) = (1, 2, 1) + \lambda(-1, -1, 1)$.

11.4. The vector $\mathbf{a}$ is perpendicular to the plane $\mathbf{r} \cdot \mathbf{a} = d$, and $\mathbf{a} \times \mathbf{u}$ is perpendicular to $\mathbf{a}$. Therefore $\mathbf{a} \times \mathbf{u}$ is parallel to the plane.

In this example $\mathbf{a} = (2, -3, -1)$. Choose two simple vectors for $\mathbf{u}$, say, $\mathbf{u}_1 = \hat{\mathbf{i}}$ and $\mathbf{u}_2 = \hat{\mathbf{j}}$, in which case two parallel vectors are

$$\mathbf{a} \times \mathbf{u}_1 = (2, -3, -1) \times (1, 0, 0) = (0, 1, -3),$$

$$\mathbf{a} \times \mathbf{u}_2 = (2, -3, -1) \times (0, 1, 0) = (-1, 0, -2).$$

11.5. Since, by (11.8),

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta,$$

the vector product will be zero if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\theta = 0$ or $\theta = 180°$.

11.6. The vectors $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{b} = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ are perpendicular since

$$\mathbf{a} \cdot \mathbf{b} = (2, 3, 6) \cdot (6, 2, -3) = 12 + 6 - 18 = 0.$$

The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right-handed system if

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = (2, 3, 6) \times (6, 2, -3) = (-21, 42, -14).$$

11.7. (a) Two sides are $\overline{AB} = \mathbf{b} - \mathbf{a}$ and $\overline{CA} = \mathbf{c} - \mathbf{a}$. As in Example 11.2 the area of the triangle ABC is

$$\begin{aligned} \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})| &= \frac{1}{2}|\mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{a}| \\ &= \frac{1}{2}|\mathbf{c} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}| \end{aligned}$$

(b) The first vertex $\mathbf{a}$ is moved to $\mathbf{a} + \lambda\mathbf{a}(\mathbf{b} - \mathbf{c})$ which is the same distance as $\mathbf{a}$ from the side BC. In other words the height of the triangle on the same base is the same as that of ABC. Hence the area will be the same as that of ABC.
(c) In (a) take $\mathbf{a} = (1, -2, -1)$, $\mathbf{b} = (1, -1, 2)$ and $\mathbf{b} = (1, -1, 2)$ and $\mathbf{c} = (1, 2, -1)$. Then

$$\begin{aligned} \text{Area} &= \frac{1}{2}|(1, -1, 2) \times (1, 2, -1) + (1, 2 - 1) \times (1, -2, -1) + \\ &\qquad (1, -2, -1) \times (1, -1, 2)|. \\ &= (6, 0, 0) \end{aligned}$$

11.8. Let E be the foot of the perpendicular from D on to the plane, and let θ be the angle between AD and DE. The vector $\mathbf{b} \times \mathbf{c}$ is a vector in the direction of ED. From a property of the scalar product

$$\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{d}||\mathbf{b} \times \mathbf{c}| \cos\theta.$$

Since $ED = |\mathbf{d}| \cos\theta$, the required perpendicular distance is (taking account of the possibility that $\cos\theta < 0$)

$$ED = |\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})|/|\mathbf{b} \times \mathbf{c}|.$$

11.9. The answer is essentially given by equation 11.11. Let

$$\mathbf{a} = \overline{QA} = (x_1 - x_0, y_1 - y_0, z_1 - z_0), \quad \mathbf{b} = \overline{QB} = (x_2 - x_0, y_2 - y_0, z_2 - z_0),$$

$$\mathbf{c} = \overline{QB} = (x_3 - x_0, y_3 - y_0, z_3 - z_0).$$

By (11.1) and (11.10), the volume V of the parallelepiped is given by

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\det A_1|,$$

where

$$A_1 = \begin{vmatrix} x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ y_1 - y_0 & y_2 - y_0 & y_3 - y_0 \\ z_1 - z_0 & z_2 - z_0 & z_3 - z_0 \end{vmatrix},$$

after a transpose between rows and columns in the determinant: this does not affect the answer.

11.10. (a) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be any three given vectors, no two of which are parallel, and let $\mathbf{v}$ be an arbitrary vector. We may express $\mathbf{v}$ in the form

$$\mathbf{v} = X\mathbf{a} + Y\mathbf{b} + Z\mathbf{c},$$

by choosing the coefficients X, Y, Z suitably. (X, Y, Z may be called the **components** of $\mathbf{v}$ in terms of the **oblique axes** $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, which need not always be unit vectors.) The components can be worked out as follows.

Form the scalar product of $\mathbf{v}$ with $\mathbf{b} \times \mathbf{c}$. Since $\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c})$ and $\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c})$ are zero by (11.10d), we obtain

$$\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c}) = X\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Therefore

$$X = \frac{\mathbf{v} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}.$$

Similarly

$$Y = \frac{\mathbf{v} \cdot (\mathbf{c} \times \mathbf{a})}{\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})}, \quad Z = \frac{\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})}.$$

(b) For the given vectors $\mathbf{a} = (1, 1, 0)$, $\mathbf{b} = (0, 1, 1)$, $\mathbf{c} = (1, 0, 1)$,

$$D = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 2, \mathbf{v} \cdot (\mathbf{b} \times \mathbf{c}) = 1, \mathbf{v} \cdot (\mathbf{c} \times \mathbf{a}) = 1,$$

$$\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b}) = 1.$$

Hence $X = Y = Z = \frac{1}{2}$.

11.11. Let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{a} = (a_1, a_2, a_3)$, etc. Hence

$$v_1 = a_1X + b_1Y + c_1Z,$$

$$v_2 = a_2X + b_2Y + c_2Z,$$

$$v_3 = a_3X + b_3Y + c_3Z.$$

Let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} v_1 & b_1 & c_1 \\ v_2 & b_2 & c_2 \\ v_3 & b_3 & c_3 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_1 & v_1 & c_1 \\ a_2 & v_2 & c_2 \\ a_3 & v_3 & c_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & b_1 & v_1 \\ a_2 & b_2 & v_2 \\ a_3 & b_3 & v_3 \end{vmatrix},$$

By elimination, it can be shown that

$$X = D_1/D, \qquad Y = D_2/D, \qquad Z = D_3/D.$$

11.12. (a) If $\mathbf{v} = X(\mathbf{b}\times\mathbf{c}) + Y(\mathbf{c}\times\mathbf{a}) + Z(\mathbf{a}\times\mathbf{b})$, then the scalar multiplication of $\mathbf{v}$ by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ leads to

$$\mathbf{a}\cdot\mathbf{v} = X\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}),$$
$$\mathbf{b}\cdot\mathbf{v} = Y\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = Y\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}),$$
$$\mathbf{c}\cdot\mathbf{v} = Z\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = Z\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}).$$

In these results we have used (11.10d) and (11.10b).
(b) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -6$, $\mathbf{v}\cdot\mathbf{v} = 1$, $\mathbf{b}\cdot\mathbf{v} = 4$, $\mathbf{c}\cdot\mathbf{v}$. Hence $X = -\frac{1}{6}$, $Y = -\frac{2}{3}$, $Z = -\frac{5}{6}$.

11.13. (a) The line L_1 is in the direction $\mathbf{u}$ and L_2 is in the direction $\mathbf{v}$. The vector product of these vectors is perpendicular to both. Therefore, let $\mathbf{w} = \mathbf{u}\times\mathbf{v}$.
(b) The position vector $(\mathbf{a}+\lambda\mathbf{u})$ is a point on L_1 and $(\mathbf{b}+\mu\mathbf{v})$ is a point on L_2. The difference is a vector joining the two points, which will be perpendicular to both if it is in the direction of $\mathbf{w}$, that is, if

$$(\mathbf{b}+\mu\mathbf{v}) - (\mathbf{a}+\lambda\mathbf{u}) = \nu\mathbf{w}.$$

This represents three equations for the three unknowns λ, μ, ν.
(c) The equation in (b) becomes

$$(1,-1,0) + \mu(1,1,1) - (-1,0,0) - \lambda(0,0,1) = \nu(0,0,1)\times(1,1,1) = \nu(-1,1,0).$$

Equating components and solving: $\lambda = -\frac{1}{2}$, $\mu = -\frac{1}{2}$, $\nu = -\frac{3}{2}$. Hence, the end-points of the perpendicular line are

$$\mathbf{a_1} = \mathbf{a}+\lambda\mathbf{u} = (-1,0,0) - \tfrac{1}{2}(0,0,1) = (-1,0,-\tfrac{1}{2}) \text{ on } L_1,$$

and

$$\mathbf{b_1} = \mathbf{b}+\mu\mathbf{v} = (1,-1,0) - \tfrac{1}{2}(1,1,1) = (\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{1}{2}) \text{ on } L_2.$$

A vector equation for L_3 is

$$\mathbf{r} = \mathbf{a_1} + \nu\mathbf{u}\times\mathbf{v} = (-1,0,-\tfrac{1}{2}) + \nu(-1,1,0).$$

The perpendicular distance is $|\mathbf{b_1}-\mathbf{a_1}| = \frac{3}{2}\sqrt{2}$.

11.14. If the position vector of P is $\mathbf{r}$ then the moment $\mathbf{M}$ of the force $\mathbf{F}$ about the point Q with position vector $\mathbf{a}$ is given by $\mathbf{M} = (\mathbf{r}-\mathbf{a})\times\mathbf{F}$.
(a)

$$\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -6\hat{\mathbf{k}}.$$

(b)

$$\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 3 & -3 \\ 2 & 0 & 0 \end{vmatrix} = -6\hat{\mathbf{j}} - 6\hat{\mathbf{k}}.$$

(c)

$$\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -3 & -3 \\ 2 & 0 & 0 \end{vmatrix} = - - 6\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

11.15. Since the magnitude of $\mathbf{F}$ is 4, and a unit vector in the given direction is $\frac{1}{3}(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$,

$$\mathbf{F} = \tfrac{4}{3}(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}).$$

(a) $\mathbf{M} = (1,-1,2)\times\frac{4}{3}(1,-2,-2) = \frac{4}{3}(6,4,-1)$.
(b) $\mathbf{M} = (3,-2,0)\times\frac{4}{3}(1,-2,-2) = \frac{4}{3}(4,6,-4)$.

(c) The component is $\hat{\mathbf{j}} \cdot (\mathbf{R} \times \mathbf{F}) = (0,1,0) \cdot \frac{4}{3}(6,4,-1) = \frac{16}{3}$ in case (a).

11.16. The moment about an axis in the direction $\hat{\mathbf{s}}$ is $\hat{\mathbf{s}} \cdot (\mathbf{R} \times \mathbf{F})$.
(a) $\hat{\mathbf{s}} = \hat{\mathbf{k}}$. Hence, using (11.10a),

$$\hat{\mathbf{s}} \cdot (\mathbf{R} \times \mathbf{F}) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -6.$$

(b) $\hat{\mathbf{s}} = -\hat{\mathbf{k}}$. Hence

$$\hat{\mathbf{s}} \cdot (\mathbf{R} \times \mathbf{F}) = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 6.$$

(c) $\hat{\mathbf{s}} = \hat{\mathbf{i}}$. Hence

$$\hat{\mathbf{s}} \cdot (\mathbf{R} \times \mathbf{F}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 0.$$

(d) $\hat{\mathbf{s}} = \hat{\mathbf{j}}$. Hence

$$\hat{\mathbf{s}} \cdot (\mathbf{R} \times \mathbf{F}) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 0$$

(e) $\hat{\mathbf{s}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$. Hence

$$\hat{\mathbf{s}} \cdot (\mathbf{R} \times \mathbf{F}) = \frac{1}{\sqrt{3}}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} = -2\sqrt{3}.$$

11.17. The axis $\overline{AB}$ is in the direction $(1,1,1) - (2,3,2) = (-1,-2,-1)$. The unit vector in the direction $\overline{AB}$ is $\hat{s} = (-1,-2,-1)/\sqrt{6}$. Let

$$\mathbf{R}_1 = \overline{AP} = (2,-3,1) - (2,3,2) = (0,-6,-1),$$

$$\mathbf{R}_2 = \overline{BP} = (2,-3,1) - (1,1,1) = (1,-4,0).$$

Then the moment about AB using point A is

$$\hat{\mathbf{s}} \cdot (\mathbf{R}_1 \times \mathbf{F}) = \frac{1}{\sqrt{6}}\begin{vmatrix} -1 & -2 & -1 \\ 0 & -6 & -1 \\ 1 & 1 & 2 \end{vmatrix} = \frac{7}{\sqrt{6}}.$$

The answer can be checked using the alternative point B:

$$\hat{\mathbf{s}} \cdot (\mathbf{R}_2 \times \mathbf{F}) = \frac{1}{\sqrt{6}}\begin{vmatrix} -1 & -2 & -6 \\ 1 & -4 & 0 \\ 1 & 1 & 2 \end{vmatrix} = \frac{7}{\sqrt{6}}.$$

Any point on point X on AB is

$$\mathbf{r} = -\overline{OA} + \lambda\overline{AB} = (2,3,2) + \lambda(-1,-2,-1).$$

A vector in the direction PX is

$$\mathbf{R}_3 = (2,3,2) + \lambda(-1,-2,-1) - (2,-3,1) = (0,6,1) + \lambda(-1,-2,-1).$$

The component of $\mathbf{F}$ in this direction will be a multiple of $\mathbf{R}_3$, and its vector product with $\mathbf{R}_3$ will be zero which means that its contribution to the moment will be zero.

11.18. The moment of $\mathbf{F}$ at $\mathbf{r}$ about an axis through the origin in the direction $\hat{\mathbf{s}}$ is

$$\mathbf{M} = \hat{\mathbf{s}} \cdot (\mathbf{r} \times \mathbf{F}).$$

In this formula $\mathbf{r} \times \mathbf{F}$ is fixed. Taking the magnitude of $\mathbf{M}$,

$$|\mathbf{M}| = |\hat{\mathbf{s}} \cdot (\mathbf{r} \times \mathbf{F})| = |\hat{\mathbf{s}}||\mathbf{r} \times \mathbf{F}| \cos\theta = |\mathbf{r} \times \mathbf{F}| \cos\theta.$$

This is a maximum when $\theta = 0$, which occurs when $\hat{\mathbf{s}}$ is in the direction of $\mathbf{r} \times \mathbf{F}$.

11.19. (a) A point P on the lamina has position vector

$$\mathbf{v} = \mathbf{r} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}},$$

where r is constant. The velocity of P is

$$\dot{\mathbf{r}} = -r\sin\theta\,\dot{\theta}\hat{\mathbf{i}} + r\cos\theta\,\dot{\theta}\hat{\mathbf{j}} = -\hat{\mathbf{i}}\omega r\sin\theta + \hat{\mathbf{j}}\omega r\cos\theta.$$

(b) If $\boldsymbol{\omega} = \omega\hat{\mathbf{k}}$, then

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega \\ r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -\hat{\mathbf{i}}\omega r\sin\theta + \hat{\mathbf{j}}\omega r\cos\theta.$$

(c) Let $\overline{OP} = \mathbf{r}$ and $\overline{OQ} = \mathbf{q}$. We are given that $\mathbf{R} = \overline{QP}$, so that, by the triangle law,

$$\mathbf{r} = \overline{OP} = \overline{OQ} + \overline{QP} = \mathbf{q} + \mathbf{R}.$$

The velocity of P relative to Q is

$$\dot{\mathbf{R}} = \dot{\mathbf{r}} - \dot{\mathbf{q}} = \boldsymbol{\omega} \times \mathbf{r} - \boldsymbol{\omega} \times \mathbf{q} = \boldsymbol{\omega} \times \mathbf{R}.$$

11.20. Let a point P of the body have position vector $\mathbf{r}$. Let Q be the foot of the perpendicular from P on to the axis of $\boldsymbol{\omega}$, and let θ $(0 \le \theta \le \pi)$ be the angle between $\mathbf{r}$ and $\boldsymbol{\omega}$. Since OP is constant, the point P must move perpendicular to OP (follows since $\mathbf{r} \cdot \dot{\mathbf{r}} = 0$), whilst since QP is constant P must move perpendicular to QP. To satisfy both conditions P must move perpendicular to $\boldsymbol{\omega}$.

Since $\mathbf{v}$ is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$, it follows that $\mathbf{v} = k\boldsymbol{\omega} \times \mathbf{r}$ for some value of the constant k. Taking the magnitude of $\mathbf{v}$,

$$|\mathbf{v}| = |k||\boldsymbol{\omega}||\mathbf{r}|\sin\theta = |k|\omega PQ.$$

Since ω is the angular rate, it follows that $|k| = 1$. in order that $\mathbf{v}$, $\boldsymbol{\omega}$, and $\mathbf{r}$ form a right-handed system of vectors, we must put $k = 1$.

Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. From the vector product expansion,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = (z\omega_2 - y\omega_3, x\omega_3 - z\omega_1, y\omega_1 - x\omega_2).$$

This can be expressed as the matrix product

$$\mathbf{v} = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \text{ or } S\boldsymbol{\omega},$$

interpreting $\mathbf{v}$ and $\boldsymbol{\omega}$ as column vectors. Note that S is a skew-symmetric matrix (see Section 7.3). Still using matrix notation

$$|\mathbf{v}|^2 = \mathbf{v}^T\mathbf{v} = \boldsymbol{\omega}^T S^T S\boldsymbol{\omega},$$

(use the rule for transpose of a product given in Section 7.3). Finally

$$S^T S = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} = \begin{bmatrix} y^2+z^2 & -xy & -zx \\ -xy & z^2+x^2 & -yz \\ -zx & -yz & y^2+z^2 \end{bmatrix}.$$

11.21. By the definition of the vector product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Also $\mathbf{a} \times (\mathbf{a} \times \mathbf{b})$ is perpendicular to both $\mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$, and must therefore lie in the plane of $\mathbf{a}$ and $\mathbf{b}$. A similar argument can be applied to $\mathbf{b} \times (\mathbf{a} \times \mathbf{b})$.

11.22. Use the identity (11.18) for the vector triple product in two different expansions. In the first

$$\mathbf{v} = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{c}]\mathbf{d},$$

which confirms that $\mathbf{v}$ lies in the plane of $\mathbf{c}$ and $\mathbf{d}$ The coefficients are

$$m = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}, \qquad n = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}.$$

Similarly

$$\mathbf{v} = -(\mathbf{c} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{b}) = -[(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}]\mathbf{a} + [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}]\mathbf{b},$$

which confirms that the vector also lies in the plane of $\mathbf{a}$ and $\mathbf{b}$.The coefficients are

$$p = (\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}, \qquad q = (\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}.$$

Suppose that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are all drawn from the origin. Then $\mathbf{v}$ lis in the direction of the line of intersection of the plane through $\mathbf{a}$ and $\mathbf{b}$, and the plane through $\mathbf{c}$ and $\mathbf{d}$.

11.23. Use (11.18) repeatedly:

$$\begin{aligned} &\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \\ &\quad (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = 0. \end{aligned}$$

11.24. (a) The vector $\mathbf{n} \times \mathbf{b}$ is perpendicular to $\mathbf{n}$ and $\mathbf{b}$. The vector $\mathbf{n} \times (\mathbf{n} \times \mathbf{b})$ is perpendicular to $\mathbf{n}$ and $\mathbf{n} \times \mathbf{b}$ and is also in the plane of $\mathbf{n}$ and $\mathbf{b}$.
(b) The straight line

$$\mathbf{r} = \mathbf{b} + \mu\mathbf{n} \times [(\mathbf{a} - \mathbf{b}) \times \mathbf{n}]$$

passes through $\mathbf{r}$ when $\mu = 0$.

Do the lines intersect? Can we find values of λ and μ for which this occurs? Consider the difference of the position vectors of the two lines:

$$\begin{aligned} &\mathbf{b} + \mu\mathbf{n} \times [(\mathbf{a} - \mathbf{b} \times \mathbf{n}] - [\mathbf{a} + \lambda\mathbf{n}] \\ &\quad = \mathbf{b} + \mu\mathbf{n} \cdot \mathbf{n}(\mathbf{a} - \mathbf{b}) - \mu[\mathbf{n} \cdot (\mathbf{a} - \mathbf{b})]\mathbf{n} - \mathbf{a} - \lambda\mathbf{n} \\ &\quad = (\mu\mathbf{n} \cdot \mathbf{n} - 1)(\mathbf{a} - \mathbf{b}) - [\lambda + \mu\mathbf{n} \cdot (\mathbf{a} - \mathbf{b})]\mathbf{n} \\ &\quad = \mathbf{0} \end{aligned}$$

if $\mu = 1/(\mathbf{n} \cdot \mathbf{n})$ and $\lambda = -\mu\mathbf{n} \cdot (\mathbf{a} - \mathbf{b})$. We can find solutions for λ and μ which means that the lines intersect. The lines must be at right angles since the scalar product of their directions is

$$\mathbf{n} \cdot \{\mathbf{n} \times [(\mathbf{a} - \mathbf{b}) \times \mathbf{n}]\} = 0,$$

by (11.10d).

11.25. Let N be the closest point to the origin on the line of intersection. The direction of the line of intersection is perpendicular to both normals $\mathbf{n}_1$ and $\mathbf{n}_2$, that is in the direction $\mathbf{n}_1 \times \mathbf{n}_2$. Also ON is perpendicular to the line of intersection so that $\overline{ON} \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0$. The two vectors

$(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_1$ and $(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_2$ are not parallel, and $\overline{ON}$ can be expressed in terms of these two vectors:

$$\overline{ON} = \alpha[(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_1] + \beta[(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_1].$$

To obtain α and β, substitute this position vector into the equations for the planes:

$$\beta[(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_2] \cdot \mathbf{n}_1 = d_1,$$

$$\alpha[(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_1] \cdot \mathbf{n}_2 = d_1.$$

Alternative forms can obtained by using formula (11.18) for the vector triple products.

11.26. Given $\mathbf{H} = (\mathbf{r} - \mathbf{q}) \times (m\mathbf{v})$,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} &= \left[\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} - \mathbf{q})\right] \times (m\mathbf{v}) + (\mathbf{r} - \mathbf{q}) \times m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\ &= [(\mathbf{v} - \mathbf{u}) \times m\mathbf{v}] + (\mathbf{r} - [\mathbf{q}) \times \mathbf{F}] \\ &= -[m\mathbf{u} \times \mathbf{v}] + [(\mathbf{r} - \mathbf{q}) \times \mathbf{F}], \end{aligned}$$

using $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ and Newton's law $\mathbf{F} = m(\mathrm{d}\mathbf{v}/\mathrm{d}t)$. If $\mathbf{u} = 0$, then

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = (\mathbf{r} - \mathbf{q}) \times \mathbf{F} = \mathbf{M},$$

the moment of $\mathbf{F}$ about Q.

Chapter 12: Linear algebraic equations

12.1. Generally Cramer's rule (Section 12.1) is not a recommended way of solving linear equations but it can occasionally be useful as a formula. For some problems just the answer is given.
(a) Given

$$\begin{array}{ccccccc} x_1 & & & + & x_3 & = & 1 \\ & & x_2 & - & x_3 & = & 3 \\ 2x_1 & + & x_2 & & & = & -1 \end{array}.$$

Let

$$D_1 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & -1 \\ -1 & 1 & 0 \end{vmatrix} = 5, \quad D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -9,$$

$$D_3 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & -1 \end{vmatrix} = -6, \quad D = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{vmatrix} = -1.$$

Hence the solution is

$$x_1 = \frac{D_1}{D} = -5, \quad x_2 = \frac{D_2}{D} = 9, \quad x_3 = \frac{D_3}{D} = 6.$$

(b) The solution is $x_1 = \frac{188}{5}$, $x_2 = -\frac{21}{5}$, $x_3 = -\frac{36}{5}$.
(c) The solution is $x_1 = 1$, $x_2 = -1$, $x_3 = -5$.
(d) Given

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1 \\ ax_1 & + & bx_2 & + & cx_3 & = & d \\ a^2x_1 & + & b^2x_2 & + & c^2x_3 & = & d^2 \end{array}.$$

Let

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ d & b & c \\ d^2 & b^2 & c^2 \end{vmatrix} = (b - c)(c - d)(d - b),$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & d & c \\ a^2 & d^2 & c^2 \end{vmatrix} = (d-c)(c-a)(a-d),$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ a^2 & b^2 & d^2 \end{vmatrix} = (b-d)(d-a)(a-b),$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$$

The solution is

$$x_1 = \frac{(c-d)(d-b)}{(c-a)(a-b)}, \quad x_2 = \frac{(d-c)(a-d)}{(b-c)(a-b)}, \quad x_3 = \frac{(b-d)(d-a)}{(b-c)(c-a)},$$

provided a, b, c are all different.

(e) The solution is $x_1 = 2$, $x_2 = -1$, $x_3 = 2$, $x_4 = 2$.

12.2. Given the equations

$$\begin{array}{rcrcrcrl} 4i_1 & - & i_2 & - & i_3 & = & 12 & \text{(i)} \\ -i_1 & - & Ri_2 & & & = & 24 & \text{(ii)} \\ i_1 & & & + & 5i_3 & = & -12 & \text{(iii)} \end{array},$$

$i_2 = (24 + i_1)/R$ from (ii), and $i_3 = (-12 - i_1)/5$, from (iii). Substitute i_1 and i_2 into (i):

$$4i_1 + \frac{1}{R}(24 + i_1) - \frac{1}{5}(-12 - i_1) = 12.$$

Hence

$$i_1 = \frac{24(2R-5)}{21R+5}, \quad i_2 = -\frac{552}{21R+5}, \quad i_3 = -\frac{12(5R-1)}{21R+5}.$$

If $i_2 = 2$ amps, then $R = 1/51$.

12.3. (a) The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -3 & 2 & 4 \\ 5 & 5 & 6 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -5 & 1 & 1 \\ 0 & -5 & 1 & -4 \end{bmatrix} \begin{matrix} (r_2' = r_2 - 5r_1) \\ (r_3' = r_3 - 5r_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -5 & 1 & 1 \\ 0 & 0 & 0 & -15 \end{bmatrix} \quad (r_3' = r_3 - r_2) \ .$$

By row 3 the equations are inconsistent.

(b) The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 & 4 \\ 0 & -1 & 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & 2 \end{bmatrix} \begin{matrix} (r_2' = r_2 - r_1) \\ (r_3' = r_3 - r_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix} \quad (r_4' = r_4 - r_2)$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad (r_4' = r_4 + 2r_3)$$

By row 4 the equations are inconsistent.

(c) The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -9 \end{bmatrix},$$

after the following sequence of row operations

$$r_5' = r_5 - r_1, \quad r_5' = r_5 - 2r_2, \quad r_5' = r_5 - 3r_3, \quad r_5' = r_5 - 4r_4.$$

From row 5 the equations are inconsistent.

12.4. The equations are

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 2 \\ 2x & + & 3y & + & z & = & 3 \\ 5x & + & 7y & + & az & = & b \end{array}.$$

Using row operations on the augmented matrix,

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & 1 & 3 \\ 5 & 7 & a & b \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & a+5 & b-10 \end{bmatrix} \begin{matrix} (r_2' = r_2 - 2r_1) \\ (r_3' = r_3 - 5r_1) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & a-1 & b-8 \end{bmatrix} \quad (r_3' = r_3 - 2r_2)$$

(i) The equations have a unique solution if $a \neq 1$, in which case $z = (b-8)/(a-1)$ and x and y can be found by back substitution.

(ii) The equations have no solutions if and only if $a = 1$ and $b \neq 8$.

(iii) There is an infinite set of solutions if $a = 1$ and $b = 8$, in which case

$$z = \lambda, \quad y = -1 - 3\lambda, \quad x = 3 + 4\lambda.$$

12.5. Row operations on the augmented matrix give

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 1 & -2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 3 & -3 & 2 \\ 0 & -1 & 4 & -1 \end{bmatrix} \begin{matrix} (r_2' = r_2 - r_1) \\ (r_3' = r_3 - r_1) \\ (r_4' = r_4 - r_1) \end{matrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -\frac{9}{2} & \frac{1}{2} \\ 0 & 0 & \frac{9}{2} & -\frac{1}{2} \end{bmatrix} \begin{matrix} (r_3' = r_3 - \frac{3}{2}r_2) \\ (r_4' = r_4 + \frac{1}{2}r_2) \end{matrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -\frac{9}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (r_4' = r_4 + r_3)$$

Row 4 is consistent, and the solution can be found by back substitution:

$$z = -\tfrac{1}{9}, \quad y = \tfrac{5}{9}, \quad x = \tfrac{16}{9}.$$

12.6. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 10 \\ 1 & -1 & -1 & 0 & 1 \\ 4 & -2 & -2 & -1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 & 10 \\ 0 & -2 & -2 & 1 & -9 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix},$$

after the sequence of row operations

$$r_2' = r_2 - r_1, \quad r_3' = r_3 - 4r_1, \quad r_3' = r_3 - 3r_2.$$

By row 3, the equations are inconsistent.

12.7. Transform the augmented matrix by row operations into echelon form:

$$\begin{bmatrix} 0 & 1 & 2 & -1 & 11 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 4 & 0 \\ 1 & -1 & 1 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 11 \\ 2 & 1 & -1 & 4 & 0 \\ 1 & -1 & 1 & -2 & 2 \end{bmatrix} \quad (r_1 \leftrightarrow r_2)$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 11 \\ 0 & -1 & -3 & 2 & -2 \\ 0 & -2 & 0 & -3 & 1 \end{bmatrix} \quad \begin{matrix} (r_3' = r_3 - 2r_1) \\ (r_4' = r_4 - r_1) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 11 \\ 0 & 0 & -1 & 1 & 9 \\ 0 & 0 & 4 & -5 & 23 \end{bmatrix} \quad \begin{matrix} (r_3' = r_3 + r_2) \\ (r_4' = r_4 + 2r_2) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 11 \\ 0 & 0 & -1 & 1 & 9 \\ 0 & 0 & o & -1 & 59 \end{bmatrix} \quad (r_4' = r_4 + 4r_3)$$

By back substitution $x_4 = -59$, $x_3 = -68$, $x_2 = 88$, $x_1 = 40$.

12.8. After interchanging rows 1 and 3 (to cover the possibility that a may be zero), the augmented matrix is

$$\begin{bmatrix} 4 & 1 & -2 & 2 \\ 1 & 2 & -a & 2 \\ a & -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 1 & -2 & 2 \\ 0 & \frac{7}{4} & \frac{1-a}{2} & \frac{3}{2} \\ 0 & -\frac{a+4}{4} & \frac{a+4}{2} & \frac{2-a}{2} \end{bmatrix} \quad \begin{matrix} (r_2' = r_2 - \frac{1}{4}r_1) \\ (r_3' = r_3 - \frac{1}{4}ar_1) \end{matrix}$$

$$\to \begin{bmatrix} 4 & 1 & -2 & 2 \\ 0 & \frac{7}{4} & \frac{1-a}{2} & \frac{3}{2} \\ 0 & 0 & \frac{16-a^2}{7} & \frac{13-2a}{7} \end{bmatrix} \quad (r_4' = r_4 + \tfrac{1}{7}(a+4)r_2)$$

Row 3 is inconsistent if $a = \pm 4$. If a is not equal to either of these values then the solution is

$$x = \frac{3}{a+4}, \quad y = \frac{2(a+1)(a-5)}{a^2-16}, \quad z = \frac{2a-13}{a^2-16}.$$

12.9. The method of Section 12.3 should be used. The inverses are
(a)

$$\begin{bmatrix} \frac{1}{18} & 0 & -\frac{1}{18} \\ -\frac{5}{54} & \frac{1}{9} & -\frac{1}{54} \\ \frac{7}{108} & \frac{1}{18} & \frac{5}{108} \end{bmatrix}.$$

(b)

$$\begin{bmatrix} \frac{1}{5} & 0 & -\frac{1}{5} \\ -\frac{6}{25} & \frac{1}{9} & \frac{1}{25} \\ \frac{7}{25} & \frac{1}{5} & \frac{3}{25} \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 4 & -4 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

(d)

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(e)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

12.10. A matrix A is singular if $\det A = 0$. For the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

rows 1 and 3 have the same elements. Hence $\det A = 0$ (Section 8.2) and the matrix is singular.

12.11. We need to find the points of intersection every set of three planes chosen from

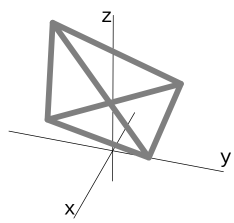

Figure 6: Problem 12.11

$$\begin{array}{rcrcrcr} 6x & - & 3y & - & z & = & -3 \\ 2x & - & y & + & 5z & = & 15 \\ & & y & + & z & = & 1 \\ 2x & + & y & - & z & = & 1 \end{array}.$$

This is equivalent to solving 4 sets of linear equations, which can be solved either by elimination or by row operations. The 4 sets of solutions are

$$(-1,-2,3), \quad (1,2,3), \quad (0,1,0), \quad (2,-1,2),$$

which are the vertices of the tetrahedron. The tetrahedron is shown in the Figure 6.

12.12. The equations of the straight lines PA, PB, PC are:

$$\frac{x-3}{2} = \frac{y-2}{1} = \frac{z-2}{1} = \lambda,$$
$$\frac{x-3}{2} = \frac{y-2}{2} = \frac{z-2}{1} = \mu,$$
$$\frac{x-3}{1} = \frac{y-2}{1} = \frac{z-2}{1} = \nu.$$

The lines PA, PB, PC meet the plane $x = 0$ where $\lambda = -\frac{3}{2}$, $\mu = -\frac{3}{2}$, $\nu = -3$. Hence the corners of the triangle projected on to the plane $x = 0$ are

$$(0, \tfrac{1}{2}, \tfrac{1}{2}), \quad (0, -1, \tfrac{1}{2}), \quad (0, -1, -1).$$

The lines PA, PB, PC meet the plane $y = 0$ where $\lambda = -2$, $\mu = -1$, $\nu = -2$. Hence the corners of the triangle projected on to the plane $y = 0$ are

$$(-1, 0, 0), \quad (1, 0, 1), \quad (1, 0, 0).$$

The lines meet the plane $z = 0$ where $\lambda = -2$, $\mu = -2$, $\nu = -2$. Hence the corners of the triangle projected on to the plane $z = 0$ are

$$(-1, 0, 0), \quad (-1, -2, 0), \quad (1, 0, 0).$$

12.13. The parabola $y = \alpha + \beta x + \gamma x^2$ passes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3) if solutions of

$$\alpha + \beta x_1 + \gamma x_1^2 = y_1,$$
$$\alpha + \beta x_2 + \gamma x_2^2 = y_2,$$
$$\alpha + \beta x_3 + \gamma x_3^2 = y_3,$$

for α, β, γ can be found. Leaving aside existence of solutions for the moment, elimination or row reduction gives

$$\alpha = \frac{x_2 x_3 y_1(x_3 - x_2) + x_3 x_1 y_2(x_1 - x_3) + x_1 x_2 y_3(x_2 - x_1)}{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)},$$
$$\beta = \frac{y_1(x_2^2 - x_3^2) + y_2(x_3^2 - x_1^2) + y_3(x_1^2 - x_2^2)}{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)},$$
$$\gamma = \frac{y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)}{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)}.$$

There are various exceptional cases. For example if $x_1 = x_2$ and $y_1 \neq y_2$ then there are no solutions. A similar result holds for the other pairs of x's.

12.14. The augmented matrix for

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 4 \\ x & - & y & + & z & = & 2 \\ 2x & + & y & - & \lambda z & = & \mu \end{array},$$

is

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -\lambda & \mu \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & -1 & -\lambda - 2 & \mu - 8 \end{bmatrix} \begin{array}{l} (r_2' = r_2 - r_1) \\ (r_3' = r_3 - r_1) \end{array}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -\lambda - 2 & \mu - 7 \end{bmatrix} \quad (r_3' = r_3 - \tfrac{1}{2} r_2)$$

(a) There is just one solution if $\lambda \neq -2$.
(b) No solutions if $\lambda = -2$ and $\mu \neq 7$.
(c) An infinite set of solutions if $\lambda = -2$ and $\mu = 7$.

12.15. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & 5 & 1 & -1 \\ 1 & 2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & -2 & -10 \\ 0 & 1 & -1 & -3 \end{bmatrix} \begin{matrix} (r_2' = r_2 - r_3) \\ (r_3' = r_3 - r_1) \end{matrix}$$
$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & -2 & -10 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

By row 3 the equations are inconsistent.
(b) The augmented matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} (r_1 \leftrightarrow r_2)$$
$$\rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \end{bmatrix} (r_3' = r_3 - r_1)$$
$$\rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equations are consistent with solution $z = \lambda$, $y = 1 - \lambda$, $x = 2 - \lambda$.
(c) The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 1 & 0 & -1 \\ 3 & 4 & -1 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & -1 & -5 \\ 0 & -2 & -4 & 0 \end{bmatrix} (r_2' = r_2 - r_1)$$
$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & -2 & 10 \end{bmatrix} (r_3' = r_3 - 2r_2)$$

There is a unique solution given by $z = -5$, $y = 10$, $x = -11$.

12.16. The determinant simplifies to

$$\begin{aligned} \Delta = \begin{vmatrix} 1-k & 2 & -1 \\ 2 & 1-k & -1 \\ -1 & -1 & 2-k \end{vmatrix} &= \begin{vmatrix} 4-k & 4-k & -4+k \\ 2 & 1-k & -1 \\ -1 & -1 & 2-k \end{vmatrix} \quad (r_1' = r_1 + r_2 - r_3) \\ &= (4-k) \begin{vmatrix} 0 & 0 & -1 \\ 1 & -k & -1 \\ 1-k & 1-k & 2-k \end{vmatrix} \quad \begin{matrix} (c_1' = c_1 + c_3) \\ (c_2' = c_2 + c_3) \end{matrix} \\ &= (4-k)(k-1)(1+k) \end{aligned}$$

Hence the determinant is zero when $k = -1, 1, 4$.

The equations

$$\begin{aligned} (1-k)x &+ 2y &- z &= 0 \\ 2x &+ (1-k)y &- z &= 0 \\ -x &- y &+ (2-k)z &= 0 \end{aligned}$$

have non-trivial solutions (that is, not all zero) ,if and only if, $\Delta = 0$, that is, if $k = -1$, $k = 1$, or $k = 4$.

Case $k=-1$. The equations are

$$\begin{array}{rcrcrcl} 2x & + & 2y & - & z & = & 0 \\ 2x & + & 2y & - & z & = & 0 \\ -x & - & y & + & 3z & = & 0 \end{array}.$$

The solution is $x=\lambda$, $y=-\lambda$, $z=0$.
Case $k=1$. the equations are

$$\begin{array}{rcrcrcl} & + & 2y & - & z & = & 0 \\ 2x & + & & - & z & = & 0 \\ -x & - & y & + & z & = & 0 \end{array}.$$

The solution is $x=\lambda$, $y=\lambda$, $z=2\lambda$.
Case $k=4$.

$$\begin{array}{rcrcrcl} -3x & + & 2y & - & z & = & 0 \\ 2x & + & -3y & - & z & = & 0 \\ -x & - & y & - & 2z & = & 0 \end{array}$$

The solutions is $x=-\lambda$, $y=-\lambda$, $z=\lambda$.

12.17. Simplifying the determinant:

$$\begin{aligned} \Delta = & \begin{vmatrix} a^2+t & ab & cd \\ ab & b^2+t & bc \\ ca & bc & c^2+t \end{vmatrix} \\ = & (t+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+t & b^2 \\ c^2 & c^2 & c^2+t \end{vmatrix} \quad (r_1'=ar_1+br_2+cr_3) \\ = & (t+a^2+b^2+c^20\begin{vmatrix} 0 & 0 & 1 \\ 0 & t & b^2 \\ -t & -t & c^2+t \end{vmatrix} \quad \begin{matrix}(c_1'=c_1-c_3) \\ (c_2'=c_2-c_3)\end{matrix} \\ = & t^2(t+a^2+b^2+c^2). \end{aligned}$$

The equations

$$\begin{array}{rcrcrcl} (1+t)x & + & 2y & + & 3z & = & 0 \\ 2x & + & (4+t)y & + & 6z & = & 0 \\ 3x & + & 6y & + & (9+t)z & = & 0 \end{array}$$

have non-trivial solutions, if and only if,

$$\Delta = \begin{vmatrix} 1+t & 2 & 3 \\ 2 & 4+t & 6 \\ 3 & 6 & 9+t \end{vmatrix} = 0.$$

Put $a=1$, $b=2$, and $c=3$ in the first determinant above. Then

$$\Delta = t^2(t+1^2+2^2+3^2) = t^2(t+14) = 0$$

when $t=0$ or $t=-14$.
Case $t=0$. The equations are

$$\begin{array}{rcrcrcl} 1x & + & 2y & + & 3z & = & 0 \\ 2x & + & 4y & + & 6z & = & 0 \\ 3x & + & 6y & + & 9z & = & 0 \end{array}.$$

There is effectively just one equation, so the solution will contain two parameters: $x=-2\mu-3\lambda$, $y=\mu$, $z=\lambda$.

Case $t = -14$. The equations become

$$\begin{array}{rcrcrcl} -13x & + & 2y & + & 3z & = & 0 \\ 2x & - & 10y & + & 6z & = & 0 \\ 3x & + & 6y & - & 5z & = & 0 \end{array}$$

The solution is $x = \frac{2}{3}\lambda$, $y = \frac{1}{3}\lambda$, $z = \lambda$.

12.18. The equations have non-trivial solutions if, and only if,

$$\begin{vmatrix} k & 4 & -1 & 3 \\ 4 & k & -1 & 3 \\ 4 & -1 & k & 3 \\ 4 & -1 & 3 & k \end{vmatrix} = \begin{vmatrix} k+6 & 4 & -1 & 3 \\ k+6 & k & -1 & 3 \\ k+6 & -1 & k & 3 \\ k+6 & -1 & 3 & k \end{vmatrix} \quad (c_1' = c_1 + c_2 + c_3 + c_4)$$

$$= (k+6)\begin{vmatrix} 1 & 4 & -1 & 3 \\ 1 & k & -1 & 3 \\ 1 & -1 & k & 3 \\ 1 & -1 & 3 & k \end{vmatrix}$$

$$= (k+6)(k+1)(k-3)(k-4) = 0,$$

after further row operations. The equations have non-trivial solutions if $k = -6, -1, 3,$ or 4.

12.19. The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 2 & 3 & 8 & -1 & 20 \\ 2 & 5 & 4 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & -1 & 2 & -1 & 12 \\ 0 & 1 & -2 & 1 & -3 \end{bmatrix} \begin{array}{l} (r_2' = r_2 - 2r_1) \\ (r_3' = r_3 - 2r_1) \end{array}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & -1 & 2 & -1 & 12 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix} \quad (r_3' = r_3 + r_2)$$

By row 3 the equations are inconsistent.

12.20. Denote the matrices by A and B respectively. By row operations reduce A to the identity matrix I_3, and then perform the same operations on I_3. Thus

$$A = \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} (r_2' = r_2 - \lambda r_3) \\ (r_1' = r_1 - \lambda r_2) \end{array}$$

whilst

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\lambda & \lambda^2 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} (r_2' = r_2 - \lambda r_3) \\ (r_1' = r_1 - \lambda r_2) \end{array}$$

$$= A^{-1}$$

Similarly

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\mu & 1 & 0 \\ \mu^2 & -\mu & 1 \end{bmatrix}.$$

Let $\lambda = 3$ and $\mu = 4$. Then

$$\begin{bmatrix} 13 & 3 & 0 \\ 4 & 13 & 3 \\ 0 & 4 & 1 \end{bmatrix} = AB.$$

Using the inverse formula $(AB)^{-1} = B^{-1}A^{-1}$,

$$\begin{aligned}(AB)^{-1} &= \begin{bmatrix} 1 & -\lambda & \lambda^2 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} (r_2' = r_2 - \lambda r_3) \\ (r_1' = r_1 - \lambda r_2) \end{matrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -\mu & 1 & 0 \\ \mu^2 & -\mu & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -3 & 9 \\ -4 & 13 & -39 \\ 16 & -52 & 157 \end{bmatrix}.\end{aligned}$$

12.21. Denote the determinant by Δ. Then

$$\begin{aligned}\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a(b+c) & b(c+a) & c(a+b) \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a(b+c) & c(b-a) & b(c-a) \end{vmatrix} \\ &\qquad (c_2' = c_2 - c_1), (c_3' = c_3 - c_1) \\ &= -(a-b)(c-a)\begin{vmatrix} a+b & c+a \\ c & b \end{vmatrix} \\ &= -(b-c)(c-a)(a-b)(a+b+c).\end{aligned}$$

The equations have non-trivial solutions if, and only if, $\Delta = 0$. From the expansion above this occurs if $b = c$, or $c = a$, or $a = b$ or $a + b + c = 0$.
Case $a + b + c = 0$. The equations can be expressed as

$$x + a^2 y - b^2 z = 0,$$

$$x + b^2 y - a^2 z = 0,$$

$$x + (a+b)^2 y - (a+b)^2 z = 0.$$

The solution is

$$x = 0, \quad y = \lambda, \quad z = \lambda.$$

12.22. Write the equations as
$x_1 = \frac{1}{3}(-x_2 - x_3 + 5)$,
$x_2 = \frac{1}{6}(2x_3 - 3x_4 + 6)$,
$x_3 = \frac{1}{4}(-x_1 + 2x_4 + 1)$,
$x_4 = \frac{1}{4}(x_2 + 2x_3 - 2)$.
Start with the values $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$ and $x_4^{(0)} = 0$, and compute $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots$ in sequence, but update variables at the first opportunity. For example, $x_1^{(1)} = \frac{5}{3}$ from the first computation and this value should replace $x_1^{(0}$ in the equation for $x_3^{(1)}$. The first two iterations and the final numerical solutions are shown in the table:

x_1	1.6667	1.3611	...	1.3981
x_2	1.0000	1.1181	...	1.0900
x_3	-0.0833	-0.2661	...	-0.2844
x_4	-0.2917	-0.3385	...	-0.3697

The matrix of coefficients in the equations is

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 6 & -2 & 3 \\ 1 & 0 & 4 & -2 \\ 0 & 1 & 2 & -4 \end{bmatrix}.$$

The system is diagonally dominant since $3 \geq 1+1=2$ in the first row, $6 \geq |-2|+3=5$ in the second row, $4 \geq 1+|-2|=3$ in the third row, and $|-4| \geq 1+2=3$ in the fourth row.

12.23. The matrix of coefficients is

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & -1 & -1 \\ 2 & 3 & -4 \end{bmatrix}.$$

Looking at the diagonal elements, $1 < |-2|+1=3$ in the first row, $|-1| < 1+|-1|=2$ in the second row, $|-4| < 2+3=5$ so that the system of equations is not diagonally dominant. In this case the Gauss-Seidel scheme may or may not converge. Let $x_1^{(0)}=0$, $x_2^{(0)}=0$, $x_3^{(0)}=0$. The equations can be written as

$x_1 = 2x_2 + x_3 + 4.$

$x_2 = x_1 - x_3 - 1.$

$x_3 = \frac{1}{4}(2x_1 + 3x_2 - 4).$

As the following table indicates, the sequences do not converge.

x_1	4	6.75	4.75	2	...
x_2	3	2.5	-0.5	0	...
x_3	3.25	4.25	1	0	...

However, the equations do have the unique solution $x=0.45$, $y=-0.55$, $z=0.75$ found by elimination.

12.24. The matrix of coefficients is

$$\begin{bmatrix} 6 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}.$$

Row 2 fails the dominant diagonal test.

The equations can be written as

$x_1 = \frac{1}{6}(x_2 - x_3 + 2),$

$x_2 = \frac{1}{2}(-3x_1 - x_3 + 1),$

$x_3 = \frac{1}{4}(-x_1 + x_2 + 5).$

Let $x_1^{(0)}=0$, $x_2^{(0)}=0$, $x_3^{(0)}=1$. Then the table shows the convergence of the Gauss-Seidel scheme:

x_1	0.1667	0.1007	...	0.1000
x_2	-0.2500	-0.2240	...	-0.2333
x_3	1.1458	1.1688	...	1.1667

12.25. Equations (12.7)-(12.9) are

$x_1 = \frac{1}{3}(-x_2 - x_3 - 1),$

$x_2 = \frac{1}{4}(x_1 - x_3 - 8),$

$x_3 = \frac{1}{5}(-2x_1 - x_2 - 14).$

Let $x_2^{(0)}=0$, $x_3^{(0)}=0$. Using the Jacobi scheme we have to specify additionally $x_1^{(0)}$. We have chosen, for comparison purposes, $x_1^{(0)} = -\frac{1}{3}$ which is the first value $x_1^{(1)}$.

x_1	-0.3333	1.2500	0.8889	...
x_2	-2.0833	-1.4167	-1.1250	...
x_3	-2.2500	-2.2500	3.0167	...

This table can be compared with the one following eqn (12.12).

Chapter 13: Eigenvalues and eigenvectors

13.1. (a) The eigenvalues are given by

$$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) - 12 = \lambda(\lambda-8) = 0.$$

Hence the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 8$.

The eigenvector $\mathbf{s}_i$ is given by $[A - \lambda_i \mathrm{I}_2]\mathbf{s}_i = 0$, $(i = 1, 2)$. If

$$\mathbf{s}_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix},$$

then, in this example,

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{matrix} 2a_1 + 3a_2 = 0 \\ 4a_1 + 6a_2 = 0 \end{matrix}.$$

A particular convenient solution is

$$\mathbf{s}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Similarly, we can choose

$$\mathbf{s}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(b) The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix} \text{ are } \lambda_1 = 4, \quad \lambda_2 = 9, \quad \mathbf{s}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c) The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix} \text{ are } \lambda_1 = 4 - 2\sqrt{2}, \quad \lambda_2 = 4 + 2\sqrt{2},$$

$$\mathbf{s}_1 = \begin{bmatrix} -1-\sqrt{2} \\ 2 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} -1+\sqrt{2} \\ 2 \end{bmatrix}.$$

(d) The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \text{ are } \lambda_1 = 3 - 2\sqrt{2}, \quad \lambda_2 = 3 + \sqrt{2},$$

$$\mathbf{s}_1 = \begin{bmatrix} -1-\sqrt{2} \\ 2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} -1+\sqrt{2} \\ 2 \end{bmatrix}.$$

(e) The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 \\ 14 & 5 \end{bmatrix} \text{ are } \lambda_1 = 3 - 4\sqrt{2}, \quad \lambda_2 = 3 + 4\sqrt{2},$$

$$\mathbf{s}_1 = \begin{bmatrix} -1-2\sqrt{2} \\ 7 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} -1+2\sqrt{2} \\ 7 \end{bmatrix}.$$

(f) The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -2 \\ 4 & 6 \end{bmatrix} \text{ are } \lambda_1 = 4 - 2\mathrm{i}, \quad \lambda_2 = 4 + 2\mathrm{i},$$

$$\mathbf{s}_1 = \begin{bmatrix} -1-\mathrm{i} \\ 2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} -1+\mathrm{i} \\ 2 \end{bmatrix}.$$

13.2. The eigenvalues of

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

are given by the equation

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - (a+c)\lambda + ac - b^2 = 0.$$

Hence

$$\lambda = \tfrac{1}{2}\{(a+c) \pm \sqrt{[(a-c)^2 + 4b^2]}\}.$$

Since $(a-c)^2 + 4b^2 \geq 0$, both eigenvalues are real.

13.3. The eigenvalues of

$$A = \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$$

are $\lambda_1 = 4$ and $\lambda_2 = 9$. The inverse of A is

$$A^{-1} = \begin{bmatrix} \frac{7}{36} & \frac{1}{12} \\ -\frac{1}{18} & \frac{1}{6} \end{bmatrix}.$$

The eigenvalues of this matrix are given by

$$\begin{vmatrix} 6-\mu & 3 \\ 2 & 7-\mu \end{vmatrix} = 0, \text{ or } 36\mu^2 - 13\mu + 1 = 0, \text{ or } (4\mu - 1)(9\mu - 1) = 0.$$

Hence the eigenvalues are $\mu_1 = \frac{1}{4} = 1/\lambda_1$ and $\mu_2 = \frac{1}{9} = 1/\lambda_2$. The eigenvalues of the inverse of a matrix are their reciprocals of the eigenvalues of the matrix.

The square of A is

$$A^2 = \begin{bmatrix} 42 & 39 \\ 26 & 55 \end{bmatrix}.$$

Its eigenvalues are $\lambda_1^2 = 16$ and $\lambda_2^2 = 81$.

13.4. (a) The eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

are given by

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 + \lambda - 4 = -(\lambda+1)(\lambda-1)(\lambda-4).$$

Hence the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 4$.

The eigenvector $\mathbf{s_i}$ associated with the eigenvalue λ_i, $(i = 1, 2, 3)$ is given by any non-zero solution of

$$[A - \lambda_i]\mathbf{s}_i = \mathbf{0}.$$

Thus $\mathbf{s}_1 = [a_1\ b_1\ c_1]^T$ satisfies

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \mathbf{0}, \text{ or } \begin{aligned} 2a_1 + b_1 + 2c_1 &= 0 \\ a_1 + 3b_1 + c_1 &= 0 \\ 2a_1 + b_1 + 2c_1 &= 0. \end{aligned}$$

The general solution is

$$\mathbf{s}_1 = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} : \text{ a convenient choice is } \mathbf{s}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Eigenvectors associated with λ_2 and λ_3 are

$$\mathbf{s}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Eigenvalues and eigenvectors are

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 5, \quad \mathbf{s}_1 = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) Eigenvalues and eigenvectors are

$$\lambda_1 = -2, \quad \lambda_2 = 2, \quad \lambda_3 = 3, \quad \mathbf{s}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{s}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

(d) Eigenvalues and eigenvectors are

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 16, \quad \mathbf{s}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{s}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

13.5. The eigenvalues are given by the solutions of

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} = \lambda^4 - 9\lambda^3 + 22\lambda^2 - 32 = (\lambda+1)(\lambda+2)(\lambda-4)^2 = 0.$$

Hence the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 4 \text{ (repeated)}.$$

The corresponding eigenvectors are

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_4 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

Note that two independent eigenvectors are associated with the repeated eigenvalue $\lambda_3 = 4$, since

$$[A - \lambda_3]\mathbf{s}_3 = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \mathbf{0}.$$

where $\mathbf{s}_3 = [a_3 \; b_3 \; c_3 \; d_3]^T$. The linear equations become

$$\begin{aligned} -3a_3 + 2b_3 &= 0, \\ 3a_3 - 2b_3 &= 0, \\ -c_3 + d_3 &= 0, \\ c_3 - d_3 &= 0. \end{aligned}.$$

The general solution contains two parameters, say α and β such that

$$a_3 = 2\alpha, \quad b_3 = 3\alpha, \quad c_3 = \beta, \quad d_3 = \beta.$$

We can obtain two independent eigenvectors, one with $\beta = 0$ and $\alpha = 1$, and one with $\alpha = 0$ and $\beta = 1$, which accounts for $\mathbf{s}_3$ and $\mathbf{s}_4$ above.

Note the relation between the eigenvalues of A and the eigenvalues of the two 2×2 submatrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

13.6. The eigenvalues are given by

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 2 & 3 \end{bmatrix} = -\lambda^3 + 8\lambda^2 - 13\lambda + 6 = -(\lambda-1)^2(\lambda-6) = 0.$$

Hence the eigenvalues are $\lambda_1 = 1$ (repeated) and $\lambda_3 = 6$. The eigenvector $\mathbf{s}_1 = [a_1,\ b_1\ c_1]^T$ satisfies

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \mathbf{0}, \text{ or } \begin{array}{c} b_1 + 2c_1 = 0 \\ 2b_1 + 4c_1 = 0 \end{array}.$$

The general solution can be expressed in terms of two parameters: $a_1 = \alpha$, $b_1 = -2\beta$, $c_1 = \beta$ so that

$$\mathbf{s}_1 = \begin{bmatrix} \alpha \\ -2\beta \\ \beta \end{bmatrix}.$$

We can associate two independent eigenvectors with λ_1, one with $\alpha = 1$, $\beta = 0$, and one with $\alpha = 0$, $\beta = 1$. Hence, three independent eigenvectors are

$$\mathbf{s}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

where $\mathbf{s}_3$ is the eigenvector associated with λ_3.

13.7. The eigenvalues of A are given by

$$\begin{vmatrix} -1-\lambda & -1 & a+1 \\ a+1 & -a-\lambda & -1 \\ -a & a+1 & -a-\lambda \end{vmatrix} = -\lambda(\lambda^2 + (2a+1)\lambda + 2a^2 + 5a + 2) = 0.$$

Hence A as a zero eigenvalue. A second solution must be $\lambda = 3$ which means that a must satisfy

$$9 + 3(2a+1) + 2a^2 + 5a + 2 = 0, \text{ or } 2a^2 + 11a + 14 = 0, \text{ or } (2a+7)(a+2) = 0.$$

The solutions are $a = -2$ and $a = -\frac{7}{2}$.

Case $a = -2$. $\det A = \lambda^2(3-\lambda)$. The third eigenvalue is a repeated $\lambda = 0$.

Case $a = -\frac{7}{2}$. $\det A = -\lambda(\lambda-3)^2$. The third eigenvalue is a repeated $\lambda = 3$.

13.8. The eigenvalues of A occur where $A\mathbf{x} = \lambda\mathbf{x}$ has non-trivial solutions. Since $A^2 = A$,

$$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A^2\mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda^2\mathbf{x} = \lambda\mathbf{x}.$$

Therefore

$$\lambda(\lambda-1)\mathbf{x} = \mathbf{0},$$

which will only have non-trivial solutions for $\mathbf{x}$ if $\lambda = 0$ or $\lambda = 1$.

A is idempotent since

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} = A.$$

The matrices A and A^2 have the same eigenvalues $0, 1, 1$. The corresponding eigenvectors of both A and A^2 are

$$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

13.9. Check that

$$A^2 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}^2 = \mathrm{I}_4.$$

Multiply $A\mathbf{x} = \lambda\mathbf{x}$ on the left by A:

$$A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}.$$

Since $A^2 = \mathrm{I}_4$, it follows that

$$\mathrm{I}_4\mathbf{x} = \lambda^2\mathbf{x}, \text{ or } (\lambda^2 - 1)\mathbf{x} = 0.$$

Non-trivial solutions for $\mathbf{x}$ only exist if $(\lambda^2 - 1) = 0$, or $\lambda = \pm 1$. The eigenvectors are

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{s}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since A has 4 independent eigenvectors it can be diagonalized.

13.10. The eigenvalues of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

are $\lambda_1 = -1$, $\lambda = 1$, $\lambda_3 = 4$. Check that

$$\text{trace A} = 1 + 1 + 2 = 4 = \lambda_1 + \lambda_2 + \lambda_3,$$

and that

$$\det A = -4 = \lambda_1\lambda_2\lambda_3.$$

13.11. The determinant

$$\begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

is zero, which means that the vectors are linearly dependent.

13.12. Use the method explained in Section 13.4. The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -4 & 1 & -2 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

are

$$\lambda_1 = -5, \quad \lambda_2 = -1, \quad \lambda_3 = 0, \quad \mathbf{s}_1 = \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Form the matrix

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -5 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Next find the inverse of C:

$$C^{-1} = \frac{1}{20} \begin{bmatrix} 2 & -1 & 1 \\ -10 & -15 & -5 \\ 4 & 8 & 12 \end{bmatrix}.$$

Now evaluate the product $C^{-1}AC$:

$$\begin{aligned} C^{-1}AC &= \frac{1}{20} \begin{bmatrix} 2 & -1 & 1 \\ -10 & -15 & -5 \\ 4 & 8 & 12 \end{bmatrix} \begin{bmatrix} -4 & 1 & -2 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -1 & -1 \\ -5 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

which is a diagonal matrix of eigenvalues of A.

13.13. The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$$

are

$$\lambda_1 = -3, \quad \lambda_2 = 5, \quad \mathbf{s}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}.$$

Then

$$C^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

Confirm that

$$C^{-1}AC = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix},$$

a diagonal matrix of the eigenvalues.

13.14. The eigenvalues of A are given by

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 2 & -1-\lambda \end{bmatrix} = -\lambda^3 + 3\lambda^2 + 4\lambda - 12 = -(\lambda+2)(\lambda-2)(\lambda-3) = 0.$$

Hence the eigenvalues are

$$\lambda_1 = -2, \quad \lambda_2 = 2, \quad \lambda_3 = 3.$$

The corresponding eigenvectors are

$$\mathbf{s}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Let

$$C = \left[\ \mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3\ \right] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$

Its inverse is

$$C^{-1} = \frac{1}{5}\begin{bmatrix} 0 & -1 & 2 \\ 5 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} C^{-1}AC &= \tfrac{1}{5}\begin{bmatrix} 0 & -1 & 2 \\ 5 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D. \end{aligned}$$

13.15. Use results outlined in Problem 8.15. Take determinants of both sides of $C^{-1}AC = D$:

$$\det[C^{-1}AC] = \det D, \text{ or } \det C^{-1}\det A \det C = \det D,$$

$$\text{or } (1/\det C)\det A \det C = \det D.$$

Hence $\det A = \det D$: the result follows, since

$$\det D = \lambda_1\lambda_2 \ldots \lambda_n.$$

13.16. The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

are

$$\lambda_1 = -\frac{1}{4},\ \lambda_2 = \frac{1}{4},\ \lambda_3 = 1,\ \mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let

$$C = \left[\ \mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3\ \right] = \begin{bmatrix} -1 & -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix}, \text{ so that } C^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Let D be the diagonal matrix of eigenvalues:

$$D = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, by Section 13.5,

$$\begin{aligned} A^n &= CD^nC^{-1} \\ &= \begin{bmatrix} -1 & -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} (-\frac{1}{4})^n & 0 & 0 \\ 0 & (\frac{1}{4})^n & 0 \\ 0 & 0 & 1^n \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} + \frac{1}{3}[1 + 3(-1)^n]2^{-2n-1} & \frac{1}{3} + \frac{1}{3}[1 - 3(-1)^n]2^{-2n-1} & \frac{1}{3}(1 - 4^{-n}) \\ \frac{1}{3} + \frac{1}{3}[1 - 3(-1)^n]2^{-2n-1} & \frac{1}{3} + \frac{1}{3}[1 + 3(-1)^n]2^{-2n-1} & \frac{1}{3}(1 - 4^{-n}) \\ \frac{1}{3}(1 - 4^{-n}) & \frac{1}{3}(1 - 4^{-n}) & \frac{1}{3} + \frac{2}{3}4^{-n} \end{bmatrix}. \end{aligned}$$

Therefore

$$A^n \to \tfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

as $n \to \infty$, since $2^{-2n-1} \to 0$ and $4^{-n} \to 0$.

13.17. A matrix A is orthogonal if $A^T = A^{-1}$. In this example

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} = A^T,$$

and so A is orthogonal. Expanding $\mathbf{X} = A\mathbf{x}$:

$$X = x. \quad Y = y\cos\alpha, \quad Z = z\sin\alpha,$$

or, inverting,

$$x = X, \quad y = Y\cos\alpha, \quad z = -Y\sin\alpha + Z\cos\alpha.$$

The x and X axes remain coincident and the $OXYZ$ coordinate frame is rotated about the x axis through an angle α relative to the $Oxyz$ frame. The points on the x axis are unaffected by the rotation.

13.18. The matrix is orthogonal since

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = A^T.$$

13.19. The matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

is orthogonal, which implies that the X and Y axes are orthogonal. Expanding

$$X = x\cos\alpha - y\sin\alpha, \quad Y = x\sin\alpha + y\cos\alpha,$$

or, inverting,

$$x = X\cos\alpha + Y\sin\alpha, \quad y = -X\sin\alpha + Y\cos\alpha.$$

A sketch of the axes drawn from these relations shows that α is the angle between the x axis and the X axis. The x axis becomes the line $Y\cos\alpha = X\sin\alpha$, and the y axis becomes the line $Y\sin\alpha = -X\cos\alpha$.

13.20. The eigenvalues are given by

$$\begin{bmatrix} -\lambda & a & b \\ -a & -\lambda & c \\ -b & -c & \lambda \end{bmatrix} = -\lambda(\lambda^2 + a^2 + b^2 + c^2).$$

Hence the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = -\surd[-a^2 - b^2 - c^2], \quad \lambda_3 = \surd[-a^2 - b^2 - c^2].$$

The two eigenvalues λ_2 and λ_3 are imaginary assuming that a, b and c are not all zero.

13.21. Given

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

then

$$\begin{aligned}
\det(A - \lambda \mathrm{I}_3) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)[(1-\lambda)(2-\lambda) - 1] - 2[2(2-\lambda) - 1] + [2 - (1-\lambda)] \\
&= -\lambda^3 + 4\lambda^2 + \lambda - 4.
\end{aligned}$$

The required powers of A are

$$A^2 = \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix}, \qquad A^3 = AA^2 = \begin{bmatrix} 21 & 22 & 21 \\ 22 & 21 & 21 \\ 21 & 21 & 22 \end{bmatrix}.$$

Hence

$$\begin{aligned}
-A^3 + 4A^2 + A - 4\mathrm{I}_3 &= -\begin{bmatrix} 21 & 22 & 21 \\ 22 & 21 & 21 \\ 21 & 21 & 22 \end{bmatrix} + 4\begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= 0
\end{aligned}$$

Mutiply both sides of this equation by A^{-1}:

$$-A^{-1}A^3 + 4A^{1}A^2 + A^{-1}A - 4A^{-1} = -A^2 + 4A + \mathrm{I}_3 + A^{1} = 0.$$

Therefore

$$A^{-1} = \frac{1}{4}[-A^2 + 4A + \mathrm{I}_3] = \frac{1}{4}\begin{bmatrix} -1 & 3 & -1 \\ 3 & -1 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

13.22. The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 5 & -1 & -3 & 3 \\ -1 & 5 & 3 & -3 \\ -3 & 3 & 5 & -1 \\ 3 & -3 & -1 & 5 \end{bmatrix}$$

are

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 4, \quad \lambda_4 = 12,$$

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

The matrix has 4 independent eigenvectors even though there is a repeated eigenvalue. Let

$$C = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \text{ and } C^{-1} = \frac{1}{4}\begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

We can then check that

$$C^{-1}AC = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

Since A has a zero eigenvalue, $\det A = 0$.

13.23. Let $\mathbf{x} = [x_1\ x_2\ x_3]^T$.
(a) In matrix form

$$x_1^2 + x_2^2 + x_3^3 + 4x_1x_2 - 4x_1x_3 + 4x_2x_3 = \mathbf{x}^T \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = -3, \quad \lambda_2 = \lambda_3 = 3, \quad \mathbf{s}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The required matrix

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(b) In matrix form

$$x_1x_2 - x_1x_3 + x_2x_3 = \mathbf{x}^T \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \mathbf{x}.$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = -1, \quad \lambda_2 = \lambda_3 = \frac{1}{2}, \quad \mathbf{s}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The required matrix is

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

13.24. Express each quadratic form as $\mathbf{x}^T A\mathbf{x}$ and find the eigenvalues of A. Then the quadratic form is positive-definite if, and only if, all its eigenvalues are positive (Section 13.7).
(a) In matrix form

$$4x_1^2 + x_2^2 - 4x_1x_2 = \mathbf{x}^T \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \mathbf{x}.$$

Hence

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix},$$

and its eigenvalues are 0 and 5. Therefore the quadratic form is not positive-definite.
(b) In matrix form

$$x_1^2 + x_2^2 + 2x_3^2 + x_2x_3 + 2x_3x_1 + 4x_1x_2 = \mathbf{x}^T \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \mathbf{x}.$$

Hence

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

and its eigenvalues are $-1, 1, 4$. Since it has a negative eigenvalue, the quadratic form is not positive-definite.

(c) In matrix form

$$6x_1^2 + 2x_2^2 - x_3x_1 = \mathbf{x}^T \begin{bmatrix} 6 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \mathbf{x}.$$

Its eigenvalues are 2 and $\frac{1}{2}\sqrt{[6 \pm \sqrt{37}]}$. One eigenvalue is negative so that the quadratic form is not positive-definite.

13.25. The equations of motion can be obtained as in Section 13.8 but with three particles. As in a generalization of Fig. 13.2, the first spring has extension x and tension $-kx$, the second extension $y-x$ and tension $k(y-x)$, the third extension $z-y$ and tension $k(z-y)$ and the fourth extension $-z$ and tension $-kz$. Applying Newton's law to each particle:

$$m\ddot{x} = -kx + k(y-x) = k(-2x+y),$$

$$m\ddot{y} = -k(y-x) + k(z-y) = k(x-2y+z),$$

$$m\ddot{z} = -k(z-y) - kz = k(y-2z).$$

Let

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad A = \frac{k}{m}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Then the equations of motion can be expressed in the matrix form

$$\ddot{\mathbf{x}} + A\mathbf{x} = \mathbf{0}.$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = 2k/m, \quad \lambda_2 = [2-\sqrt{2}]k/m, \quad \lambda_2 = [2+\sqrt{2}]k/m.$$

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}.$$

Introduce the normal coordinates $\mathbf{X} = [X\ Y\ Z]^T$ where

$$\mathbf{x} = C\mathbf{X} = \begin{bmatrix} -X+Y+Z \\ \sqrt{2}Y - \sqrt{2}Z \\ X+Y+Z \end{bmatrix}.$$

The equations of motion become

$$C\ddot{\mathbf{X}} + AC\mathbf{X} = \mathbf{0} \text{ or } \ddot{\mathbf{X}} + D\mathbf{X} = \mathbf{0},$$

where

$$C^{-1}AC = D = \frac{k}{m}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{bmatrix}.$$

13.26. Calculating A^2 and A^3:

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^3 = AA^2 = \mathrm{I}_3.$$

Therefore

$$A^4 = A^3A = A, \quad A^5 = A^3A^2 = A^2,$$

and, in general,

$$A^{3m} = \mathrm{I}_3, \quad A^{3m+1} = A, \quad A^{3m+2} = A^2, \quad (m = 1, 2, 3, \ldots).$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{2}(-1 - \mathrm{i}\sqrt{3}), \quad \lambda_3 = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3}),$$

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} -\frac{1}{2}(1 + \mathrm{i}\sqrt{3}) \\ -\frac{1}{2}(1 - \mathrm{i}\sqrt{3}) \\ 1 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} -\frac{1}{2}(1 - \mathrm{i}\sqrt{3}) \\ -\frac{1}{2}(1 + \mathrm{i}\sqrt{3}) \\ 1 \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}(1 + \mathrm{i}\sqrt{3}) & -\frac{1}{2}(1 - \mathrm{i}\sqrt{3}) \\ 1 & -\frac{1}{2}(1 - \mathrm{i}\sqrt{3}) & -\frac{1}{2}(1 + \mathrm{i}\sqrt{3}) \\ 1 & 1 & 1 \end{bmatrix}$$

Its inverse is given by

$$C^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ -1 + \mathrm{i}\sqrt{3} & -1 - \mathrm{i}\sqrt{3} & 2 \\ -1 - \mathrm{i}\sqrt{3} & -1 + \sqrt{3} & 2 \end{bmatrix}.$$

By (13.5)

$$\begin{aligned} A^n &= CD^nC^{-1} \\ &= \begin{bmatrix} 1 & -\frac{1}{2}(1 + \mathrm{i}\sqrt{3}) & -\frac{1}{2}(1 - \mathrm{i}\sqrt{3}) \\ 1 & -\frac{1}{2}(1 - \mathrm{i}\sqrt{3}) & -\frac{1}{2}(1 + \mathrm{i}\sqrt{3}) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(-1)^{\frac{1}{3}n} & 0 \\ 0 & 0 & -(-1)^{\frac{2}{3}n} \end{bmatrix}^n \\ &\quad \times \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ -1 + \mathrm{i}\sqrt{3} & -1 - \mathrm{i}\sqrt{3} & 2 \\ -1 - \mathrm{i}\sqrt{3} & -1 + \sqrt{3} & 2 \end{bmatrix} \\ &= \begin{cases} \mathrm{I}_3 & n = 3m \\ A & n = 3m + 1 \\ A^2 & n = 3m + 2 \end{cases} \end{aligned}$$

where $m = 1, 2, 3, \ldots$.

13.27. Multiply S on the right by A:

$$\begin{array}{rcccl} S & = & A & + & A^2 + \cdots + A^n \\ SA & = & & & A^2 + \cdots + A^n + A^{n-1} \end{array},$$

and take the difference $S - SA$, so that

$$S - SA = A - A^{n-1}, \text{ or } S(\mathrm{I}_3 - A) = A(\mathrm{I}_3 - A^n).$$

Therefore

$$S = A(\mathrm{I}_3 - A^n)(\mathrm{I}_3 - A)^{-1}.$$

The method fails if $\mathrm{I}_3 - A$ is a singular matrix, which is equivalent to A having a unit eigenvalue.

The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

are

$$\lambda_1 = -1, \quad \lambda_2 = 4, \quad \mathbf{s}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then

$$A^n = CD^nC^{-1} = \frac{1}{5}\begin{bmatrix} 3(-1)^n + 2^{2n+1} & -3(-1)^n + 3 \times 4^n \\ -2(-1)^n + 2^{2n+1} & 2(-1)^n + 3 \times 4^n \end{bmatrix}$$

Finally

$$[\mathrm{I}_2 - A]^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix}$$

so that

$$\begin{aligned} S &= A(\mathrm{I}_2 - A^n)(\mathrm{I}_2 - A)^{-1} \\ &= \frac{1}{5}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 5 - 3(-1)^n - 2^{2n+1} & 3(-1)^n - 3 \times 4^n \\ 2(-1)^n - 2^{2n+1} & 5 - 2(-1)^n - 3 \times 4^n \end{bmatrix}\begin{bmatrix} \frac{1}{6} & -\frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix} \\ &= \frac{1}{30}\begin{bmatrix} 5 - 3(-1)^n - 2^{2n+1} & -27[(-1)^n - 4^n] \\ -8[(-1)^n - 4^n] & 0 \end{bmatrix}. \end{aligned}$$

13.28. As in Example 13.5, the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

are

$$\lambda_1 = -4, \quad \lambda_2 = 1, \quad \lambda_3 = -1.$$

We now choose different eigenvectors (compare Example 13.5):

$$\mathbf{s}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}.$$

Its inverse is

$$C^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}.$$

We can now check that

$$C^{-1}AC = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The problem illustrates the point that the choice of eigenvectors and consequently the choice of C does not affect the matrix $C^{-1}AC$: it always equals D.

13.29. The remaining 3×3 minors of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 11 & 12 & 5 \\ 9 & 8 & 7 & 6 \end{bmatrix},$$

are

$$\begin{vmatrix} 1 & 2 & 3 \\ 10 & 11 & 12 \\ 9 & 8 & 7 \end{vmatrix} = 0, \qquad \begin{vmatrix} 1 & 3 & 4 \\ 10 & 12 & 5 \\ 9 & 7 & 6 \end{vmatrix} = -160, \qquad \begin{vmatrix} 2 & 3 & 4 \\ 11 & 12 & 5 \\ 8 & 7 & 6 \end{vmatrix} = -80.$$

The matrix has 18 (2×2) minors. Their values are:

with elements from rows 1 and 2: $-9, -18, -35, -9, -34, -33$;

with elements from rows 1 and 3: $-10, -20, -30, -10, -20, -10$;

with elements from rows 2 and 3: $-19, -38, 15, -19, 26, 37$.

13.30. (a) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

Then $\det A = 0$ which means that $\operatorname{rank} A < 3$. However, the 2×2 minor

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \neq 0.$$

Therefore the rank of A is 2.

(b) Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}.$$

In this case $\det A = 12$, which means that the matrix is non-singular and has rank 3.

(c) The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 \end{bmatrix}$$

is of rank 2.

13.31. Apply row operations to the matrix in Problem 13.30c:

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \begin{matrix} (\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{r}_1) \\ (\mathbf{r}_3' = \mathbf{r}_3 - 2\mathbf{r}_1) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\mathbf{r}_3' = \mathbf{r}_3 + \mathbf{r}_2)$$

In echelon form the matrix has one row of zeros which means that its rank is 2.

13.32. The eigenvalues of

$$A_1 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

are $\lambda_1 = 0$ and the repeated eigenvalue $\lambda_2 = 1$. The rank of $\lambda_1 \mathrm{I}_3 - A_1$ is 2, and the rank of $\lambda_2 \mathrm{I}_3 - A_1$ is also 2.

The eigenvalues of

$$A_2 = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

are $\lambda_3 = 2$ and the repeated eigenvalue $\lambda_4 = 1$. The rank of $\lambda_3 \mathrm{I}_3 - A_2$ is 2, but the rank of $\lambda_4 \mathrm{I}_3 - A_2$ is 1. The vector space associated with this eigenvalue is 2, which means that we can define two independent eigenvalues.

The eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

are $\lambda_1 = -1$, $\lambda_2 = 1$ (repeated), $\lambda_3 = 2$. The rank of A is 4 (since $\det A \neq 0$). The ranks of $\lambda_1 \mathrm{I}_4 - A$ and $\lambda_3 \mathrm{I}_4 - A$ are both 3. The rank of $\lambda_2 \mathrm{I}_2 - A$ is 3. Hence in this problem we can find only three independent eigenvectors, which can be expressed as

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$