PART I: Elementary methods, differentiation, complex numbers

Chapter 1: Standard functions and techniques

1.1. (a) $y = x^4$ for $-1.5 \leq x \leq 1$:

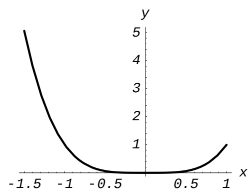

Figure 1: Problem 1.1a

(b) $y = x(1 - x)$ for $-1 \leq x \leq 2$:

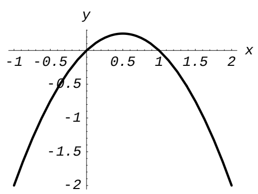

Figure 2: Problem 1.1b

(c) $y = 1 + x + x^2$ for $|x - 1| \leq 2$:

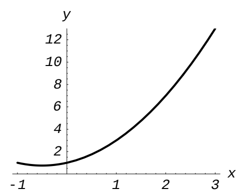

Figure 3: Problem 1.1c

(d) $y = |x - 1|$ for $-3 \leq x \leq 3$:

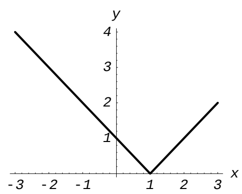

Figure 4: Problem 1.1d

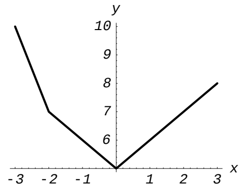

Figure 5: Problem 1.1e

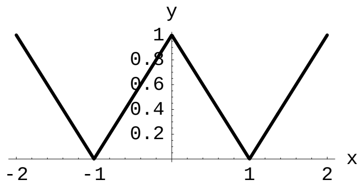

Figure 6: Problem 1.1f

(e) $y = |x| + |x - 3| + |x + 2|$ for $-3 \leq x \leq 4$:
(f) $y = ||x| - 1|$ for $-2 \leq x \leq 2$:
(g) $y = \sqrt{(x^2 + 1)}$ for $|x| \leq 2$:

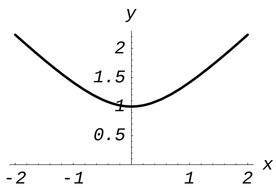

Figure 7: Problem 1.1g

1.2. (a) $y = -2x + 3$; (b) $y = 1$; (c) $y = \frac{2}{3}x - \frac{1}{3}$. The intersections occur at $A : (2, 1)$, $B : (\frac{5}{4}, \frac{1}{2})$, $C : (1, 1)$. The side lengths are: $AB = \frac{1}{4}\sqrt{13}$, $BC = \frac{1}{4}\sqrt{5}$, $CA = 1$.

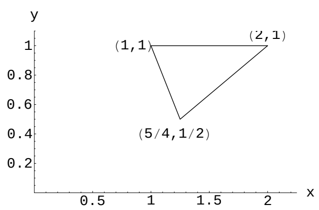

Figure 8: Problem 1.2

1.3. (a) Slope is 1, and the line cuts the axes at $(0, -1)$ and $(1, 0)$.
(b) Slope is $\frac{1}{3}$, and the line cuts the axes at $(0, -\frac{2}{3})$ and $(2, 0)$.
(c) Slope is $-\frac{2}{5}$, and the line cuts the axes at $(0, -\frac{4}{5})$ and $(2, 0)$.

1.4. (a) $y = x + 1$; (b) $y = -2x - 4$; (c) $y = 0.5x - 0.5$; (d) $y = 3x - 1$; (e) the slope of the line must be $-\frac{1}{4}$: $y = -\frac{1}{4}x + \frac{11}{4}$.

1.5. The products of the slopes in each case must be -1. The slopes are: (a) $-\frac{3}{2}$ and $\frac{2}{3}$; (b) 2 and $-\frac{1}{2}$; (c) 2 and $-\frac{1}{2}$; (d) 1 and -1.

1.6. At the point of intersection, $x + y + 1 = 0$ and $2x - 3y - 2 = 0$, so the line

$$(x + y + 1) + \alpha(2x - 3y - 2) = 0$$

must pass through this point, which has coordinates $(-\frac{1}{5}, -\frac{4}{5})$. The straight line joining this point to $(1,1)$ is

$$2y = 3x - 1,$$

with $\alpha = \frac{1}{2}$.

1.7. (a) Centre at $(0,0)$, radius 3; (b) centre at $(1,0)$, radius 2; (c) centre at $(1,1)$, radius $\sqrt{23}$; (d) centre at $(\frac{1}{2}, -\frac{1}{2})$, radius $\frac{1}{2}\sqrt{11}$.

1.8. $(x-1)^2 + (y+2)^2 = 9$.

1.9. Eliminate one of the variables in each case and solve the resulting quadratic equation.
(a) $(2,2)$ and $(2,-2)$;
(b) Eliminate y, so that x satisfies the equation

$$x^2 + (2x+1)^2 - 2x + 2(2x+1) - 4 = 0, \text{ or } 5x^2 + 6x - 1 = 0.$$

The points of intersection are

$$(\tfrac{1}{5}(-3-\sqrt{14}, -1-2\sqrt{14}) \text{ and } (\tfrac{1}{5}(-3+\sqrt{14}, -1+2\sqrt{14})$$

(c) $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, one point only since the line is tangential to the circle.

1.10. To three decimal places, the distances of the points from the origin are

$$1.060, \quad 0.993, \quad 1.011, \quad 0.896, \quad 1.124.$$

The average value of these distances is $r = 1.017$. The equation of the circle is

$$x^2 + y^2 = r^2 = 1.034.$$

1.11. (a) $x = \mathrm{H}(t+1) - \mathrm{H}(t-1)$.

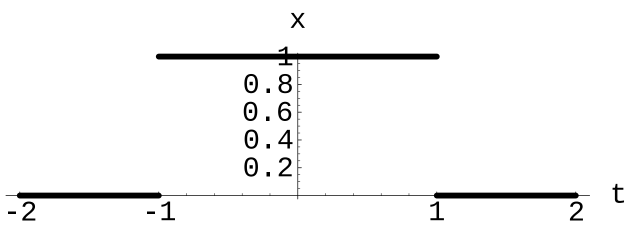

Figure 9: Problem 1.11a

(b) $x = \mathrm{sgn}(1+t) + \mathrm{sgn}(1-t)$.

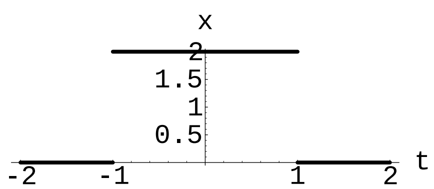

Figure 10: Problem 1.11b

(c) $x = t\mathrm{H}(t-1)$.
(d) $(t^2-1)[\mathrm{sgn}(t+1) + \mathrm{sgn}(1-t)]$.

1.12. (a) $f(t) = \mathrm{H}(2-t) + \mathrm{H}(t+1) - 1$; (b) $f(t) = 2t\mathrm{H}(t)$;
(d) $f(t) = (3-t)\mathrm{H}(3-t) + (t-2)\mathrm{H}(2-t) + (t-1)\mathrm{H}[1-t) - t\mathrm{H}(-t)$.

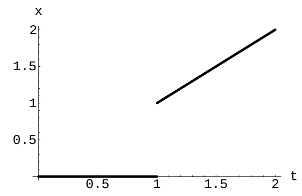

Figure 11: Problem 1.11c

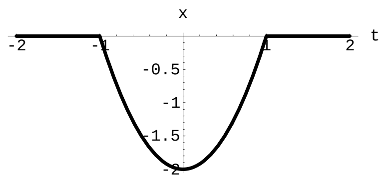

Figure 12: Problem 1.11d

1.13. (a) $\frac{1}{6}\pi$ radians; (b) $\frac{2}{3}\pi$ radians.

1.14. (a) $1/\sqrt{2}$; (b) 1; (c) 0; (d) $-1/\sqrt{2}$; (e) $\sqrt{3}/2$; (f) $-\sqrt{3}/2$; (g) $-\sqrt{3}/2$; (h) $-\sqrt{3}/2$.

1.15. (a) Using the identity $\cos^2 B = \frac{1}{2}(1+\cos 2B)$

$$\begin{aligned}\cos^4 A &= \frac{1}{4}(1+\cos 2A)^2 = \frac{1}{4}(1+2\cos 2A+\cos^2 2A)\\ &= \frac{1}{4}(1+2\cos 2A+\frac{1}{2}(1+\cos 4A)) = \frac{1}{8}(3+4\cos 2A+\cos 4A).\end{aligned}$$

(b) Use the identities $\sin^2 B = \frac{1}{2}(1-\cos 2B)$ and $\cos^2 2B = \frac{1}{2}(1+\cos 4B)$:

$$\sin^4 A = \frac{1}{4}(1-2\cos 2A+\cos^2 2A) = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A.$$

1.16. (a) $\cos(x+\frac{1}{2}\pi) = \cos x \cos\frac{1}{2}\pi - \sin x \sin\frac{1}{2}\pi = \sin x$; (b) $\cos x$; (c) $-\cos x$; (d) $-\cos x$ (for both); (e) $\sin x$ (for both).

1.17. (a)

$$\begin{aligned}\cos x+\cos y &= \cos[\frac{1}{2}(x+y)+\frac{1}{2}(x-y)]+\cos[\frac{1}{2}(x+y)-\frac{1}{2}(x-y)]\\ &= \cos[\frac{1}{2}(x+y)]\cos[\frac{1}{2}(x-y)]-\sin[\frac{1}{2}(x+y)]\sin[\frac{1}{2}(x-y)]+\\ &\quad \cos[\frac{1}{2}(x+y)]\cos[\frac{1}{2}(x-y)]+\sin[\frac{1}{2}(x+y)]\sin[\frac{1}{2}(x-y)]\\ &= 2\cos[\frac{1}{2}(x+y)]\cos[\frac{1}{2}(x-y)]\end{aligned}$$

(b) $\sin x - \sin y = 2\sin[\frac{1}{2}(x-y)]\cos[\frac{1}{2}(x+y)]$.
(c) $\cos x - \cos y = -2\sin[\frac{1}{2}(x+y)]\sin[\frac{1}{2}(x-y)]$.

1.18. (a) $x = n\pi$, $(n = 0, \pm 1, \pm, 2, \ldots)$;
(b) $x = \frac{1}{2}(2n+1)\pi$, $(n = 0, \pm 1, \pm, 2, \ldots)$;
(c) $x = 2n\pi$, $(n = 0, \pm 1, \pm, 2, \ldots)$;
(d) $x = \frac{1}{6}(2n+1)$, $(n = 0, \pm 1, \pm, 2, \ldots)$;
(e) $x = \frac{1}{2}n\pi$, $(n = 0, \pm 1, \pm, 2, \ldots)$;
(f) $x = 2n$, $(n = 0, \pm 1, \pm, 2, \ldots)$.

1.19.

	amplitude	angular frequency	period	phase
(a)	2	0.2	10π	3.2
(b)	1.5	0.2	10π	-0.48
(c)	3.87	0.2	10π	-0.135
(d)	1	1	2π	π

1.20. (a) $F(x) = \frac{1}{2}\sqrt{-x}$; (b) $F(x) = \frac{1}{2}(x-3)$; (c) $F(x) = \frac{1}{2}\arcsin x$;
(d) $F(x) = \arcsin(\frac{1}{2}x)$; (e) $F(x) = [\arccos x]^{\frac{1}{2}}$;
(f) $F(x) = \arccos\left[\frac{2}{\pi}\arcsin x\right]$;
(g) $F(x) = x^{-4}$; (h) $F(x) = -\frac{1}{2} + \sqrt{(x+\frac{1}{4})}$.

1.21. The graph shows $y = x^3 - x + 1$ (the dashed curve) and its inverse.

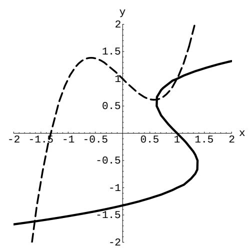

Figure 13: Problem 1.21

1.22. (a) $x = \frac{1}{2}\ln 3$; (b) $x = \frac{1}{3}\mathrm{e}^2$; (c) $x = \mathrm{e}^{-3}$; (d) $x = -\frac{1}{3}\ln 3$;
(e) the equation is the same as $(\mathrm{e}^x - 1)^2 = 0$: hence $x = 0$;
(f) $x = 2$; (g) $x = 2/17$; (h) $x = \pm\sqrt{2}$; (i) $x = \pm\sqrt{(1+\mathrm{e}^{\mathrm{e}})}$;
(j) $x = (\ln 3)/(\ln 2)$; (k) $x = -(\ln 2)/(2\ln 3)$; (l) $x = \frac{1}{2}\ln[4+\sqrt{17}]$;
(m) no solutions.

1.23. $2^x = \mathrm{e}^{x\ln 2}$.

1.24. Consider two values of x, say x_1 and x_2, where $x_1 > x_2$. Then if $10^{x_1} = 2 \times 10^{x_2}$, it follows that

$$10^{x_1 - x_2} = 2, \text{ or } x_1 - x_2 = \frac{\ln 2}{\ln 10},$$

an interval which is independent of x_1 and x_2.

1.25. (a) $(x-1)^2 + y^2 \leq 9$.

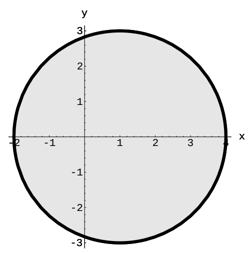

Figure 14: Problem 1.25a

(b) $x \geq 0$, $y \geq 0$, and $x + y \leq 1$.
(c) $(x^2/4) + (y^2/9) \leq 1$.
(d) $x^2 + y^2 \leq 1$ and $x \geq 0$.
(e) $|x| + |y| \leq 1$.

1.26. Let

$$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{\mathrm{e}^y - \mathrm{e}^{-y}}{\mathrm{e}^y + \mathrm{e}^{-y}},$$

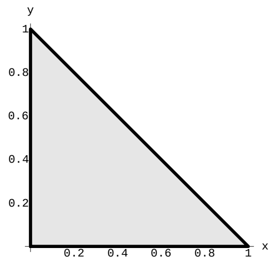

Figure 15: Problem 1.25b

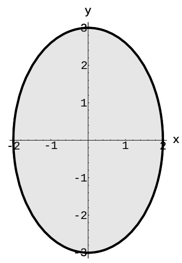

Figure 16: Problem 1.25c

where $-1 < x < 1$. Hence

$$(1-x)e^{2y} = 1 + x \text{ so that } y = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$$

as required.

1.27. From triangle ABC

$$\begin{aligned} AC &= AB\sin\theta + \sqrt{(BC^2 - AB^2\cos^2\theta)} \\ &= 2.5[\sin\omega t + \sqrt{(4-\cos^2\omega t)}] \text{ cm}, \end{aligned}$$

where $\theta = \omega t$. The angular frequency $\omega = 400\pi/3$.

1.28. $x = 5\cos(\omega t - 0.927)$. The amplitude $c = 5$ and the phase angle is $\phi = -0.927$.

1.29. $f(0) = 2$ implies $C = 2$ and $f(1) = 0.5$ implies $0.5 = Ce^{-\alpha} = 2e^{-\alpha}$. Hence $\alpha = \ln 4$. Also $f(2) = \frac{1}{8}$.

1.30. The tidal period is $2\pi/0.5 = 12.57$ hours. We require the times when the depth is 2m in one period, which are given by solutions of

$$2 = 5 + 4.5\sin 0.5t \text{ so that } \sin 0.5t = -\tfrac{2}{3}.$$

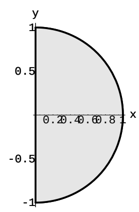

Figure 17: Problem 1.25d

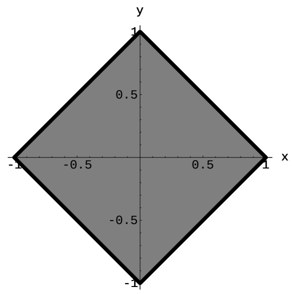

Figure 18: Problem 1.25e

Two consecutive times are 11.11 hours and 7.74 hours. Hence the yacht can float free for 9.20 hours in each tidal period. The yacht floats when $\sin 0.5t > -\frac{2}{3}$. It is helpful to sketch $y = \sin 0.5t$ and $y = -\frac{2}{3}$ and plot their intersections.

1.31. (a) The cardioid $r = 0.5(1 + \cos\theta)$:

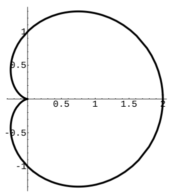

Figure 19: Problem 1.31a

(b) The folium $r = (4\sin^2\theta - 1)\cos\theta$

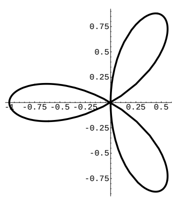

Figure 20: Problem 1.31b

(c) $r = \sin 2\theta$:

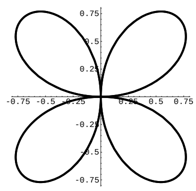

Figure 21: Problem 1.31c

(d) The Archimedean spiral $r = 0.04\theta$:
]
(e) The equiangular spiral $r = 0.1\mathrm{e}^{0.1\theta}$:

1.32. (a) $\operatorname{sgn}(\sin x)$:
(b) $\operatorname{sgn}\cos 2x$:

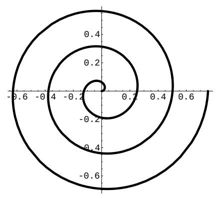

Figure 22: Problem 1.31d

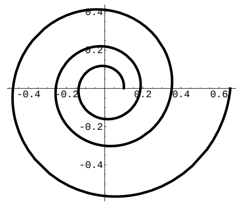

Figure 23: Problem 1.31e

(c) $\mathrm{H}(x)\sin x$:
(d) $\sin^2 x$:
(e) $|\sin x|$:
(f) $\sin|x|$:
(g) $\mathrm{H}(x-\pi)\sin x$:

1.33. Let the points be $A:(-7,3)$, $B:(1,-3)$ and $C:(4,1)$. The slope of AB is $-\frac{3}{4}$ and the slope of BC is $\frac{4}{3}$; the product of the slopes is -1 which means that $\widehat{ABC}$ is a right angle. Let D be the fourth vertex. Then the equations of the lines AD and DC are

$$y-3=\tfrac{4}{3}(x+7) \text{ and } y-1=-\tfrac{3}{4}(x-4),$$

or

$$3y-4x-37=0 \text{ and } 4y+3x-16=0.$$

These lines intersect at the point $D:(-4,7)$

There is a general formula buried here, if you notice that the coordinates of D are $(-7+4-1, 3+1-(-3))$.

1.34. (a) periodic, period $\frac{1}{2}\pi$; (b) periodic, period 2π; (c) periodic, period 2π; (d) not periodic; (e) periodic, period 2π; (f) periodic, period π; (g) not periodic; (h) periodic, period π; (i) periodic, period π; (j) periodic, period 8π; (k) periodic, period $\frac{2}{3}$, since $\sin 3t$ has period $\frac{2}{3}\pi$ and $\cos 9t$ has period $\frac{2}{9}\pi$ but has the period of $\sin 3t$; (l) not periodic.

1.35. (a) neither odd nor even; (b) even; (c) odd since $\sin x$ is odd; (d) odd since product of odd and even functions; (e) even; (f) even; (g) neither odd nor even.

1.36. (a) $\frac{1}{5(x-2)}-\frac{1}{5(x+3)}$; (b) $-\frac{1}{x+1}+\frac{2}{x+2}$; (c) $\frac{1}{x}+\frac{1}{x-1}$;

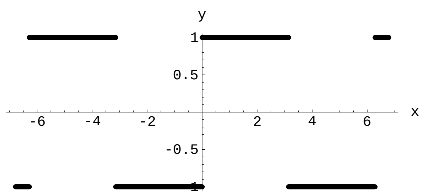

Figure 24: Problem 1.32a

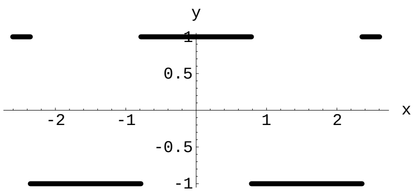

Figure 25: Problem 1.32b

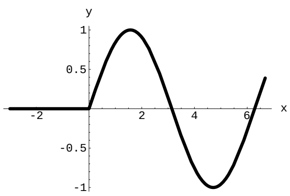

Figure 26: Problem 1.32c

(d) $\frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$; (e) $\frac{1}{2(x-1)} - \frac{1}{x} + \frac{1}{2(x+1)}$;
(f) $\frac{1}{4x} - \frac{1}{2(x+2)^2} - \frac{1}{4(x+2)}$; (g) $\frac{1}{x+1} = \frac{4}{(x+2)^2}$;
(h) $\frac{1}{2(x-3)} + \frac{1}{2(x+1)}$; (i) $\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}$;
(j) $\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}$.

1.37. (a) $\frac{1}{x} - \frac{x+1}{x^2+x+1}$; (b) $\frac{1}{2(x-1)} + \frac{1-x}{2(x^2+1)}$; (c) $-\frac{1}{5(x+1)} + \frac{x+6}{5(x^2+2x+6)}$.

1.38. (a) $\frac{1}{x^2} - \frac{1}{x^2+1}$; (b) $x - 3 - \frac{1}{x+1} + \frac{8}{x+2}$;
(c) $1 + \frac{9}{8(x-3)} + \frac{1}{8(x+1)} - \frac{9}{4(x+3)}$.

1.39. (a) $4 + 8 + 16 + 32$; (b) $1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17}$;
(c) $x + 2x^2 + 3x^3 + 4x^4$.

1.40. For (a), (b), (c), (e) and (f) proceed as in Example 1.17.
(a) $2[1 - (\frac{1}{2})^8] = \frac{255}{128}$.
(b) $\frac{1}{2} \cdot \frac{1}{3}[1 - (\frac{1}{3})^5] = \frac{121}{729}$.
(c) $(1 - \mathrm{e}^{-12})/(1 - \mathrm{e}^2)$.
(d) The sum is 642. More generally, let

$$x + 2x^2 + \cdots + nx^n = T.$$

Then

$$T - xT = (1 - x)T = xS - nx^{n+1},$$

where S is the sum of the geometric series

$$1 + x + \cdots + x^{n-1}.$$

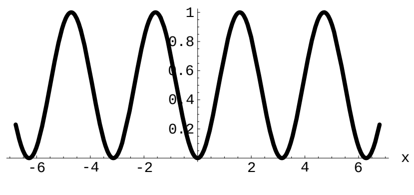

Figure 27: Problem 1.32d

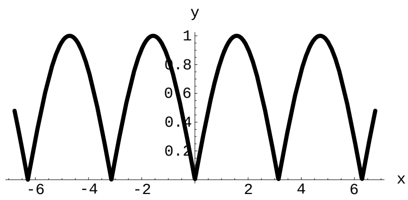

Figure 28: Problem 1.32e

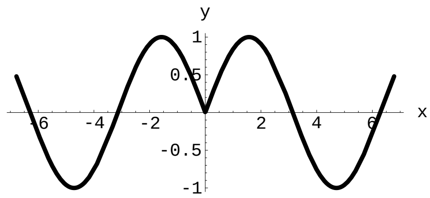

Figure 29: Problem 1.32f

For the given problem $x = 2$.
(e) $-\frac{1}{2}\frac{2}{3}[1 - (-\frac{1}{2})^{10} = -\frac{341}{1024}$.
(f) $2[\frac{1-(0.5)^7}{0.5}] + 3[(\frac{1-(0.6)^7}{0.4}] = 11.258\ldots$.

1.41. The series can be expressed as

$$x + x^5 + x^9 + \cdots + x^{41} = x\sum_{n=0}^{10}(x^4)^n.$$

Using (1.33), the sum of the series is

$$x\frac{1-x^{44}}{1-x^4}.$$

1.42. Let D be the foot of the perpendicular on to the side AC. Then

$$c^2 = DB^2 + DA^2 = DB^2 + (AC - DC)^2.$$

But $DB = a\sin C$ and $DC = a\cos C$. Therefore

$$\begin{aligned} c^2 &= a^2\sin^2 C + (b - a\cos C)^2 \\ &= a^2\sin^2 C + a^2\cos^2 C + b^2 - 2ab\cos C \\ &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

1.43. The ratio of any pair of successive terms is

$$\frac{f(t_0 + (n+1)T)}{f(t_0 + nT)} = \frac{Ae^{c(t_0+(n+1)T)}}{Ae^{c(t_0+nT)}} = e^{cT},$$

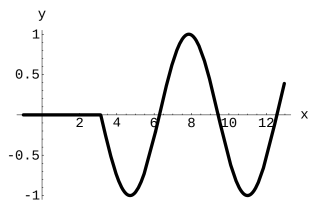

Figure 30: Problem 1.32g

which is independent of n. The common ratio is e^{cT}.

1.44. (a) $1.111\ldots = 1 + \frac{1}{10} + \frac{1}{100} + \cdots$ is an infinite geometric series with common ratio $\frac{1}{10}$ and sum $1/(1 - \frac{1}{10}) = 11/9$. (b) The common ratio is $1/10$, and as a fraction the sum is 1; (c) the common ratio is $1/100$, and as a fraction the sum is $1/99$; (d) the common ration is $1/100$, and as a fraction the sum is $1/11$; (e) the common ration is $1/10$ and as a fraction the sum is $2/3$; (f) the notation means $2.\dot{7}\dot{2} = 2.727272\ldots$: the common ratio is $1/100$ and the sum represents the fraction $30/11$.

1.45. The sum of the infinite geometric series is

$$\sum_{n=m}^{\infty} x^n = \frac{x^m}{1-x}, \qquad |x| < 1.$$

(a) 2; (b) $10/9$; (c) $e/(e-1)$; (d) $\frac{2}{3}$; (e) $3/5$.

1.46. (a) 24, 720, 5040; (b) 12; (c) 720; (d) 220; (e) 120; (f) 1, 3, 3, 1.

1.47. (a) (i) $n(n-1)$; (ii) $(n+1)n$; (b) (i) $2^m m!$; (ii) $(2m+1)!/(2^m m!)$.

1.48. (a) (i) 120; (ii) 504; (iii) 120; (iv) 35; (v) 35; (vi) 252; (vii) 4950; (viii) $\binom{10}{7} ={}_{10} C_7 = 120$.
(b) ${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ Also ${}_nP_{n-1} = \frac{n!}{(n-n+1)!} = \frac{n!}{1!} = n!$. Consider a collection of n different letters. The number of different words of length n which can be made without repetition is ${}_nP_n = n!$. Suppose that the letters are $A, B, C, \ldots$. Suppose that the first letter of an n character word is A. Then the remaining $n-1$ letters can be chosen in $(n-1)!$ ways. Repeat the procedure for words with first letters B, C and so on. We obtain all words with n characters again, and there are $n(n-1)! = n!$ of them.

1.49. (a) ${}_4P_1 = 4!$; (b) ${}_4C_3 = 4$; (c) $4^4 = 256$; (d) 20; (e) ${}_4P_4+{}_4P_3+{}_4P_2+{}_4P_1 = 4+12+24+24 = 64$.
(f) Without repetitions the number of combinations is

$${}_4C_1 +_4 C_2 +_4 C_3 +_4 C_4 = 4 + 6 + 4 + 1 = 15.$$

With 2 letters the same there are $4 + 12 + 12 = 28$ possibilities, and with 3 letters the same there are $4 + 12 = 16$. Hence the total number of combinations is $15 + 28 + 16 = 59$

1.50. (a) With no E's there are ${}_4P_3 = 24$ words, with 1 E there are $3 \times_4 P_2 = 36$, and with 2 E's there are $3 \times_4 P_1 = 12$. Hence there are $24 + 36 + 12 = 72$ words.
(b) Label six letters A, B, C, D, E_1, E_2. Then the number of words treating E_1 and E_2 as distinct is $6! = 720$. The letters E_1 and E_2 can be interchanged in $2! = 2$ ways. Hence the number of six-letter words is $720/2 = 360$.

1.51. (a) There are ${}_5P_4 = 5!/1! = 120$ distinct four-digit numbers.
(b) To be divisible by 5, the last digit must be 5. The preceding 3 digits can be chosen in ${}_4P_3 = 24$ ways. Hence there are 24 numbers divisible by 5.
(c) To be divisible by 2 the final digit must be 2 or 4. As in (b) the number of numbers is $2{}_4P_3 = 48$.
(d) The numbers contain either 1, 2, 3 or 4 digits. There are 4 one-digit numbers (excluding zero). For two-digit numbers we must exclude those starting with zero since they are the same as the one-digit numbers. Hence there are 16 distinct two-digit numbers. Similarly there are $4{}_4P_2 = 48$ three-digit numbers and $4{}_4P_3 = 96$ four-digit numbers. Hence the total number is $4 + 16 + 48 + 96 = 164$ words.

1.52. (a) Without restriction, the number of distinct combinations of personnel (no distinction being made as to which particular post is assigned to each person) is ${}_7C_4 = 7!/(4!3!) = 35$.
(b) There is one selection with 4 females, 12 with 3 females and one male, 18 with 2 females and 2 males and 4 with one female and three males. (b) The posts can be filled in the following ways: ${}_4C_4 = 1$ with 4 females; ${}_4C_3\,{}_3C_1 = 12$ with 3 females and one male; ${}_4C_2\,{}_3C_2 = 18$ with 2 females and 2 males; ${}_4C_1{}_3C_3 = 4$ with one female and 3 males. This confirms the 35 combinations of personnel.

1.53. (a) We may model the problem by thinking of an ordered line of N pool balls, of various colours (types) denoted by $A, B, \ldots$, the number of each colour being $N_A, N_B, \ldots$. The number of possible orders (permutations) for the individual balls is $N!$, but we cannot distinguish visually between balls having the same colour, so many of the $N!$ orders will look identical.

Suppose that the number of *distinguishable* arrangements is M. Each one of of these corresponds to a possible $N_A!N_B!\ldots$ permutations within the separate colours, so that

$$N! = M[N_A!N_B!\ldots], \text{ or } M = \frac{N!}{N_A!N_B!\ldots}.$$

(b) We require the total number of different combinations, involving every number $1, 2, \ldots, N$ of balls. Consider any one of these: it contains 0 or 1 or 2... or N_A (that is, $(1 + N_A)$ possibilities) of type A; 0 or 1 or 2... or N_B of type B; and so on. The number of possible combinations is therefore

$$(1 + N_A)(1 + N_B)\ldots - 1,$$

in which the term -1 is introduced to exclude the case of an all-zero 'combination'.

1.54. (a) The national groups may be ordered (permuted) in 4! ways. By allowing for 5! permutations possible within each group we obtain

$$5!5!5!5!4! = 5!^4 4! = 2880$$

distinct line-ups.

(b) The number of distinct orderings of the 4 groups around a circular table is $(4 - 1)! = 3!$ (see Example 1.23). All possible permutations within the groups are then to be allowed for, so the total number of arrangements is $5!3! = 720$.

1.55. (a) (Prizes identical) The number of combinations of 3 distinct prizewinners out of 10 eligibles is ${}_{10}C_3 = 120$.
(b) (Prizes different) Call the Prizes P_1, P_2, P_3. P_1 may go to any of 10 people; with each allocation P_2 may go to any of the remaining 9; then P_3 to any of the remaining 8; all of these distributions being distinct. The total number of possibilities is $10 \times 9 \times 8 = 720$.
(c) (Prizes equal, distribution arbitrary) There are 3 types of distribution which can occur:

(i) One person gets all the prizes: there are 10 possibilities.

(ii) There are 10 persons who might get 2 prizes. With each of these there are 9 persons eligible for the other prize. There are therefore $9 \times 10 = 90$ possibilities.

(iii) Three different people get prizes. Part (a) gives the number: there are 120 possibilities.

Therefore the total number of possibilities is

$$10 + 90 + 120 = 220$$

(d) P_1 may go to any of the 10; similarly with P_2 and P_3. Therefore the total number is $10\times10\times10 = 1000$.

1.56. (a) The table shows the permissible numbers in the 3 categories. The number of combinations possible within each category are given in brackets.

Accountants	Lawyers	Doctors	Committees
-	1(2)	3(1)	2
-	2(1)	2(${}_3C_2$)	3
1(2)	-	3(1)	2
1(2)	1(2)	2(${}_3C_2$)	12
1(2)	2(1)	1(3)	6
2(1)	-	2(${}_3C_2$)	3
2(1)	1(2)	1(3)	6
2(1)	2(1)	-	1
			Check: ${}_7C_4 = 35$

Committees with exactly 1 accountant: $2 + 12 + 6 = 20$.
Committees with exactly 1 doctor: $6 + 6 = 12$.
(b) To locate the fallacy consider combinations of the 7 letters A, B, C, D, E, F, G, and take $n = 4$ and $r = 3$. Take, say, the $r = 3$-fold combination ABC and supplement it by, say, the unused letter D, to form the combination $ABCD$. In the fallacious construction this will be counted several times; for example, the same combination is counted again when it arises from supplementing BCD by A.

The result is shown to be false by simply substituting the given numbers: only one contradiction is sufficient to dispose of it.

1.57. (a) Refer back to (1.44).
(b) $(1-x)^6 = 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
(c)

$$\begin{aligned}(x + x^{-1})^5 &= x^5 + 5x^4x^{-1} + 10x^3x^{-2} + \\ &\quad 10x^2x^{-3} + 5xx^{-4} + x^{-5} \\ &= x^5 + 5x^3 + 10x + 10x^{-1} + 5x^{-3} + x^{-5}\end{aligned}$$

$$\begin{aligned}(x - x^{-1})^5 &= x^5 + 5x^4(-x)^{-1} + 10x^3(-x)^{-2} + 10x^2(-)x^{-3} + \\ &\quad 5x(-x)^{-4} + (-x)^{-5} \\ &= x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} - x^{-5}\end{aligned}$$

1.58.

$$\begin{aligned}(1.01)^{10} &= (1 + 0.01)^{10} \\ &= 1 + 10 \times (0.01) + 45 \times (0.01)^2 + 120 \times (0.01)^3 + \cdots \\ &= 1 + 0.1 + 0.0045 + 0.00012 + \cdots = 1.105\end{aligned}$$

to three decimal places.

Similarly

$$\begin{aligned}(0.99)^8 &= (1 - 0.01)^8 \\ &= 1 - 8 \times (0.01) + 28 \times (0.01)^2 - 56 \times (0.01)^3 + \cdots \\ &= 1 - 0.08 + 0.0028 - 0.00056 + \cdots = 0.923\end{aligned}$$

to 3 decimal places.

1.59. Use the binomial theorem in the form

$$(1 + x)^n = 1 +_n C_1 x +_n C_2 + \cdots +_n C_n x^n.$$

(a) Put $x = 2$, so that

$$3^n = 1 + 2_n C_1 + 2^2 \ _n C_2 + \cdots + 2^n \ _n C_n x^n.$$

For the second result put $x = -1$:

$$0 = 1 -_n C_1 +_n C_2 - \cdots + (-1)^n \ _n C_n x^n.$$

(b) Obtain two series with $x = 1$ and $x = -1$. Then add and subtract the series.

1.60. $F(n, k)$ is defined for $n = 0, 1, 2, \ldots$, and $k = 0, 1, 2, \ldots, n$ by

$$F(n, k) = {}_nC_0 + {}_{n+1}C_1 + {}_{n+2}C_2 + \cdots + {}_{n+k}C_k. \qquad \text{(i)}$$

A certain formula, namely

$$F(n, k) = {}_{n+k+1}C_k \qquad \text{(ii)}$$

is suggested for the sum in (i), and its truth for small values of k can be confirmed by calculation; for example, from (i)

$$F(n,0) = {}_nC_0 = \frac{n!}{0!n!} = 1 \equiv {}_{n+1}C_0, \tag{iii}$$

$$F(n,1) = {}_nC_0 + {}_{n+1}C_1 = 1 + \frac{(n+1)!}{1!n!} = n+2 \equiv {}_{n+2}C_1;$$

and so on.

To prove the truth of (ii) for all values of $0 \leq k \leq n$, recast (i) into the form

$$F(n,k+1) \equiv F(n,k) + {}_{n+k+1}C_{k+1}, \tag{iv}$$

a 'recurrence formula' enabling us to advance one step at a time in k, starting, for example, with $F(n,0)$ and finding $F(n,1), F(n,2), \ldots,$ successively.

Now suppose we have verified the formula (ii) for any one particular value of k, say for $k = K$; that is, we know somehow that

$$F(n,K) = {}_{n+K+1}C_K = \frac{(n+K+1)!}{K!(n+1)!} \tag{v}$$

(for all n). Then from (iv)

$$\begin{aligned}
F(n,K+1) &= F(n,K) + {}_{n+K+1}C_{K+1} \\
&= {}_{n+K+1}C_K + {}_{n+K+1}C_{K+1} \text{ from (v)} \\
&= \frac{(n+K+1)!}{K!(n+1)!} + \frac{(n+K+1)!}{(K+1)!n!} \\
&= \frac{(n+K+1)!}{K!n!}\left(\frac{1}{n+1} + \frac{1}{K+1}\right) \\
&= \frac{(n+K+2)!}{(K+1)!(n+1)!} = {}_{n+K+2}C_{K+1}
\end{aligned}$$

We have proved that if (iv) is true for $k = K$, it is true for $k = K+1$, where K may take any value in $0 \leq K < n$.

But we verified in (iii) that (iv) holds good when $k = K = 0$. Therefore by (vi) it is true when $k = K+1 = 1$, so by (vi) again it is true when $k = K+2 = 2$, and so on. It is therefore true for all k.

1.61. Using partial fractions

$$\frac{1}{x^2+3x+2} = \frac{1}{(1+x)(2+x)} = \frac{1}{1+x} - \frac{1}{2+x}.$$

Write as

$$\frac{1}{x^2+3x+2} = (1+x)^{-1} - \frac{1}{2}(1+\frac{1}{2}x)^{-1},$$

and expand both terms using (1.37) for infinite geometric series. Hence

$$\begin{aligned}
\frac{1}{x^2+3x+2} &= (1-x+x^2-x^3+\cdots) - \frac{1}{2}(1-\frac{x}{2}+\frac{x^2}{2^2}-\frac{x^3}{2^3}+\cdots) \\
&= \frac{1}{2} - (1-\frac{1}{2.2})x + (1-\frac{1}{2.2^2})x^2 - (1-\frac{1}{2.2^3})x^3 + \cdots.
\end{aligned}$$

1.62. $V_1 = A(1+R)$, $V_2 = V_1(1+R) = A(1+R)^2$, etc. In pounds: for 1000 @ 3% p.a.;

$$V_5 = 1000(1+0.03)^5 = 1159.27, \quad V_{10} = 1343.92; \quad V_{15} = 1557.97.$$

(b) Let the period start at an arbitrary time T_0. Then

$$\frac{V_{T_0+T}}{V_{T_0}} = \frac{(1+R)^{t_0+T}}{(1+R)^{T_0}} = (1+R)^T.$$

(c) Let T_2 be the doubling period, so that from (b)

$$(1+R)^{T_2} = 2 \text{ and } T_2 = \frac{\ln 2}{\ln(1+R)}.$$

If $R = 3\%$, $T_2 = 23.4$ yr; if $R = 6\%$, $T_2 = 11.9$ yr; if $R = 9\%$, $T_2 = 8.0$ yr.

For the ten-times period, $T_{10} = \ln 10/\ln(1+R)$. If $R = 6\%$, then $T_{10} + \ln 10/\ln 1.06 = 39.6$ yr.

1.63. If the income is withdrawn annually, it has been allowed to accrue to the fund through the previous year with interest at the going rate R annually, or r monthly, the relation being

$$A(1+R) = A(1+r)^{12}$$

where A is the value of the fund at the start of that year. By the binomial theorem,

$$(1+r)^{12} = 1 + 12r + \cdots\cdots > 1 + 12r,$$

so $R > 12r$.

1.64. If the interest is payable monthly at the rate of r_M per month, the interest on a fixed debt D over any 12-month period is $D(1+r_M)^{12}$. This is equal to $D(1+R)$ where R is the annual equivalent rate (AER). Therefore $R = (1+r_M)^{12} - 1$. If $r_M = 1\%$, $R = 1.01^{12} - 1 = 0.126\,(12.6\%)$. If $r_M = 3\%$, $R = 0.425\,(42.5\%)$.

1.65. (a) After N complete years the initial payment A has drawn interest for N yrs, the second payment for $N-1$ yrs, and so on, and the $(N-1)$th payment for 1 yr. The value V_N of the fund is then given by the geometric series

$$\begin{aligned} & A(1+R)^N + A(1+R)^{N-1} + \cdots + A(1+R) \\ = \;& A(1+R)\{1 + (1+R) + \cdots + (1+R)^{N-1}\} \\ = \;& A(1+R)\{(1+R)^N - 1\}/R. \end{aligned}$$

(b) $N = 10$, $R = 5\%$. We obtain

$$V_{10} = \frac{100(1.05)(1.05^{10} - 1)}{0.05} = £1320.68,$$

equivalent to a gain of 32% on the total investment of £1000.

(c) M investments of $2A$, at 2-year intervals. Formula (a), with the fund value increasing by a factor $(1+R)^2$ in each interval, becomes

$$V_{2M} = \frac{(2A)(1+R)^2)\{[(1+R)^2]^M - 1\}}{\{(1+R)^2 - 1\}}.$$

Using the data in (b) we obtain

$$V_{10} = \frac{200(1.05)^2(1.05^{10} - 1)}{1.05^2 - 1} = £1352.88.$$

Chapter 2: Differentiation

2.1. Below are some sample values for three values of x on either side of the point where the tangent is required. (The exact value of the slope is also given here.)

(a) $y = x^3$ at $(1, 1)$.

x	0.94	0.96	0.98	1.02	1.04	1.06
chord slope	2.82	2.88	2.94	3.06	3.12	3.18

The slope is 3.

(b) $y = \sqrt{x}$ at $(1, 1)$.

x	0.85	0.90	0.95	1.025	1.10	1.15
chord slope	0.520	0.513	0.506	0.494	0.488	0.483

The slope is 0.5

(c) $y = \cos x$ at $(\frac{1}{4}\pi, 1/\sqrt{2})$.

$x - \frac{1}{4}\pi$	-0.09	-0.06	-0.03	$+0.03$	0.06	0.09
chord slope	0.674	0.685	0.696	0.718	0.728	0.738

The slope is $1/\sqrt{2} = 0.707$

(d) $y = e^x$ at $(0, 1)$.

x	-0.15	-0.10	-0.05	0.05	0.10	0.15
chord slope	0.929	0.952	0.975	1.025	1.052	1.079

The slope is 1.

(e) $y = e^{2x}$ at $(0, 1)$.

x	-0.15	-0.10	-0.05	0.05	0.10	0.15
chord slope	1.728	1.813	1.903	2.103	2.214	2.332

The slope is 2.

(f) $y = x^3 + x^{\frac{1}{2}}$ at $(1, 2)$.

x	0.94	0.96	0.98	1.02	1.04	1.06
chord slope	3.33	3.38	3.44	3.56	3.62	3.68

The slope is 3.5, the sum of the slopes in (a) and (b).

(g) $y = \ln x$ at $(1, 0)$.

x	0.94	0.96	0.98	1.02	1.04	1.06
chord slope	1.031	1.0206	1.010	0.990	0.981	0.971

The slope is 1

2.2. (a) For $y = 3x$ at $(2, 6)$,

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[\frac{3(2 + \delta x) - 6}{\delta x} \right] = 3.$$

(b) For $y = 3 - 2x$ at $(1, 1)$,

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[\frac{[3 - 2(1 + \delta x) - (3 - 2)]}{\delta x} \right] = -2.$$

(c) For $y = 3x^2$ at $(1, 3)$,

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \to 0} \left[\frac{3(1 + \delta x)^2 - 3(1)^2}{\delta x} \right] \\ &= \lim_{\delta x \to 0} [6 + 3\delta x] = 6. \end{aligned}$$

(d) For $y = x^3$ at $(1, 1)$,

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \to 0} \left[\frac{(1 + \delta x)^3 - 1^3}{\delta x} \right] \\ &= \lim_{\delta x \to 0} \left[\frac{3\delta x + 3(\delta x)^2 + (\delta x)^3}{\delta x} \right] \\ &= \lim_{\delta x \to 0} [3 + 3\delta x + (\delta x)^2] = 3 \end{aligned}$$

(e) For $y = 1/x$ at $(2, \frac{1}{2})$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x} \left[\frac{1}{2 + \delta x} - \frac{1}{2} \right] \\ &= \lim_{\delta x \to 0} \left[\frac{-1}{2(2 + \delta x)} \right] = -\frac{1}{4}. \end{aligned}$$

(f) For $y = 3x + 2x^2$ at $(1, 5)$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim \delta x \to 0 \left[\frac{3(1 + \delta x) + 2(1 + \delta x)^2 - 3 - 2}{\delta x} \right] \\ &= \lim_{\delta x \to 0} [3 + 4 + 2\delta x] = 7. \end{aligned}$$

(g) For $y = (1 + 2x)^2 = 1 + 4x + 4x^2$ at $(-1, 1)$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x} [1 + 4(-1 + \delta x) + 4(-1 + \delta x)^2 - 1 + 4 - 4] \\ &= \lim_{\delta x \to 0} [4 + 4(-2 + (\delta x)^2] = -4. \end{aligned}$$

2.3. (a) $y = 3x^2$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x} [3(x + \delta x)^2 - 3x^2] \\ &= \lim_{\delta x \to 0} [6x + 3\delta x] \\ &= 6x. \end{aligned}$$

(b) $y = x^3$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x} [(x + \delta x)^3 - x^3] \\ &= \lim_{\delta x \to 0} [3x^2 + 3x\delta x + (\delta x)^2] \\ &= 3x^2. \end{aligned}$$

(c) $y = 1/x$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x} \left[\frac{1}{x + \delta x} - \frac{1}{x} \right] \\ &= \lim_{\delta x \to 0} \left[\frac{-1}{x(x + \delta x)} \right] \\ &= -1/x^2. \end{aligned}$$

(d) $y = x + \frac{1}{2}$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x} [(x + \delta x + \frac{1}{2}) - (x + \frac{1}{2})] \\ &= 1. \end{aligned}$$

(e) $y = x + (1/x)$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x} \left[\left(x + \delta x + \frac{1}{x + \delta x} \right) - \left(x + \frac{1}{x} \right) \right] \\ &= \lim_{\delta x \to 0} \left[1 - \frac{1}{x^2 + x\delta x} \right] \\ &= 1 - \frac{1}{x^2}. \end{aligned}$$

(f) $y = 2x^2 - 3$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{1}{\delta x}[(2(x+\delta x)^2 - 3) - (2x^2 - 3)] \\ &= \lim_{\delta x \to 0}[4x + 2\delta x] \\ &= 4x.\end{aligned}$$

2.4. Let $x = f(t)$ be the displacement function in each case. The average velocity over the interval t to $t + \delta t$ equals

$$[f(t+\delta t) - f(t)]/\delta t.$$

(a) $x = f(t) = 3t$. When $t = 1$

Interval δt	0.5	0.25	0.1	0.01
$f(1+\delta t)$	4.5	3.75	3.3	3.03
Average velocity	3	3	3	3

(Since $f(t)$ is linear in t the velocity 3 at all t.)
(b) $x = f(t) = 5t^2$. When $t = 3$.

Interval δt	0.5	0.25	0.1	0.01
$f(3+\delta t)$	61.25	52.81	48.05	45.35
Average velocity	32.5	31.25	30.5	30.05

The values are approaching the limit 30.
(c) $x = f(t) = 2t - 5t^2$. When $t = 1$.

Interval δt	0.5	0.25	0.1	0.01
$f(1+\delta t)$	−8.25	3.75	−3.85	−3.08
Average velocity	−10.5	−9.25	−8.5	−8.05

The limit is -8.
(d) $x = 2t - 5t^2$. When $t = 0.2$.

Interval δt	0.5	0.25	0.1	0.01
$f(0.2+\delta t)$	−1.25	−0.3125	−0.05	−0.0005
Average velocity	−25	−1.25	−0.5	−0.05

In the limit the velocity is zero.

2.5. (a) $\mathrm{d}y/\mathrm{d}x = 1$ for all x; (b) $\mathrm{d}y/\mathrm{d}x = 3x^2$ so that $\mathrm{d}y/\mathrm{d}x = 27$ at $x = 3$; (c) $\mathrm{d}y/\mathrm{d}x = 4x^3$ so that $\mathrm{d}y/\mathrm{d}x = 32$ at $x = 2$ and -32 at $x = -2$.

2.6. (a) $y = x$; $\mathrm{d}y/\mathrm{d}x = 1$: the graph is a straight line at 45° to the x axis.

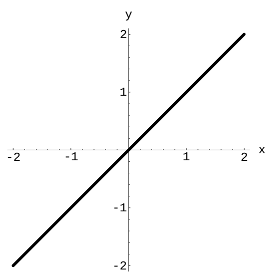

Figure 31: Problem 2.6a

(b) $y = x^2$; $\mathrm{d}y/\mathrm{d}x = 2x$: the slope is negative for $x < 0$ and positive for $x > 0$, and increases from $-\infty$ to $+\infty$: the curve is a parabola.

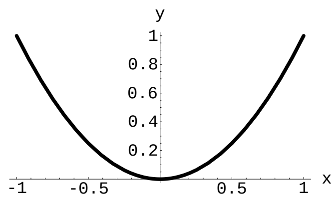

Figure 32: Problem 2.6b

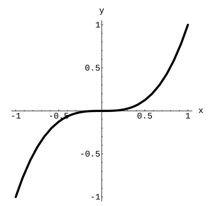

Figure 33: Problem 2.6c

(c) $y = x^3$; $\mathrm{d}y/\mathrm{d}x = 3x^2$: the slope is positive except at $x = 0$ where it is zero.
(d) $y = x^4$; $\mathrm{d}y/\mathrm{d}x = 4x^3$.

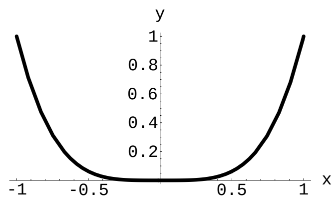

Figure 34: Problem 2.6d

(e) $y = x^5$; $\mathrm{d}y/\mathrm{d}x = 5x^4$.

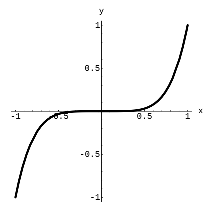

Figure 35: Problem 2.6e

2.7. For the displacement $x = t^3$, the velocity of the point is $\mathrm{d}x/\mathrm{d}t = 3t^2$ and its acceleration is $\mathrm{d}^2x/\mathrm{d}t^2 = 6t$. The graph of acceleration against time is a straight line.

2.8. (a) If $V = \frac{4}{3}\pi r^3$ then $\mathrm{d}V/\mathrm{d}r = 4\pi r^2$.
(b) If $S = \pi d^2$ then $\mathrm{d}S/\mathrm{d}d = 2\pi d$.
(c) If $E = kT^4$ then $\mathrm{d}E/\mathrm{d}t = 4kT^3$.
(d) If $I = V/R$ then $\mathrm{d}I/\mathrm{d}V = 1/R$.
(e) If $H = RI^2$ then $\mathrm{d}H/\mathrm{d}I = 2RI$.
(f) If $V = RT/P$ then $\mathrm{d}V/\mathrm{d}T = R/P$.

2.9.

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 2x + 1) = 3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) - 2\frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(1) = 6x - 2.$$

(b) $$\frac{d}{dx}(x^7 - 3x^6 + x + 1) = 7x^6 - 18x^5 + 1.$$

(c) $$\frac{d}{dx}(x + C) = 1.$$

(d) $$\frac{d}{dx}[x(x-1)] = \frac{d}{dx}(x^2 - x) = 2x - 1.$$

(e) $$\frac{d}{dx}[x^2(x^2+1) - 1] = \frac{d}{dx}[x^4 + x^2 + 1] = 4x^3 + 2x.$$

(f) $$\frac{d}{dx}(ax^2 + bx + c) = 2ax + b.$$

(g). $$\frac{d}{dx}[(x-1)^2] = \frac{d}{dx}(x^2 - 2x + 1) = 2x - 2.$$

2.10. Let m_1 and m_2 be the slopes of the curves at the point of intersection, and check that $m_1 m_2 = -1$. Then
(a) $m_1 = (d/dx)(1 + x - x^2) = 1 - 2x = -1$ at $x = 1$,
$m_2 = (d/dx)(1 - x + x^2) = -1 + 2x = 1$ at $x = 1$. Hence $m_1 m_2 = -1$ as required.
(b) $m_1 = -x = -1$, $m_2 = 1$ at $x = 1$.
(c) $m_1 = -x = -1$, $m_2 = x = 1$ at $x = 1$.

2.11. (a) The curves $y = x^2$ and $y = 1 - x^2$ intersect where $x^2 = 1 - x^2$ or where $x^2 = \frac{1}{2}$. Hence the points of intersection occur at $A : (\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $B; (-\frac{1}{\sqrt{2}}, \frac{1}{2})$.

The slopes of the curves at A are

$$m_1 = 2x = 2/\sqrt{2} = \sqrt{2} \text{ and } m_2 = -2x = -2/\sqrt{2} = -\sqrt{2}.$$

Let

$$\tan\alpha_1 = \sqrt{2} \quad (0 < \alpha_1 < \frac{1}{2}\pi) \text{ and } \tan\alpha_2 = -\sqrt{2} \quad (-\frac{1}{2}\pi < \alpha_1 < 0).$$

Using the identity from (1.17a):

$$\begin{aligned}\tan(\alpha_1 - \alpha_2) &= \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{\sqrt{2} + \sqrt{2}}{1 - \sqrt{2}\sqrt{2}} \\ &= -2\sqrt{2}\end{aligned}$$

We choose a positive value for the angle (a sketch of the intersection of the curves is helpful). Hence $\alpha_1 - \alpha_2 = \arctan(-2\sqrt{2}) = 109.47°$.

The slopes of the curves at B are

$$n_1 = -2x = -2/\sqrt{2} = -\sqrt{2} \text{ and } n_2 = 2x = 2/\sqrt{2} = \sqrt{2}.$$

The two slopes at B are interchanged but otherwise the same. Hence the angle between the tangents will also be $109.47°$. (Note that in both these cases you might obtain the alternative angles $(180 - 109.47)°$.)
(b) The curves $y = \frac{1}{3}x^3$ and $y = x^2 - 2x + \frac{4}{3}$ intersect where

$$x^3 = 3x^2 - 6x + 4 \text{ or where } (x-1)(x^2 - 2x + 4) = 0.$$

The only real root is $x = 1$. Hence the point of intersection is at $(1, \frac{1}{3})$. The slopes of the curves at this point are $m_1 = 1$ and $m_2 = 0$. Let $\tan\alpha_1 = 1$ and $\tan\alpha_2 = 0$. Then we can choose $\alpha_1 = \frac{1}{4}\pi$ and $\alpha_2 = 0$. The required angle is $\frac{1}{4}\pi$.

2.12. Use the limits in Section 2.6. Note that, in all the following, ε is never zero, so cancellation is legitimate.

(a)
$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon}=\lim_{\varepsilon\to 0}1=1.$$

(b)
$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{2\varepsilon}=\lim_{\varepsilon\to 0}\frac{1}{2}=\frac{1}{2}.$$

(c)
$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{\varepsilon}=\lim_{\varepsilon\to 0}\varepsilon=0.$$

(d)
$$\lim_{\varepsilon\to 0}\frac{\mathrm{e}^{2\varepsilon}-1}{2\varepsilon}=\lim_{\mu\to 0}\frac{\mathrm{e}^{\mu}-1}{\mu}=1,\quad(\mu=2\varepsilon)\text{ (from (2.11))}.$$

(e)
$$\lim_{\varepsilon\to 0}\frac{\mathrm{e}^{2\varepsilon}-1}{\varepsilon}=\lim_{\mu\to 0}2\frac{\mathrm{e}^{\mu}-1}{\mu}=2,\quad(\mu=2\varepsilon).$$

(f)
$$\lim_{\varepsilon\to 0}\frac{\sin 2\varepsilon}{2\varepsilon}=\lim_{\mu\to 0}\frac{\sin\mu}{\mu}=1,\quad(\mu=2\varepsilon)\text{ (from (2.13)}.$$

(g)
$$\lim_{\varepsilon\to 0}\frac{\sin 2\varepsilon}{\varepsilon}=\lim_{\mu\to 0}2\frac{\sin\mu}{\mu}=2,\quad(\mu=2\varepsilon).$$

(h)
$$\lim_{\varepsilon\to 0}\frac{\ln(1+\varepsilon^2)}{\varepsilon^2}=\lim_{\mu\to 0}2\frac{\ln(1+\mu)}{\mu}=1,\quad(\mu=\varepsilon^2)\text{ (from (2.14))}.$$

(i) Note that (2.13) is only true if ε is measured in radians. Therefore replace ε degrees by $180\varepsilon/\pi$ radians. Hence

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\sin\varepsilon}{\varepsilon}&=\lim_{\varepsilon\to 0}\frac{\sin(\pi\varepsilon/180)}{\varepsilon}=\lim_{\mu\to 0}\frac{\pi\sin\mu}{180\mu}\quad(\mu=\pi\varepsilon/180)\\&=\pi/180\text{ (from (2.13))}.\end{aligned}$$

(j)

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\tan\varepsilon}{\varepsilon}&=\lim_{\varepsilon\to 0}\frac{\sin\varepsilon}{\varepsilon}\frac{1}{\cos\varepsilon}=\lim_{\varepsilon\to 0}\frac{\sin\varepsilon}{\varepsilon}\lim_{\varepsilon\to 0}\frac{1}{\cos\varepsilon}\\&=1\times 1=1\end{aligned}$$

(k)

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\sinh\varepsilon}{\varepsilon}&=\lim_{\varepsilon\to 0}\frac{\mathrm{e}^{\varepsilon}-\mathrm{e}^{-\varepsilon}}{2\varepsilon}=\lim_{\varepsilon\to 0}\frac{\mathrm{e}^{2\epsilon-1}}{2\varepsilon}\lim_{\varepsilon\to 0}\mathrm{e}^{-\varepsilon}\\&=1\times 1=1\end{aligned}$$

(l)
$$\lim_{\varepsilon\to 0}\frac{\mathrm{e}^{-\varepsilon-1}}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\mathrm{e}^{\varepsilon}-1}{\varepsilon}\lim_{\varepsilon\to 0}[-\mathrm{e}^{-\varepsilon}]=-1.$$

2.13.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)&=\lim_{\delta x\to 0}\left[\frac{\cos(x+\delta x)-\cos x}{\delta x}\right]\\&=\lim_{\delta x\to 0}\left[\frac{-2\sin\frac{1}{2}(2x+\delta x)\sin\frac{1}{2}(\delta x)}{\delta x}\right]\\&=\lim_{\delta x\to 0}\left[-\sin(x+\frac{1}{2}\delta x)\frac{\sin(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x}\right]\\&=-\sin x\end{aligned}$$

2.14.

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) = \lim_{\varepsilon\to 0}\left[\frac{\mathrm{e}^{2(x+\varepsilon)} - \mathrm{e}^{2x}}{\varepsilon}\right] = 2\mathrm{e}^{2x}\lim_{\varepsilon\to 0}\left[\frac{\mathrm{e}^{2\varepsilon} - 1}{2\varepsilon}\right] = 2\mathrm{e}^{2x},$$

(from (2.11)).
(b)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) &= \lim_{\varepsilon\to 0}\left[\frac{\sin[2(x+\varepsilon)] - \sin 2x}{\varepsilon}\right] \\ &= \lim_{\varepsilon\to 0}\left[\frac{2\sin\varepsilon}{\varepsilon}\cos\frac{1}{2}(4x+2\varepsilon)\right] \\ &= 2\cos 2x\end{aligned}$$

(c). $$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}) = \lim_{\varepsilon\to 0}\left[\frac{\mathrm{e}^{-(x+\varepsilon)} - \mathrm{e}^{-x}}{\varepsilon}\right] = -\mathrm{e}^{-x}\lim_{\varepsilon\to 0}\left[\frac{\mathrm{e}^{-\varepsilon} - 1}{-\varepsilon}\right] = -\mathrm{e}^{-x}.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} - \mathrm{e}^{-x}) = \frac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x}) = \cosh x.$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} + \mathrm{e}^{-x}) = \frac{1}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}) = \sinh x.$$

2.15.

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}(2\sin x - 3\cos x) = 2\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) - 3\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 2\cos x + 3\sin x.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}(\ln 3x) = \frac{\mathrm{d}}{\mathrm{d}x}(\ln 3 + \ln x) = \frac{1}{x}.$$

(c) $$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x^{3}) = \frac{\mathrm{d}}{\mathrm{d}x}(3\ln x) = \frac{3}{x}.$$

(d) $$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x - x) = \cos x - 1.$$

(e) $$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} - 1 - x - \frac{1}{2}x^{2}) = \mathrm{e}^{x} - 1 - x.$$

2.16. The required tangent lines are
(a) $y = 3x - 2$; (b) $y = 24x - 39$; (c) $y = -x + \frac{1}{2}\pi$; (d) $y = x/\mathrm{e}$; (e) $y = 1$; (f) $y = -x + 3$.

2.17.

(a) $y = x^{6}, \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^{5}, \quad \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 30x^{4}, \quad \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = 120x^{3}.$

(b) $y = 3x^{2} - 2x + 2, \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x - 2, \quad \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6, \quad \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0.$

(c) $y = x^{6} - x^{2}, \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^{5} - 2x, \quad \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 30x^{4} - 2, \quad \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = 120x^{3}.$

(d) $y = 2\sin x - 3\cos x, \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x + 3\sin x,$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\sin x + 3\cos x, \quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -2\cos x - 3\sin x.$$

(e) $y = \mathrm{e}^x - 1 - x - \dfrac{1}{2}x^2, \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - 1 - x, \quad \dfrac{\mathrm{d}^2}{dx^2} = \mathrm{e}^x - 1, \quad \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = \mathrm{e}^x.$

2.18. To prove that $(\mathrm{d}x^N/\mathrm{d}x^N)(x^N) = N!$ for all integers $N \geq 1!$. We can confirm the formula for the case $N = 1$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1 = 1! \tag{i}$$

as a starting-point in a step-by-step argument.

Suppose that we have somehow established that the result is true for any one particular value of N, say for $N = K$, so that

$$\frac{\mathrm{d}^N(x^N)}{\mathrm{d}x^N} = K! \text{ when } N = K. \tag{ii}$$

Next, consider the transition to $N = K + 1$:

$$\begin{aligned}
\frac{\mathrm{d}^{K+1}}{\mathrm{d}x^{K+1}}(x^{K+1}) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^K}{\mathrm{d}x^K}(x^{K+1})\right] = \frac{\mathrm{d}^K}{\mathrm{d}x^K}\left[\frac{\mathrm{d}}{\mathrm{d}x}(x^{K+1})\right] \\
&= \frac{\mathrm{d}^K}{\mathrm{d}x^K}\{(K+1)x^K\} \text{ (by (2.9))} \\
&= (K+1)K! \text{ (by (ii))} \\
&= (K+1)!
\end{aligned}$$

In other words, if (ii) is true for some integer $N = K$, it follows that it is also true for $N = K + 1$. Since we *now* know it is true for $N = K + 1$, the same argument implies that it is true for $N = (K + 1) + 1 = K + 2$; and so on for all subsequent values of N.

But we have verified its truth in the case $N = 1$ (in equation (i)); therefore (ii) is true for all values of N. This is proof by induction.

2.19. If $y = x^2(x^2 - 1)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 6x, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 6.$$

(a) The slope of the curve is positive where $\mathrm{d}y/\mathrm{d}x > 0$ or where $x(2x + \sqrt{3})(2x - \sqrt{3}) > 0$. This occurs where $-\frac{1}{2}\sqrt{3} < x < 0$ and $\frac{1}{2}\sqrt{3} < x$.
(b) The slope of the curve is negative where $x(2x + \sqrt{3})(2x - \sqrt{3}) < 0$, that is where $x < -\frac{1}{2}\sqrt{3}$ and $0 < x < \frac{1}{2}\sqrt{3}$.
(c) The second derivative is positive where $12(x + \frac{1}{2}\sqrt{2})(x - \frac{1}{2}\sqrt{2}) > 0$, that is where $x > \frac{1}{2}\sqrt{2}$ and $x < -\frac{1}{2}\sqrt{2}$.
(d) The second derivative is negative where $-\frac{1}{2}\sqrt{2} < x < \frac{1}{2}\sqrt{2}$.

2.20. At $x = x_0$ the tangent has slope $m_0 = 2x_0$. Hence the slope of the normal is $-1/m_0 = -1/(2ax_0)$. The equation of the normal is therefore

$$y - ax_0^2 = -\frac{1}{2ax_0}(x - x_0).$$

Chapter 3: Further techniques for differentiation

3.1.

(a)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^x) = x\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) + \frac{\mathrm{d}}{\mathrm{d}x}(x)\mathrm{e}^x = x\mathrm{e}^x + \mathrm{e}^x.$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x) = x\cos x + \sin x.$$

(c)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x\cos x) = -x\sin x + \cos x$$

(d)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x\sin x) = \mathrm{e}^x\cos x + \mathrm{e}^x\sin x.$$

(e)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x) = 1 + \ln x.$$

(f)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\ln x) = x^2x^{-1} + 2x\ln x = x + 2x\ln x.$$

(g)
$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x\ln x) = \frac{\mathrm{e}^x}{x} + \mathrm{e}^x\ln x.$$

(h)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^x) = x^2\mathrm{e}^x + 2x\mathrm{e}^x.$$

(i)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x\cos x) = -\sin^2 x + \cos^2 x = \cos 2x.$$

(j)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2x^3) = x^2.3x^2 + 2x.x^3 = 5x^4 = \frac{\mathrm{d}}{\mathrm{d}x}(x^5).$$

3.2. All these problems illustrate the reciprocal and quotient rules given in (3.2).
(a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} = -\mathrm{cosec}\,^2 x\end{aligned}$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x+1}\right) = \frac{1}{(x+1)^2}\left((x+1)\frac{\mathrm{d}}{\mathrm{d}x}(x) - x\frac{\mathrm{d}}{\mathrm{d}x}(x+1)\right) = \frac{1}{(x+1)^2}.$$

(c)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) = \frac{1}{x^2}(x\cos x - \sin x).$$

(d)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^x}{x}\right) = \frac{\mathrm{e}^x}{x^2}(x-1).$$

(e)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right) = \frac{1}{(x^2+1)^2}[(x^2+1)(2x) - (x^2-1)(2x)] = \frac{4x}{(x^2+1)^2}.$$

(f) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tan x}{x^2}\right) = \frac{1}{x^4}(x^2\sec^2 x - 2x\tan x) = \frac{1}{x^3}(x\sec^2 x - 2\tan x).$$

(g)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)\\ &= \frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}\\ &= -\frac{2}{(\sin x-\cos x)^2}.\end{aligned}$$

(h) $$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos^2 x} = \sec x\tan x.$$

(i) $$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}\, x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} = -\cot x\,\mathrm{cosec}\, x.$$

(j) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{3x^2-2}\right) = \frac{-2-3x^2}{(-2+3x^2)^2}.$$

(k) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x(x^3+1)}\right) = \frac{-1-4x^3}{x^2(x^3+1)}.$$

(l) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\ln x}\right) = -\frac{1}{x(\ln x)^2}.$$

(m) $$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^{-n}}\right) = nx^{n-1} \text{ (by the quotient rule)}.$$

(n) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x+1}\right) = -\frac{1}{(x+1)^2}.$$

(o) $$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\mathrm{e}^x}\right) = -\frac{1}{\mathrm{e}^{2x}}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = -\mathrm{e}^{-x}.$$

(p) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\tan x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\mathrm{cosec}^{\,2} x \text{ (as in (a))}.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}\ln x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{x^2}\right) = \frac{1}{x^4}\left(\frac{x^2}{x} - 2x\ln x\right) = \frac{1-2\ln x}{x^3}.$$

3.3.

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2};\quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{1}{(1-x)}\right] = \frac{2}{(1-x)^2},$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\frac{1}{(1-x)}\right] = \frac{6}{(1-x)^4}.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x) = x\cos x + \sin x; \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}(x\sin x) = 2\cos x - x\sin x;$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x\sin x) = -x\cos x - 3\sin x.$$

(c) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x-1}\right) = -\frac{1}{(x-1)^2}; \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{x}{x-1}\right) = \frac{2}{(x-1)^3};$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{x}{x-1}\right) = -\frac{6}{(x-1)^4}.$$

(d) Let $y = f(x)g(x)$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g\frac{\mathrm{d}f}{\mathrm{d}x} + f\frac{\mathrm{d}g}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}^2f}{\mathrm{d}x^2} + f\frac{\mathrm{d}^2g}{\mathrm{d}x^2},$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 3\frac{\mathrm{d}g}{\mathrm{d}x}\frac{\mathrm{d}^2f}{\mathrm{d}x} + 3\frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}^2g}{\mathrm{d}x^2} + g\frac{\mathrm{d}^3f}{\mathrm{d}x^3} + f\frac{\mathrm{d}^3g}{\mathrm{d}x^3}.$$

3.4. These problems use the chain rule (3.3) in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

(a) Let $u = \sin x$. Then $y = u^2$, and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = 2u\cos x = 2\sin x\cos x = \sin 2x.$$

(b) Let $u = \cos x$, $y = u^2$. Then $(\mathrm{d}/\mathrm{d}x)(\cos^2 x) = -2\sin x\cos x = -\sin 2x$.
(c) Let $u = x^2$, $y = \sin u$. Then $(\mathrm{d}/\mathrm{d}x)(\sin x^2) = 2x\cos x^2$.
(d) Let $u = x^2$, $y = \cos u$. Then $(\mathrm{d}/\mathrm{d}x)(\cos x^2) = -2x\sin x^2$.
(e) Let $u = \tan x$, $y = u^2$. Then $\mathrm{d}/\mathrm{d}x)(\tan^2 x) = 2\sec^2 x\tan x$.
(f) Let $u = x^2$, $y = \tan u$. Then $(\mathrm{d}/\mathrm{d}x)(\tan x^2) = 2x\sec^2 x^2$.
(g) Let $u = 1/x$, $y = \cos u$. Then $\mathrm{d}/\mathrm{d}x)[\cos(1/x)] = 2\sin(1/x)/x^2$.
(h) Let $u = -x$, $y = \mathrm{e}^u$. Then $(\mathrm{d}/\mathrm{d}x)(\mathrm{e}^{-x}) = -\mathrm{e}^{-x}$.
(i) Let $u = 1/(x+1)$, $y = u^5$. Then $(\mathrm{d}/\mathrm{d}x)(1/(x+1)^5) = -5/(x+1)^6$.
(j) Let $u = x^3+1$, $y = u^4$. Then $(\mathrm{d}/\mathrm{d}x)[(x^3+1)^4] = 12x^2(x^3+1)^3$.
(k) Let $u = 3x$, $y = \sin u$. Then $(\mathrm{d}/\mathrm{d}x)(\sin 3x) = 3\cos 3x$.
(l) Let $u = \frac{1}{2}x$, $\cos u$. Then $(\mathrm{d}/\mathrm{d}x)(\cos\frac{1}{2}x) = -\frac{1}{2}\sin\frac{1}{2}x$.
(m) Let $u = \frac{1}{2}x$, $y = \tan u$. Then $(\mathrm{d}/\mathrm{d}x)(\tan\frac{1}{2}x) = \frac{1}{2}\sec^2 x$.
(n) Let $u = -3x$, $y = \mathrm{e}^u$. Then $(\mathrm{d}/\mathrm{d}x)(\mathrm{e}^{-3x}) = -3\mathrm{e}^{-3x}$.
(o) Let $u = 2x+1$, $y = \sin u$. Then $(\mathrm{d}/\mathrm{d}x)[\sin(2x+1)] = 2\cos(2x+1)$.
(p) Let $u = 3x-2$, $y = \cos u$. Then $(\mathrm{d}/\mathrm{d}x)[\cos(3x-2)]) = -3\sin(3x-2)$.
(q) Let $u = 1-2x$, $y = \tan u$. Then $(\mathrm{d}/\mathrm{d}x)[\tan(1-2x)] = -2\sec^2(1-2x)$.
(r) Let $u = 1/x$, $y = \mathrm{e}^u$. Then $(\mathrm{d}/\mathrm{d}x)(\mathrm{e}^{1/x}) = -\mathrm{e}^{1/x}/x^2$.

3.5. All these problems use the result that $(\mathrm{d}/\mathrm{d}x)x^\alpha = \alpha x^{\alpha-1}$.

((a)) $$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}) = -2x^{-3}.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-1}) = -\frac{1}{x^2}.$$

(c)
$$\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}}.$$

(d)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-\frac{1}{3}}) = -\frac{1}{3}x^{-\frac{4}{3}}.$$

(e)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}.$$

(f)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x}) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}.$$

(g)
$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^3} = \frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}.$$

(h)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{-1}) = -\frac{1}{x^2}.$$

(i)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sqrt{x}}\right] = \frac{\mathrm{d}}{\mathrm{d}x}(x^{-\frac{1}{2}}) = \frac{-1}{2x^{\frac{3}{2}}}.$$

3.6. (a)
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}}\sin x) = x^{\frac{1}{2}}\cos x + \frac{\sin x}{2x^{\frac{1}{2}}}.$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{\frac{1}{3}} x) = \frac{\cos x}{3\sin^{\frac{2}{3}} x}.$$

(c)
$$\frac{\mathrm{d}}{\mathrm{d}x}[(x^2+1)^{-\frac{1}{2}}] = -\frac{x}{(x^2+1)^{\frac{3}{2}}}.$$

(d)
$$\frac{\mathrm{d}}{\mathrm{d}t}[\sin^2(3t+1)] = 6\cos(3t+1)\sin(3t+1).$$

(e)
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t}\cos t) = -\mathrm{e}^{-t}(\cos t + \sin t).$$

(f)
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t}\sin t) = \mathrm{e}^{-t}(\cos t - \sin t).$$

(g)
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-2t}\cos 3t) = -\mathrm{e}^{-2t}(2\cos 3t + 3\sin 3t).$$

(h)
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-3t}\cos 2t) = -\mathrm{e}^{-3t}(3\cos 2t + 2\sin 2t).$$

(i)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x\cos^2 x) = \cos^3 x - 2\cos x\sin^2 x.$$

(j)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^2 x\cos x) = 2\cos^2 x\sin x - \sin^3 x.$$

(k)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\sin x}{x}\right)^2\right] = \frac{2\cos x\sin x}{x^2} - \frac{2\sin^2 x}{x^3}.$$

(l)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x(\sin^3 x)\right] = 3x\cos x\sin^2 x + \sin^3 x.$$

(m)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x(\cos^3 x)\right] = -3x\cos^2 x\sin x + \cos^3 x.$$

3.7. (a)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^2 x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(1+\cos 2x) = -\sin 2x.$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^2 x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(1-\cos 2x) = \sin 2x.$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^2 x) = \frac{\mathrm{d}}{\mathrm{d}x}(\cos x\cos x) = -\cos x\sin x - \sin x\cos x = -\sin 2x.$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^2 x) = \frac{\mathrm{d}}{\mathrm{d}x}(\sin x\sin x) = \sin x\cos x + \cos x\sin x = -\sin 2x.$$

(c) To apply the chain rule let $u = \cos x$, $y = u^2$. Then
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^2 x) = \frac{\mathrm{d}}{\mathrm{d}u}(u^2)\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -2u\sin x = -2\cos x\sin x = -\sin 2x.$$

Let $u = \sin x$ Then
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^2 x) = \frac{\mathrm{d}}{\mathrm{d}u}(u^2)\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 2u\cos x = 2\sin x\cos x = \sin 2x.$$

3.8.

(a)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = (-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 0$$

(b)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + n^2x = (-n^2A\cos nt - n^2B\sin nt) + n^2(A\cos nt + B\sin nt) = 0.$$

(c)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 9x = (9A\mathrm{e}^{nt} + 9B\mathrm{e}^{-nt}) - 9(A\mathrm{e}^{nt} + B\mathrm{e}^{-nt}) = 0.$$

(d)
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - n^2x = (n^2A\mathrm{e}^{nt} + n^2B\mathrm{e}^{-nt}) - n^2(A\mathrm{e}^{nt} + B\mathrm{e}^{-nt}) = 0.$$

(e)
$$\frac{\mathrm{d}x}{\mathrm{d}t} = (-A+B)\mathrm{e}^{-t}\cos t - (A+B)\mathrm{e}^{-t}\sin t.$$
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -2B\mathrm{e}^{-t}\cos t + 2A\mathrm{e}^{-t}\sin t.$$

Hence

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}+2x=$$
$$[-2B\mathrm{e}^{-t}\cos t+2A\mathrm{e}^{-t}\sin t]+$$
$$2[(-A+B)\mathrm{e}^{-t}\cos t-(A+B)\mathrm{e}^{-t}\sin t]+2[A\mathrm{e}^{-t}\cos t+B\mathrm{e}^{-t}\sin t]=0.$$

(f) The fourth derivative of each term in y returns the same function in each case. Hence

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}-y=0.$$

3.9. Use the chain rule in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

The intermediate variables are defined in each problem.
(a) Let $u=\cos x$, $v=u^2$, so that $y=\mathrm{e}^v$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^v.2u.(-\sin x)=-2\mathrm{e}^{\cos^2 x}\cos x\sin x.$$

(b) Let $u=x^2$, $v=\cos u$, so that $y=\mathrm{e}^{-v}$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(-\mathrm{e}^{-v}).(-\sin u).2x=2x\sin(x^2)\mathrm{e}^{-\cos x^2}.$$

(c) Let $u=x^2$, $v=\cos u$, so that $y=\ln v$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{v}.(-\sin u).2x=-2x\tan(x^2).$$

(d) Let $u=x^2$, $v=\mathrm{e}^u-1$, so that $y=v^4$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4v^3.\mathrm{e}^u.2x=8x\mathrm{e}^{x^2}(\mathrm{e}^{x^2}-1)^3.$$

3.10. Use the result (3.7) which states that if $y=u(x)v(x)w(x)$ then $\ln y=\ln u+\ln v+\ln w$ which when differentiated gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}=vw\frac{\mathrm{d}u}{\mathrm{d}x}+wu\frac{\mathrm{d}v}{\mathrm{d}x}+uv\frac{\mathrm{d}w}{\mathrm{d}x}.$$

(a) Let $u=x$, $v=\mathrm{e}^x$, $w=\sin x$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x.\sin x.1+\sin x.x.\mathrm{e}^x+x.\mathrm{e}^x.\cos x=\mathrm{e}^x[\sin x+x\sin x+x\cos x].$$

(b) Different variables are used. Let $x=t\mathrm{e}^t\cos t$, and let $u=t$, $v=\mathrm{e}^t$, $w=\cos t$. Hence

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^t.\cos t.1+\cos t.t.\mathrm{e}^t+t.\mathrm{e}^t.(-\sin t)=\mathrm{e}^t[\cos t+t\cos t-t\sin t].$$

(c) Let $u=x^{\frac{1}{2}}$, $v=\mathrm{e}^{2x}$, $w=\sin^{\frac{1}{2}}3x$. Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\mathrm{e}^{2x}.\sin^{\frac{1}{2}}3x.(\frac{1}{2}x^{-\frac{1}{2}})+\sin^{\frac{1}{2}}3x.x^{\frac{1}{2}}.2\mathrm{e}^{2x}+x^{\frac{1}{2}}.\mathrm{e}^{2x}.\frac{3}{2}\cos 3x\sin^{-\frac{3}{2}}3x\\&=\frac{\mathrm{e}^{2x}}{2x^{\frac{1}{2}}\sin^{\frac{1}{2}}x}[3x\cos 3x+(4x+1)\sin 3x].\end{aligned}$$

This function and its derivative will only be real for restricted values of x—for example, for $0 \leq x \leq \frac{1}{3}\pi$.

3.11. (a) Treating y as a function of x, and using the chain rule for $y(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2)=0, \text{ or } 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0.$$

Hence $\mathrm{d}y/\mathrm{d}x = -x/y$ as required. Solving the equation $x^2+y^2=4$ for y, it follows that $y = \pm(4-x^2)^{\frac{1}{2}}$. Differentiating with respect to x, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mp x(4-x^2)^{-\frac{1}{2}} = -\frac{x}{y},$$

after substitution back in terms of y. This agrees with the answer obtained by the implicit method. Note that will always be two points on a circle which have the same slope. The tangent is always perpendicular to the radius to the point which has slope $m = y/x$.

(b) In this case for $x > 0$, $y \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}}+y^{\frac{1}{2}})=0, \text{ or } \frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=0.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{\frac{y}{x}}.$$

(c) In this case

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3+xy-y^3)=0 \text{ or } 3x^2+y+x\frac{\mathrm{d}y}{\mathrm{d}x}-3y^2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y+3x^2}{x-3y^2}.$$

(d) In this case

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin y - y\sin x)=0 \text{ or } \sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{\mathrm{d}y}{\mathrm{d}x}\sin x - y\cos x = 0.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos x-\sin y}{x\cos y-\sin x}.$$

3.12. The expression for $\mathrm{d}y/\mathrm{d}x$ obtained from the implicit relation $f(x,y) = c$ does not depend on c. For example for $x^2+y^2=c$, we always have $\mathrm{d}y/\mathrm{d}x = -x/y$. However, the *value* of $\mathrm{d}y/\mathrm{d}x$ will depend indirectly on c since x and y must always satisfy $x^2+y^2=c$.

3.13. If $xy^2 - x^2y = 1$, then

$$y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}-2xy-x^2\frac{\mathrm{d}y}{\mathrm{d}x}=0. \qquad \text{(i)}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy-y^2}{2xy-x^2}. \qquad \text{(ii)}$$

Differentiate (i) again with respect to x:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}+2y\frac{\mathrm{d}y}{\mathrm{d}x}+2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2xy\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2y-2x\frac{\mathrm{d}y}{\mathrm{d}x}-2x\frac{\mathrm{d}y}{\mathrm{d}x}-x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0. \qquad \text{(iii)}$$

Eliminate $\mathrm{d}y/\mathrm{d}x$ between (ii) and (iii): the answer is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{6xy(-x^3+2x^2y-2xy^2+y^3)}{(2xy-x^2)^3}.$$

3.14. (a) Let $y = \arcsin x$. Then $x = \sin y$. Differentiating with respect to y,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y = \sqrt{(1 - \sin^2 y)} = \sqrt{(1 - x^2)}.$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\sqrt{(1 - x^2)}}.$$

(b) Let $y = \arccos x$ so that $x = \cos y$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{(1 - x^2)}}.$$

(c) Let $y = \arctan x$ so that $x = \tan y$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{1 + x^2}.$$

(d) Let $y = \sinh^{-1} x$ so that $x = \sinh y$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(1 + \sinh^2 y)}} = \frac{1}{\sqrt{(1 + x^2)}}.$$

(e) Let $y = \cosh^{-1} x$ so that $x = \cosh y$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\sinh y} = \frac{1}{\sqrt{(\cosh^2 - 1)}} = \frac{1}{\sqrt{(x^2 - 1)}}.$$

(f) Let $y = \tanh^{-1} x$ so that $x = \tanh y$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}.$$

3.15. (a) If $r = \sin \frac{1}{2}\theta$, then the (x, y) coordinates are

$$x = r\cos\theta = \sin\frac{1}{2}\theta\cos\theta, \qquad y = r\sin\theta = \sin\frac{1}{2}\theta\sin\theta.$$

Using parametric differentiation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\bigg/\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\frac{1}{2}\cos\frac{1}{2}\theta\sin\theta + \sin\frac{1}{2}\theta\cos\theta}{\frac{1}{2}\cos\frac{1}{2}\theta\cos\theta - \sin\frac{1}{2}\theta\sin\theta}.$$

At $\theta = \frac{1}{2}\pi$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{2\sqrt{2}} + 0}{0 - \frac{1}{\sqrt{2}}} = -\frac{1}{2}.$$

(b) If $r = 1 + \sin^2\theta$, then

$$x = r\cos\theta = (1 + \sin^2\theta)\cos\theta, \qquad y = r\sin\theta = (1 + \sin^2\theta)\sin\theta.$$

Hence

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta - \sin^3\theta + 2\cos^2\theta\sin\theta = -\frac{1}{2\sqrt{2}} \text{ at } \theta = \tfrac{1}{4}\pi,$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta + 3\sin^2\theta\cos\theta = \frac{1}{\sqrt{2}} + \frac{3}{2\sqrt{2}} = \frac{5}{2\sqrt{2}} \text{ at } \theta = \frac{1}{4}\pi.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = -5$$

.

3.16. (a) For $x = t^3$ and $y = t^2$,

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2t}{3t^2} = \frac{2}{3t} = \frac{2}{3x^{\frac{1}{3}}}.$$

(b) For $x = 2\cos t$ and $y = 2\sin t$,

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2\cos t}{-2\sin t} = \pm\frac{x}{2\sqrt{(4-x^2)}},$$

assuming $0 \le t \le \frac{1}{2}\pi$.

3.17. Elimination of t between x and y, using the identity $\cos^2 t + \sin^2 t = 1$, gives the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The derivative is given by

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t.$$

The speed of the point is

$$\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} = \sqrt{[a^2\sin^2 t + b^2\cos^2 t]}.$$

3.18. In exponential form $a^x = e^{x\ln a}$. Hence

$$\frac{d}{dx}(a^x) = (\ln a)e^{x\ln a} = x^a \ln a.$$

Chapter 4: Applications of differentiation

4.1. $f(u) = u^2$, so $f'(u) = 2u$ and $f''(u) = 2$ for all arguments u.
(a) Let $u = t$. Then $f'(t) = 2t$.
(b) Let $u = t^2$. Then $f'(t^2) = 2u = 2t^2$.
(c) $(d/dt)f(t^2) = (d/dt)t^4 = 4t^3$.
(d) Let $u = t^{\frac{1}{2}}$. Then $f'(t^{\frac{1}{2}}) = 2u = 2t^{\frac{1}{2}}$.
(e) $(d/dt)f(t^{\frac{1}{2}}) = (d/dt)t = 1$.
(f) Let $u = t^{\frac{1}{2}}$. Then $f''(t^{\frac{1}{2}}) = 2$.

4.2. Denote the function in each case by $f(x)$. The stationary points are given by $f'(x) = 0$. If $A : x = a$ is a stationary point, then A is a minimum if $f''(a) > 0$, or a maximum if $f''(a) < 0$. If $f''(a) = 0$, then the stationary point can be a minimum, maximum or point of inflection depending on the sign of $f'(x)$ on either side of $x = a$.
(a) Since $f(x) = x^2 - x$, then

$$f'(x) = 2x - 1, \qquad f''(x) = 2.$$

Stationary point: $x = \frac{1}{2}$.

Test: $f''(\frac{1}{2}) = 2 > 0$ so $x = \frac{1}{2}$ is a minimum.
(b) Since $f(x) = x^2 - 2x - 3$, then

$$f'(x) = 2x - 2, \qquad f''(x) = 2.$$

Stationary point: $x = 1$.
Test: $f''(1) = 2 > 0$ so $x = 1$ is a minimum.
(c) Since $f(x) = x \ln x$, then

$$f'(x) = 1 + \ln x, \qquad f''(x) = \frac{1}{x}.$$

Stationary point: $x = \mathrm{e}^{-1}$.
Test: $f''(\mathrm{e}^{-1}) = \mathrm{e} > 0$ so $x = \mathrm{e}^{-1}$ is a minimum.
(d) Since $f(x) = x\mathrm{e}^{-x}$, then

$$f'(x) = (1 - x)\mathrm{e}^{-x}, \qquad f''(x) = (-2 + x)\mathrm{e}^{-x}.$$

Stationary point: $x = 1$.
Test: $f''(1) = -\mathrm{e}^{-1} < 0$ so $x = 1$ is a maximum.
(e) Since $f(x) = 1/(x^2 + 1)$, then

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}, \qquad f''(x) = \frac{2(3x^2 - 1)}{(x^2 + 1)^3}.$$

Stationary point: $x = 0$.
Test: $f''(0) = -2 < 0$ so $x = 0$ is a maximum.
(f) Since $f(x) = x^2 - 3x + 2$, then

$$f'(x) = 2x - 3, \qquad f''(x) = 2.$$

Stationary point: $x = \frac{3}{2}$.
Test: $f''(\frac{3}{2}) = 2 > 0$ so $x = \frac{3}{2}$ is a minimum.
(g) Since $f(x) = \mathrm{e}^x + \mathrm{e}^{-x}$, then

$$f'(x) = \mathrm{e}^x - \mathrm{e}^{-x}, \qquad f''(x) = \mathrm{e}^x + \mathrm{e}^{-x}.$$

Stationary point: $x = 0$.
Test: $f''(0) = 2 > 0$ so $x = 0$ is a minimum.
(h) Since $f(x) = x^2 + 4x + 2$, then

$$f'(x) = 2x + 4, \qquad f''(x) = 2.$$

Stationary point: $x = -2$.
Test: $f'' - 2) = 2 > 0$ so $x = -2$ is a minimum.
(i) Since $f(x) = x - x^3$, then

$$f'(x) = 1 - 3x^2, \qquad f''(x) = -6x.$$

Stationary points: $x = \pm 1/\sqrt{3}$.
Tests: $f''(1/\sqrt{3}) = -6/\sqrt{3} < 0$, so $x = 1/\sqrt{3}$ is a maximum.
$f''(-1/\sqrt{3}) = 6/\sqrt{3} > 0$ so $x = -1/\sqrt{3}$ is a minimum.
(j) Since $f(x) = x^2(x - 1)$, then

$$f'(x) = 3x^2 - 2x, \qquad f''(x) = 6x - 2.$$

Stationary points: $x = 0$ and $x = \frac{2}{3}$.

Tests: $f''(0) = -2 < 0$ so $x = 0$ is a maximum.
$f''(\frac{2}{3}) = 2 > 0$ so $x = \frac{2}{3}$ is a minimum.
(k) Since $f(x) = \sin x - \cos x$, then

$$f'(x) = \cos x + \sin x, \qquad f''(x) = -\sin x + \cos x.$$

Stationary points: $x = \frac{3}{4}\pi$ and $\frac{7}{4}\pi$ for $0 < x < 2\pi$.
Tests: $f''(\frac{3}{4}\pi) = -\surd 2 < 0$ so $x = \frac{3}{4}\pi$ is a maximum;
$f''(\frac{7}{4}\pi) = \surd 2 > 0$ so $x = \frac{7}{4}\pi$ is a minimum.
(l) Since $f(x) = \sin x \cos x = \frac{1}{2}\sin 2x$, then

$$f'(x) = \cos 2x, \qquad f''(x) = -2\sin 2x.$$

Stationary points: $x = -\frac{3}{4}\pi$, $x = -\frac{1}{4}\pi$, $x = \frac{1}{4}\pi$, $x = \frac{3}{4}\pi$ for $-\pi < x < \pi$.
Tests: $f''(-\frac{3}{4}\pi) = -2 < 0$ so $x = -\frac{3}{4}\pi$ is a maximum;
$f''(-\frac{1}{4}\pi) = 2 > 0$ so $x = -\frac{1}{4}\pi$ is a minimum;
$f''(\frac{1}{4}\pi) = -2 < 0$ so $x = \frac{1}{4}\pi$ is a maximum;
$f''(\frac{3}{4}\pi) = 2 > 0$ so $x = \frac{3}{4}\pi$ is a minimum.
(m) Since $f(x) = \mathrm{e}^{-x}\sin x$, then

$$f'(x) = \mathrm{e}^{-x}(-\sin x + \cos x), \qquad f''(x) = -2\mathrm{e}^{-x}\cos x.$$

Stationary points occur where $\tan x = 1$, at $x = (n + \frac{1}{4})\pi$, $(n = 0, \pm 1 \pm 2, \ldots)$.
Tests: $f''[(n + \frac{1}{4})\pi] = -\surd 2(-1)^n \mathrm{e}^{-(n+\frac{1}{4})\pi} < 0$ or > 0 according as n is even or odd. Hence the stationary point is a maximum $x = (n + \frac{1}{4})\pi$ is a maximum if n is even, and a minimum if n is odd.
(n) Since $f(x) = \mathrm{e}^{-\frac{1}{3}x}\sin 2x$, then

$$f'(x) = \tfrac{1}{3}\mathrm{e}^{-\frac{1}{3}x}(-\sin 2x + 6\cos 2x), \qquad f''(x) = \tfrac{1}{9}\mathrm{e}^{-\frac{1}{3}x}(-12\cos 2x - 35\sin 2x).$$

Stationary points occur where $\tan 2x = 6$ at $x = \alpha + \frac{1}{2}n\pi$, $(n = 0, \pm 1 \pm 2, \ldots)$, where $\alpha = \frac{1}{2}\arctan 6$.
Tests: $f''(\alpha + \frac{1}{2}n\pi) = -222\mathrm{e}^{-\frac{1}{3}(\alpha+\frac{1}{2}n\pi)}(-1)^n/[9\surd 37] < 0$ or > 0 according as n is even or odd. Hence the stationary point $x = \alpha + \frac{1}{2}n\pi$ is a maximum if n is even, and a minimum if n is odd.
(o) Since $f(x) = x - \cos x$, then

$$f'(x) = 1 + \sin x, \qquad f''(x) = \cos x.$$

Stationary points occur where $\sin x = -1$ at $x = (2n - \frac{1}{2})\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$.
Tests: $f''[(2n - \frac{1}{2})\pi] = 0$ for all n. Hence the test fails. But $f'(x) = 1 + \sin x \geq 0$ for all x. Hence all the stationary points must be points of inflection.
(p) Since $f(x) = 2\mathrm{e}^x - \frac{1}{2}\mathrm{e}^{2x}$, then

$$f'(x) = 2\mathrm{e}^x - \mathrm{e}^{2x}, \qquad f''(x) = 2\mathrm{e}^x - 2\mathrm{e}^{2x}.$$

Staionary point occurs where $\mathrm{e}^x = 2$ at $x = \ln 2$.
Test: $f''(\ln 2) = -4 < 0$ so $x = \ln 2$ is a maximum.
(q) Since $f(x) = x^2\mathrm{e}^{-x}$, then

$$f'(x) = x\mathrm{e}^{-x}(2 - x), \qquad f''(x) = \mathrm{e}^{-x}(2 - 4x + x^2).$$

Stationary points at $x = 0$ and $x = 2$.
Tests: $f''(0) = 2 > 0$ so that $x = 0$ is a minimum; $f''(2) = -2\mathrm{e}^{-2} < 0$ so that $x = 2$ is a maximum.
(r) Since $f(x) = (\ln x)/x$, then

$$f'(x) = \frac{1}{x^2}(1 - \ln x), \qquad f''(x) = \frac{1}{x^3}(-3 + 2\ln x).$$

Stationary point where $\ln x = 1$ at $x = \mathrm{e}$.
Test: $f''(\mathrm{e}) = -\mathrm{e}^{-3} < 0$ so $x = \mathrm{e}$ is a maximum.
(s) Since $f(x) = (1-x)^3$, then

$$f'(x) = -3(1-x)^3, \qquad f''(x) = 6(1-x)^2.$$

Stationary point: $x = 1$.
Test: $f''(1) = 0$ and the test fails. However, $f'(x) \le 0$ for all x. Hence $x = 1$ is a point of inflection.
(t) Since $f(x) = \sin^3 x$, then

$$f'(x) = 3\sin^2 x \cos x, \qquad f''(x) = 3\sin x(2 - 3\sin^2 x).$$

Stationary points occur where $\sin x = 0$, at $x = n\pi$, and where $\cos x = 0$, at $x = (n + \frac{1}{2})\pi$ for $n = 0, \pm 1, \pm 2, \ldots$.
Tests: $f''(n\pi) = 0$ so that the test fails. However, if n is even, then $f'(x)$ is positive in a small interval including $x = n\pi$, and if n is odd, then $f'(x)$ is negative in a small interval including $x = n\pi$. In both cases, therefore, the stationary point is a point of inflection.

$f''[(n + \frac{1}{2})\pi] = -3(-1)^n$ so that $x = (n + \frac{1}{2})\pi$ is a maximum if n is even, and a minimum if n is odd.
(u) Since $f(x) = \mathrm{e}^{-x^2}$, then

$$f'(x) = -2x\mathrm{e}^{-x^2}, \qquad f''(x) = 2\mathrm{e}^{-x^2}(2x^2 - 1).$$

Stationary point: $x = 0$.
Test: $f''(0) = -2 < 0$ so $x = 0$ is a maximum.
(v) Since $f(x) = \mathrm{e}^{x^2 - x}$, then

$$f'(x) = (2x-1)\mathrm{e}^{x^2-x}, \qquad f''(x) = (4x^2 - 4x + 3)\mathrm{e}^{x^2-x}.$$

Stationary point: $x = \frac{1}{2}$.
Test: $f''(\frac{1}{2}) = 2\mathrm{e}^{-\frac{1}{4}} > 0$ so $x = \frac{1}{2}$ is a minimum.
(w) Since $f(x) = x + x^{-1}$, then

$$f'(x) = 1 - \frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}.$$

Stationary points: $x = \pm 1$.
Tests: $f''(1) = 2 > 0$ so $x = 1$ is a minimum; and $f''(-1) = -2 < 0$ so $x = -1$ is a maximum.
(x) Since $f(x) = x^3\mathrm{e}^{-x}$, then

$$f'(x) = x^2\mathrm{e}^{-x}(3-x), \qquad f''(x) = x\mathrm{e}^{-x}(6 - 6x + x^2).$$

Stationary points: $x = 0$ and $x = 3$.
Tests: $f''(0) = 0$ and the test fails. In a small interval which includes the origin $f'(x) > 0$ which means that $x = 0$ is a point of inflection.
$f''(3) = -9\mathrm{e}^{-3} < 0$ so $x = 3$ is a maximum.

4.3. If $y = f[u(x)]$, then by the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = f'(u)u'(x),$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}[f'(u)u'(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f'(u)]u'(x) + f'(u)u''(x) \text{ (product rule)} \\ &= f''(u)[u'(x)]^2 + f'(u)u''(x)\end{aligned}$$

Since $f'(u) > 0$ for all u, then $\mathrm{d}y/\mathrm{d}x$ can only be zero if $u'(x) = 0$ (by the chain rule (3.3)). Hence $f[u(x)]$ and $u(x)$ have stationary points only at the same values of x.

In 4.2(v), $f(u) = \mathrm{e}^u$ and $u = x^2 - x$.

4.4. If the sides have lengths $x > 0$ and $y > 0$, then the given area $A = xy$. The length of the perimeter is $P = 2x + 2y$. Eliminate y so that

$$P = 2x + \frac{2A}{x}.$$

The first and second derivatives of P are

$$\frac{\mathrm{d}P}{\mathrm{d}x} = 2 - \frac{2A}{x^2}, \qquad \frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{4A}{x^3}.$$

Hence the perimeter length is stationary where $\mathrm{d}P/\mathrm{d}x = 0$: at $x = \pm\sqrt{A}$. Since $x > 0$, choose the stationary value $x = \sqrt{A}$. For this value

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{4A}{A^{\frac{3}{2}}} > 0,$$

so that the perimeter is a minimum when $x = y = \sqrt{A}$. The piece of ground must be a square.

4.5. Let the the base of the cross-section be $x > 0$ which will also be the diameter of the semicircle, and let the height of the rectangle be $y > 0$. The given area A of the tunnel cross-section is

$$A = xy + \tfrac{1}{8}\pi x^2.$$

The length of the perimeter is $P = x + 2y + \frac{1}{2}\pi x$. Eliminate y, so that

$$P = \frac{2}{x}\left(A - \frac{1}{8}\pi x^2\right) + x(1 + \frac{1}{2}\pi) = \frac{2A}{x} + (1 + \frac{1}{4}\pi)x.$$

This is stationary where

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{2A}{x^2} + 1 + \frac{1}{4}\pi = 0,$$

which occurs at $x = \sqrt{\frac{8A}{4+\pi}}$ (choosing the positive root). The perimeter is a minimum since

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{4A}{x^3} > 0$$

at the stationary point.

4.6. Let r be the radius of the base and h the height of the drum. The volume V of the drum is given by $V = \pi r^2 h$ and its prescribed surface area by $A = 2\pi r^2 + 2\pi r h$. We are given that A is a constant, so eliminate h in the expression for V:

$$V = \frac{1}{2}[Ar - 2\pi r^3].$$

Differentiating

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{2}[A - 6\pi r^2], \quad \frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -6\pi r.$$

The volume is stationary where

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 0, \text{ at } r = \sqrt{\frac{A}{6\pi}},$$

choosing the positive root. Obviously $\mathrm{d}^2V/\mathrm{d}r^2 < 0$ which proves that this radius gives a minimum volume. The height of this drum is $h = \sqrt{[2A/(3\pi)]}$ which is equal to its diameter.

4.7. Similar to 4.6: the volume is given by the same formula but the prescribed A is different:

$$V = \pi r^2 h, \quad A = \pi r^2 + 2\pi r h.$$

Elimination of h leaves

$$V = \frac{1}{2} r(A - \pi r^2).$$

Differentiating

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{2}[A - 3\pi r^2], \quad \frac{\mathrm{d}^2 V}{\mathrm{d}r^2} = 3\pi r.$$

Hence the radius and height of the drum of minimum volume are

$$r = h = \sqrt{\frac{A}{3\pi}}.$$

4.8. (a) $y = 1/(x^2 + 1)$:

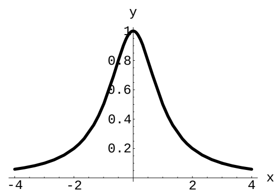

Figure 36: Problem 4.8a

(b) $y = \mathrm{e}^{x^2}$:

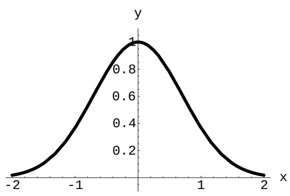

Figure 37: Problem 4.8b

(c) $y = x/(x - 1)$:

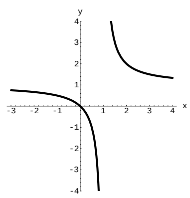

Figure 38: Problem 4.8c

(d) $y = x\mathrm{e}^{-x}$:
(e) $y = x^2\mathrm{e}^{-x}$:
(f) $y = x^3\mathrm{e}^{-x}$:
(g) $y = \mathrm{e}^{2x} - 4\mathrm{e}^{x}$:
(h) $y = (\ln x)/x$ for $x > 0$:
(i) $[\ln(-x)]/x$ for $x < 0$:
(j) $y = x \ln x - x$ for $x > 0$:

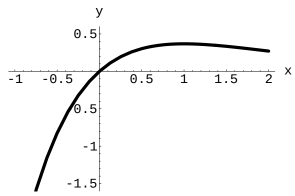

Figure 39: Problem 4.8d

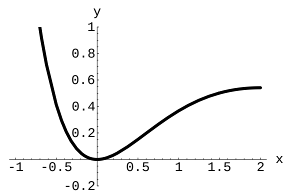

Figure 40: Problem 4.8e

(k) $y = \sin(1/x)$:
(l) $y = (x^2 - 1)^2$:
(m) $y = x(x^2 - 1)^2$:
(n) $y = (\sin x)/x$:
4.9. (a) $y = 1/(x^2 - 1)$:
(b) $y = x/(x^2 - 1)$:
(c) $1/[x(x - 2)]$:
(d) $y = x^3/(1 - x)$:
(e) $y = (x + 2)/(x - 1)$:
(f) $y = 1/(x + 1) + 1/(x + 2)$:
4.10. The incremental formula given by (4.4) is

$$\delta y \approx f'(a)\delta x \text{ at } x = a.$$

The exact value is given by

$$\delta y = f(a + \delta x) - f(a).$$

(a) $f(x) = x^3$: $\delta y \approx 3x^2\delta x$. With $x = 2$ and $\delta x = 0.1$, the approximate and exact values are given by

$$\delta y \approx 1.200, \qquad \delta y = (2.1)^3 - 2^3 = 1.157\ldots.$$

(b) $f(x) = x \sin x$: $\delta y \approx (\sin x + x \cos x)\delta x$. With $x = \frac{1}{2}\pi$ and $\delta x = -0.2$ the approximate and exact values are given by

$$\delta y \approx (\sin\frac{1}{2}\pi + \frac{1}{2}\pi\cos\frac{1}{2}\pi)(-0.2) = -0.2,$$

$$\delta y = (\frac{1}{2}\pi - 0.2)\sin(\frac{1}{2}\pi - 0.2) - \frac{1}{2}\pi\sin\frac{1}{2}\pi = -0.227\ldots.$$

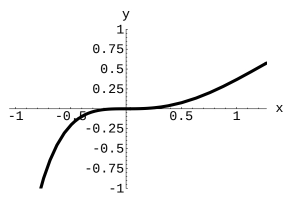

Figure 41: Problem 4.8f

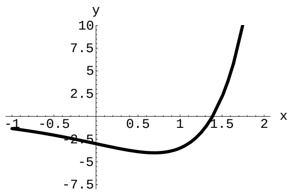

Figure 42: Problem 4.8g

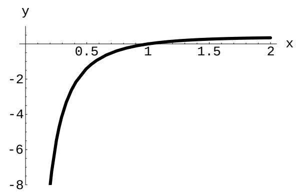

Figure 43: Problem 4.8h

(c) $f(x) = \cos x$: $\delta y \approx -\sin x \delta x$. With $x = \frac{1}{4}\pi$ and $\delta x = 0.1$ the approximate and exact values are given by

$$\delta y \approx (-\sin \frac{1}{4}\pi)(0.1) = -0.0707\ldots,$$

$$\delta y = \cos(\frac{1}{4}\pi + 0.1) - \cos(\frac{1}{4}\pi) = -0.0741\ldots.$$

(d) $f(x) = (1 + x)/(1 - x)$: $\delta y = 2/(1 - x)^2 \delta x$. With $x = 2$ and $\delta x = -0.2$ the approximate and exact values are given by

$$\delta y \approx \frac{2}{(1-2)^2}(-0.2) = -0.4, \quad \delta y = -0.5.$$

(e) $y = \tan x$: $\delta y \approx \sec^2 x \delta x$. With $x = \frac{1}{4}\pi$ and $\delta x = 0.1$ the approximate and exact values are given by

$$\delta y \approx (\sec^2 \frac{1}{4}\pi)(0.1) = 0.2, \quad \delta y = \tan(\frac{1}{4}\pi + 0.1) - \tan \frac{1}{4}\pi = 0.223\ldots.$$

(f) $f(x) = 1/(1 - x^2)$: $f'(x) = 2x/(1 - x^2)^2$. With $x = 0.5$ and $\delta x = \pm 0.1$ the approximate and exact values are given by

$$\delta y \approx \frac{1}{(1-0.5^2)^2}(\pm 0.1) = \pm 0.177\ldots,$$

$$\delta y = \frac{1}{1-(0.05 \pm 0.1)^2} - \frac{1}{1-(0.05)^2} = 0.229\ldots \text{ or } -0.142\ldots.$$

4.11. (a) With f fixed,

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{f^2}{(u-f)^2}.$$

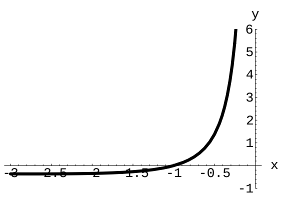

Figure 44: Problem 4.8i

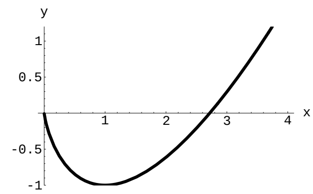

Figure 45: Problem 4.8j

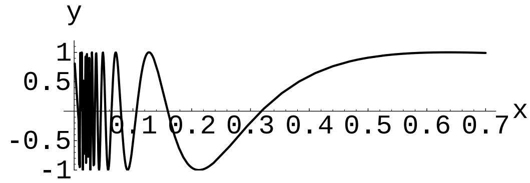

Figure 46: Problem 4.8k

Hence, with $f = 0.75$, $u = 1.25$ and $\delta u = 0.05$,

$$\delta v \approx -\frac{f^2 \delta u}{(u-f)^2} = \frac{-(0.75)^2(0.05)}{(1.25-0.75)^2} = -0.112\ldots.$$

(b) The voltage is given by

$$v = \frac{E(R_1R_4 - R_2R_3)}{(R_1+R_2)(R_3+R_4)}.$$

Its derivative with respect to R_1 is

$$\frac{\mathrm{d}v}{\mathrm{d}R_1} = \frac{ER_2}{(R_1+R_2)^2}.$$

Hence

$$\delta v \approx \frac{ER_2}{(R_1+R_2)^2} = \frac{5}{18}\delta R_1.$$

(c) With b and A constant in $a = b\sin A/(\sin B)$,

$$\frac{\mathrm{d}a}{\mathrm{d}B} = \frac{-b\sin A\cos B}{\sin^2 B} = -a\cot B.$$

Hence

$$\delta a \approx -a\cot B\,\delta B.$$

(d) In terms of a, b, c,

$$A = \frac{1}{4}\sqrt{[(a+b+c)(-a+b+c)(a-b+c)(a+b-c)]}.$$

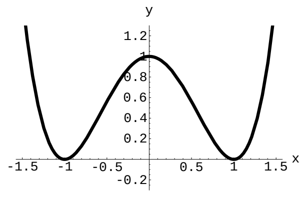

Figure 47: Problem 4.8l

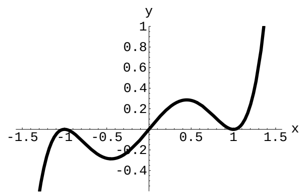

Figure 48: Problem 4.8m

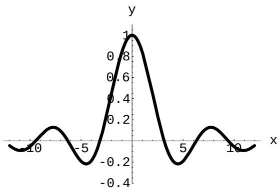

Figure 49: Problem 4.8n

Logarithmic differentiation (see equation (3.7)) gives

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}c} = \frac{1}{2}\left[\frac{1}{a+b+c} + \frac{1}{-a+b+c} + \frac{1}{a-b+c} + \frac{1}{a+b-c}\right].$$

The incremental formula for δA becomes, at $a = 2$, $b = 4$, $c = 5$

$$\delta A \approx \frac{\mathrm{d}A}{\mathrm{d}c}\delta c = -\frac{25}{2\sqrt{231}}(0.1) = 0.0822\ldots.$$

4.12. Given $C = P(1+r)^n$.
(a) With n and P fixed,

$$\frac{\mathrm{d}C}{\mathrm{d}r} = Pn(1+r)^{n-1}, \text{ so that } \delta C \approx Pn(1+r)^{n-1}\delta r.$$

(b) With r and P fixed,

$$\frac{\mathrm{d}C}{\mathrm{d}n} = P\frac{\mathrm{d}}{\mathrm{d}n}\mathrm{e}^{n\ln(1+r)} = P(1+r)^n\ln(1+r) \quad \text{see Problem 3.18.}$$

Hence

$$\delta C \approx P(1+r)^n\ln(1+r)\delta n.$$

(c) Suppose that $P = £100$, $r = 0.05$ (5%) and $n = 10$ years. The tables below show comparisons between the approximate increments δC for decreasing values of δr (n fixed) and δn (r fixed).

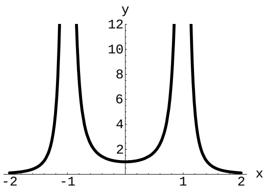

Figure 50: Problem 4.9a

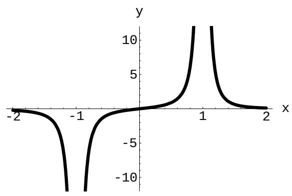

Figure 51: Problem 4.9b

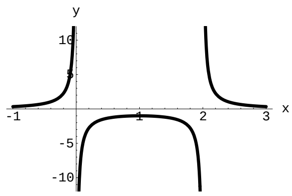

Figure 52: Problem 4.9c

δr	approximate increment in C $Pn(1+r)^{n-1}\delta r$	exact increment in C $P(1+r+\delta r)^n - P(1+r)^n$
0.01	$15.513\ldots$	$16.195\ldots$
0.005	$7.756\ldots$	$7.925\ldots$
0.001	$1.551\ldots$	$1.557\ldots$
0.0001	$0.155\ldots$	$0.155\ldots$

δn	approximate increment in C $P(1+r)^n \ln(1+r)\delta n$	exact increment in C $P(1+r)^{n+\delta n} - P(1+r)^n$
1	$7.947\ldots$	$8.144\ldots$
0.1	$0.795\ldots$	$0.797\ldots$
0.01	$0.080\ldots$	$0.080\ldots$

4.13. The iterations in Newton's method are

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots,$$

for a given initial value x_0. (a) Let $f(x) = x^4 + 2x^2 - x - 1$. Then

$$f'(x) = 4x^3 + 4 - 1.$$

For example, if we start with $x_0 = 0.75$, we obtain

$$x_1 = 0.75 - \frac{f(0.75)}{f'(0.75)} = 0.833\ldots.$$

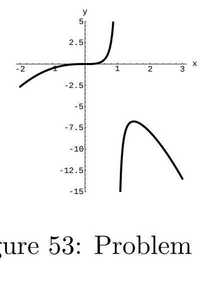

Figure 53: Problem 4.9d

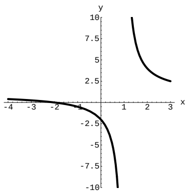

Figure 54: Problem 4.9e

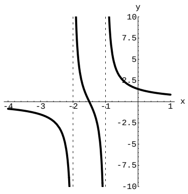

Figure 55: Problem 4.9f

Repeat the process starting with x_1 to obtain x_2, and so on. The solution is $x = 0.825\ldots$.
(b) Let $f(x) = x^4 + x^{\frac{1}{3}} - 1$. Then

$$f'(x) = 4x^3 + \frac{1}{3}x^{-\frac{2}{3}}.$$

The solution is $x = 0.619\ldots$.
(c) Let $f(x) = x \ln x + 0.3$. Then

$$f'(x) = 1 + \ln x.$$

The solution is $x = 0.168\ldots$.
(d) Let $f(x) = \mathrm{e}^x - 4x^3$. Then

$$f'(x) = \mathrm{e}^x - 12x^2.$$

The solution is $x = 0.831\ldots$.
(e) Let $f(x) = \tan x - 2x$. Then

$$f'(x) = \sec^2 x - 2.$$

By Newton's method the solution is $x = 1.165\ldots 4$
(f) Let $f(x) = \mathrm{e}^x \sin x/(1+x)$. Then

$$x - \frac{f(x)}{f'(x)} = x + \frac{(1+x)[2(1+x)\mathrm{e}^{-x} - \sin x]}{(1+x)\cos x + x \sin x}.$$

Let $x_0 = 1.85$. Then

$$x_1 = 1.663, \quad x_2 = 1.689, \quad x_3 = 1.690$$

to three decimal places.

4.14. Since $f(x) = x\mathrm{e}^{-x} + 1$ then $f'(x) = (1-x)\mathrm{e}^{-x}$. The function has its only stationary point at $x = 1$ which is a maximum. The slope of $y = f(x)$ in the neighbourhood of the solution of $f(x) = 0$ is therefore positive whilst that for any value of x greater than 1 will be negative. By the geometrical construction of Newton's method illustrated in Figure 4.15, any tangent which starts for $x > 1$ will produce iterations which diverge from the required solution. The graph of $y = x\mathrm{e}^{-x} - 1$ is shown in the figure.

4.15. (a) The graph shows a continuous function in which $f(a)$ and $f(b)$ have opposite signs.
(b) Let $g(x) = \mathrm{e}^x - 3x$. The table gives a sequence of values for $g(x)$ at intervals 0.25.

x	0	0.25	0.5	0.75	1.0	1.25	1.5
$g(x)$	1	0.534	0.149	-0.133	-0.282	-0.260	-0.018

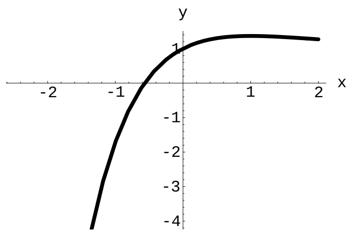

Figure 56: Problem 4.14

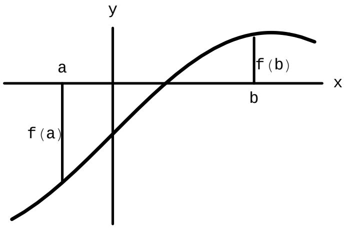

Figure 57: Problem 4.15

x	1.75	2.0	2.25	2.5
$g(x)$	0.505	1.389	2.738	4.682

Evidently the solutions of the equation lie between $x = 0.5$ and $x = 0.75$, and between $x = 1.5$ and $x = 1.75$. Note also that the function has a minimum value at $x = \ln 3 = 1.098\ldots$, which means, for example, that any initial value for the smaller solution must start at a value of $x < \ln 3$ for the reasons outlined in Problem 4.14. Similar conditions apply to the other solution.

The solutions are $x = 0.6190\ldots$, and $x = 1.5121\ldots$.

4.16. (a) Calculate $f(a+nE)$ for $n = 1, 2, \ldots$. Stop the program at $n = N$, when $f(a+NE)$ and $f(a+(N-1)E)$ have different signs.
(b) In the following table the interval is bisected four times with $E = 0.125$ and $N = 8$.

x	0	0.125	0.250	0.375	0.500	0.625
$f(a+NE)$	-1	-0.983	-0.929	-0.829	-0.676	-0.457

x	0.750	0.875	1
$f(a+NE)$	-0.162	0.224	0.718

The solution of the equation lies between $x = 0.75$ and $x = 0.875$. The computed solution is $x = 0.806\ldots$.
(c) Four decimal accuracy is obtained after 10 iterations using the bisection method, whilst Newton's method achieve the same accuracy after just 4 iterations

4.17. The slope of the normal at $x = x_0$ is $-1/f'(x_0)$ and at $x = x_0 + \delta x_0$ is $-1/f'(x_0 + \delta x_0)$. Hence their equations are

$$y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0),$$

$$y - f(x_0 + \delta x_0) = -\frac{1}{f'(x_0 + \delta x_0)}(x - x_0 - \delta x_0).$$

Solving these equations for x and y:

$$x = x_0 - \frac{f'(x_0)[f'(x_0 + \delta x_0)\{f(x_0 + \delta x_0) - f(x_0)\} + \delta x_0]}{f'(x_0 + \delta x_0) - f'(x_0)},$$

$$y = f(x_0) + \frac{[f'(x_0 + \delta x_0)\{f(x_0 + \delta x_0) - f(x_0)\} + \delta x_0]}{f'(x_0 + \delta x_0) - f'(x_0)},$$

Divide the numerators and denominators by δx_0 and let the increment tend to zero, so that the centre of curvature (x_c, y_c) is located at

$$\left(x_0 - \frac{f'(x_0)[1 + f'(x_0)^2]}{f''(x_0)},\ f(x_0) + \frac{[1 + f'(x_0)^2]}{f''(x_0)}\right).$$

The radius of curvature

$$R = \sqrt{[(x_c - x_0)^2 + (y_c - y_0)^2)]} = \frac{[1 + f'(x_0)^2]^{\frac{3}{2}}}{f''(x_0)}.$$

For the parabola $y = x^2$,

$$f(x) = x^2, \quad f'(x) = 2x, \quad f''(x) = 2.$$

Hence the centre of curvature of the point (x_0, x_0^2) is located at

$$[x_0 - x_0(1 + 4x_0^2), x_0^2 + \frac{1}{2}(1 + 4x_0^2)],$$

and its radius of curvature is $R = \frac{1}{2}(1 + 4x_0^2)^{\frac{3}{2}}$.

4.18. We shall prove Leibniz's formula by induction. For $n = 1$, the formula is true since

$$(fg)^{(1)} = f^{(1)}g + fg^{(1)}$$

by the product rule: note that ${}_1C_1 = 1$. Assume that the given formula is true for $n = k$ and all x. Then

$$(fg)^{(k)} = f^{(k)}g + {}_kC_1 f^{(k-1)}g^{(1)} + {}_kC_2 f^{(k-2)}g^{(2)} + \cdots + {}_kC_k fg^{(k)}.$$

Differentiate both sides with respect to x:

$$\begin{aligned} (fg)^{(k+1)} &= \\ &\quad (f^{(k+1)}g + f^{(k)}g^{(1)}) + ({}_kC_1 f^{(k)}g^{(1)} + {}_kC_1 f^{(k-1)}g^{(2)}) \\ &\quad + ({}_kC_2 f^{(k-1)}g^{(2)} + {}_kC_2 f^{(k-2)}g^{(2)}) + \cdots + ({}_kC_k f^{(1)}g^{(k)} + {}_kC_k fg^{(k+1)}) \\ &= f^{(k+1)}g + (1 + {}_kC_1)f^{(k)}g^{(1)} + ({}_kC_1 + {}_kC_2)f^{(k-1)}g^{(2} + \cdots + {}_kC_k fg^{(k+1)}. \end{aligned}$$

The coefficients can be written as

$$1 + {}_kC_1 = 1 + \frac{k!}{1!(k-1)!} = k + 1 = {}_{k+1}C_1,$$

$${}_{k-1}C_1 + {}_kC_2 = \frac{k!}{1!(k-1)!} + \frac{k!}{2!(k+2)!} = k + \frac{k(k-1)}{2!} = \frac{k(k+1)}{2!} = {}_{k+1}C_2,$$

and, in general,

$$\begin{aligned} {}_kC_r + {}_kC_{r+1} &= \frac{k!}{r!(k-r)!} + \frac{k!}{(r+1)!(k-r-1)!} \\ &= \frac{k!}{r!(k-r-1)!}\left[\frac{1}{k-r} + \frac{1}{r+1}\right] \\ &= \frac{k!(k+1)}{r!(k-r-1)!(k-r)(r+1)} = \frac{(k+1)!}{(r+1)!(k-r)!} \\ &= {}_{k+1}C_{r+1} \end{aligned}$$

Hence

$$(fg)^{(k+1)} = f^{(k+1)}g + {}_{k+1}C_1 f^{(k)}g^{(1)} + {}_{k+1}C_2 f^{(k-1)}g^{(2)} + \cdots + {}_{k+1}C_{k+1} fg^{(k+1)}.$$

Hence if the result is true for $n = k$, then it is true for $n = k + 1$. We have shown that it is true for $n = 1$ (the product rule); therefore it is true for $n = 2, 3, \ldots$.

Chapter 5: Taylor series and approximations

5.1. (a) For $f(x) = \mathrm{e}^{\frac{1}{2}x}$,

$$f'(x) = \frac{1}{2}\mathrm{e}^{\frac{1}{2}x}, \quad f''(x) = \frac{1}{4}\mathrm{e}^{\frac{1}{2}x}, \quad \frac{1}{8}\mathrm{e}^{\frac{1}{2}x},$$

so that

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = \frac{1}{4}, \quad f''(0) = \frac{1}{8}.$$

The Taylor polynomial approximation to four terms becomes

$$f(x) \approx 1 + \frac{1}{2}x + \frac{1}{2!}\frac{1}{4}x^2 + \frac{1}{3!}\frac{1}{8}x^3.$$

We can estimate that the three term approximation will be accurate to two decimal places if for the fourth term

$$\left|\frac{1}{8.3!}x^3\right| < 0.005, \text{ or } |x| < 0.621\ldots.$$

(b) For $f(x) = (1+x)^{\frac{1}{2}}$, the Taylor approximation is

$$(1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

The three-term approximation will be accurate to two decimal places if

$$\left|\frac{x^3}{16}\right| < 0.005 \text{ or } |x| < 0.432\ldots.$$

(c) For $f(x) = (1+x)^{-\frac{1}{3}}$, the four-term Taylor polynomial is

$$\begin{aligned}(1+x)^{-\frac{1}{3}} &\approx 1 + \left(-\tfrac{1}{3}\right)x + \left(-\tfrac{1}{3}\right)\left(-\tfrac{4}{3}\right)\left(\tfrac{x^2}{2!}\right) + \left(-\tfrac{1}{3}\right)\left(-\tfrac{4}{3}\right)\left(-\tfrac{7}{3}\right)\left(\tfrac{x^3}{3!}\right)\\ &\approx 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3\end{aligned}$$

The three-term approximation will be accurate to two decimal places if

$$\frac{14}{81}|x|^3 < 0.005 \text{ or } |x| < 0.306\ldots$$

.

(d) The Taylor approximation to four terms for $\sin 2x$ can be obtained form the series for $\sin y$ where $y = 2x$ (use (5.4c)):

$$\begin{aligned}\sin 2x &\approx (2x) - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7\\ &\approx 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7\end{aligned}$$

The three-term approximation will be accurate to two decimal places if

$$\frac{8}{15}|x|^7 < 0.005 \text{ or } |x| < 0.196\ldots.$$

(e) Using the expansion for $\cos z$, where $z = \frac{1}{2}x$ (see (5.4d)):

$$\cos\frac{1}{2}x \approx 1 - \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6.$$

The three-term polynomial will be accurate to two decimal places if

$$\tfrac{1}{46080}x^6 < 0.005 \text{ or } |x| < 2.475\ldots.$$

(f) The four-term expansion is (see(5.4e)):

$$\ln(1+x) \approx x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4.$$

The three-term polynomial will be accurate to two decimal places if

$$\frac{1}{4}x^4 < 0.005 \text{ or } |x| < 0.376\ldots.$$

(g) Let $f(x) = (1+x^2)^{\frac{1}{2}}$. Put $u = x^2$. Then, as in (b),

$$\begin{aligned}(1+x^2)^{\frac{1}{2}} &= (1+u)^{\frac{1}{2}} \approx 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 \\ &\approx 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6.\end{aligned}$$

The three-term polynomial will be accurate to two decimal places if

$$\frac{1}{16}x^6 < 0.005 \text{ or } |x| < 0.656\ldots.$$

(h) The four-term Taylor polynomial for $\ln(1+3x)$ is (put $u = 3x$, etc.),

$$\begin{aligned}\ln(1+3x) &\approx (3x) - \frac{1}{2}(3x)^2 + \frac{1}{3}(3x)^3 - \frac{1}{4}(3x)^4 \\ &\approx 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4.\end{aligned}$$

The three-term approximation will be accurate to two decimal places if

$$\frac{81}{4}x^4 < 0.005 \text{ or } |x| < 0.125\ldots.$$

5.2. The Taylor expansion for $f(x)$ about $x = 0$ is

$$f(x) = f(0) + f'(0) + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots.$$

(a) Let $f(x) = \mathrm{e}^x$. Then $f'(x) = f''(x) = \cdots = \mathrm{e}^x$. Hence

$$f(0) = f'(0) = f''(0) = \cdots = 1.$$

(b) Let $f(x) = \sin x$. Then

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x, \text{ etc.}$$

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^{(4)}(0) = 0, \ \ldots,$$

the Taylor coefficients being

$$1, \quad \frac{1}{1!}, \quad \frac{1}{2!}, \ \ldots .$$

(c) Let $f(x) = \cos x$. Then

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x, \ldots,$$

so that

$$f(0)=1,\quad f'(0)=0\quad f''(0)=-1\quad f'''(0)=0,\quad f^{(4)}(0)=1,\ldots.$$

(d) Let $f(x)=(1+x)^{\alpha}$. Then

$$f'(x)=\alpha(1+x)^{\alpha-1},\quad f''(x)=\alpha(\alpha-1)(1-x)^{\alpha-2},\ldots.$$

so that

$$f(0)=1,\quad f'(0)=\alpha,\quad f''(0)=\alpha(\alpha-1),\ldots.$$

(e) Let $f(x)=\ln(1+x)$. Then

$$f'(x)=\frac{1}{1+x},\quad f''(x)=-\frac{1}{(1+x)^2},$$

$$f'''(x)=\frac{2\times 1}{(1+x)^3},\quad f^{(4)}(x)=\frac{3\times 2\times 1}{(1+x)^4)},\ \ldots.$$

so that

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=-1,\quad f'''(0)=2!,\quad f^{(4)}(0)=3!,\ldots.$$

Therefore the coefficient of x^n for $n\geq 1$ is

$$(-1)^{n-1}\frac{(n-1)!}{n!}=\frac{(-1)^{n-1}}{n}.$$

5.3. (a) The general term for e^x is $x^n/n!$. Hence, for four-decimal point accuracy we require n such that, for $|x|=2$,

$$\frac{x^n}{n!}=\frac{2^n}{n!}<0.00005.$$

For $n=11$, $2^n/n!=0.000051\ldots$ and for $n=12$, $2^n/n!=0.0000085\ldots<0.00005$. Hence terms up to x^{11} are required.

(b) The general term for $\sin x$ is $(-1)^n x^{2n-1}/(2n-1)!$. Hence for four-decimal accuracy we require the smallest n such that, for $|x|=2$,

$$\left|\frac{x^{2n-1}}{(2n-1)!}\right|=\frac{2^{2n-1}}{(2n-1)!}<0.00005.$$

For $n=6$, $2^{2n-1}/(2n-1)!=0.000051\ldots$ and for $n=7$, $2^{2n-1}/(2n-1)!=0.0000031\ldots$. Hence terms up to and including x^{11} are required.

(c) The general term for $\cos x$ is $(-1)^n x^{2n}/(2n)!$. Hence for four decimal accuracy we require the smallest n such that, for $|x|=2$,

$$\left|\frac{x^{2n}}{(2n)!}\right|=\frac{2^{2n}}{(2n)!}<0.00005.$$

For $n=5$, $2^{2n}/(2n)!=0.00028\ldots$ and for $n=6$, $2^{2n}/(2n)!=0.0000085\ldots$. Hence terms up to and including x^{10} are required.

(d) For $(1+x)^{\frac{1}{2}}$ the general term is

$$\frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}x^n,$$

where $\alpha=-\frac{1}{2}$. For four-decimal accuracy we require the smallest n such that, for $|x|=0.5$,

$$\left|\frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}(0.5)^n\right|<0.00005.$$

For $n = 8$ the last term has magnitude $0.000051\ldots$, and for $n = 9$ the magnitude is $0.000021\ldots$.
(e) For $\ln(1+x)$, the general term in its Taylor series is $(-1)^{n+1}x^n/n$. For four-decimal accuracy we require the smallest n such that, for $|x| = 0.5$,

$$\left|(-1)^{n+1}\frac{x^n}{n}\right| = \frac{0.5^n}{n} < 0.00005.$$

For $n = 10$, $\frac{0.5^n}{n} = 0.000097\ldots$, whilst for $n = 11$, $\frac{0.5^n}{n} = 0.000044\ldots$. Hence terms up to and including x^{10} are required.

5.4. (a) Let $f(x) = \arcsin x$. Then

$$f'(x) = \frac{1}{\sqrt{(1-x^2)}}, \quad f''(x) = \frac{x}{(1-x^2)^{\frac{3}{2}}}, \quad f'''(x) = \frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}}$$

so that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 1$. Hence

$$\arcsin x = x + \frac{1}{6}x^3 + \cdots.$$

(b) Let $f(x) = \arccos x$. Then

$$f'(x) = -\frac{1}{\sqrt{(1-x^2)}}, \quad f''(x) = -\frac{x}{(1-x^2)^{\frac{3}{2}}}$$

so that $f(0) = \frac{1}{2}\pi$, $f'(0) = -1$, $f''(0) = 0$. The Taylor series starts with

$$\arccos x = \frac{1}{2}\pi - x + \cdots.$$

(c) Let $f(x) = \arctan x$. Then

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = -\frac{2x}{(1+x^2)^2}, \quad f'''(x) = \frac{-2+6x^2}{(1+x^2)^3}.$$

Hence $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -2$. The Taylor series starts with

$$\arctan x = x - \frac{1}{3}x^3 + \cdots.$$

(d) Let $f(x) = \mathrm{e}^{-x}\sin x$. Then

$$f'(x) = \mathrm{e}^{-x}(\cos x - \sin x), \quad f''(x) = -2\mathrm{e}^{-x}\cos x.$$

Hence $f(0) = 0$, $f'(0) = 1$, $f''(0) = -2$. The Taylor series starts with

$$\mathrm{e}^{-x}\sin x = x - x^2 + \cdots.$$

(e) Let $f(x) = \mathrm{e}^{-x}\cos x$. Then

$$f'(x) = -\mathrm{e}^{-x}(\cos x + \sin x).$$

Hence $f(0) = 1$, $f'(0) = -1$ so that the Taylor series starts with

$$\mathrm{e}^{-x}\cos x = 1 - x.$$

5.5. (a) Let $f(x) = 1/(1+3x)$. Then $f'(x) = -3/(1+3x)^2$, $f''(x) = 18/(1+3x)^3$ so that

$$f(0) = 1, \quad f''(0) = -3, \quad f''(0) = 18.$$

The first three terms of its Taylor series are

$$\frac{1}{1+3x} = 1 - 3x + 9x^2 + \cdots.$$

Alternatively, the binomial expansion (5.4f) can be used. Also from (5.4f) the expansion will be valid for

$$-1 < 3x < 1 \text{ or } -\frac{1}{3} < x < \frac{1}{3}.$$

(b) Adapting (5.4f),

$$\begin{aligned}\frac{1}{2-x} &= 2^{-1}\left[1+\left(-\frac{x}{2}\right)\right]^{-1} = \frac{1}{2}\left[1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2+\cdots\right]\\ &= \frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+\cdots\end{aligned}$$

the series is valid for $-1 < \frac{1}{2}x < 1$ or $-2 < x < 2$.

(c) Using (5.4f) again

$$(3-x)^{\frac{1}{3}} = 3^{\frac{1}{3}}(1-\tfrac{1}{3}x) = 3^{\frac{1}{3}}\left[1-\tfrac{1}{9}x-\tfrac{1}{81}x^2-\cdots\right].$$

The series is valid for $-3 < x < 3$.

(d) Using (5.4f) again

$$\begin{aligned}(x-3)^{\frac{1}{3}} &= (-3)^{\frac{1}{3}}\left[1+\left(-\frac{x}{3}\right)\right]^{\frac{1}{3}}\\ &= (3)^{\frac{1}{3}}\left[-1+\frac{1}{9}x+\frac{1}{81}x^2+\cdots\right].\end{aligned}$$

The series is valid for $-3 < x < 3$.

(e) Adapting (5.4e),

$$\ln(9-x) = \ln\left[9\left(1-\frac{1}{9}x\right)\right] = \ln 9 + \ln\left(1-\frac{1}{9}x\right) = 2\ln 3 - \frac{1}{9}x - \frac{1}{162}x^2 - \cdots.$$

The series is valid for $-9 < x < 9$.

(f) From (5.4d),

$$\begin{aligned}\cos(\frac{1}{2}x) &= 1-\frac{1}{2!}\left(\frac{x}{2}\right)^2+\frac{1}{4!}\left(\frac{x}{2}\right)^4-\cdots\\ &= 1-\frac{1}{8}x^2+\frac{1}{384}x^4-\cdots.\end{aligned}$$

The series is valid for all x.

(g) Put $u = x^{\frac{1}{2}}$ for $x > 0$, and use (5.4c):

$$\sin(x^{\frac{1}{2}}) = \sin u = u - \frac{1}{6}u^3 + \frac{1}{120}u^5 - \cdots = x^{\frac{1}{2}} - \frac{1}{6}x^{\frac{3}{2}} + \frac{1}{120}x^{\frac{5}{2}} - \cdots.$$

Since $x^{\frac{1}{2}}$ is not real for $x < 0$ the series will be valid only for $x \geq 0$.

(h) Put $u = x^{\frac{1}{2}}$ for $x > 0$, and use (5.4d):

$$\cos(x^{\frac{1}{2}}) = 1 - \tfrac{1}{2}x + \tfrac{1}{24}x^2 + \cdots.$$

The series is valid for all $x \geq 0$.

5.6. Multiply standard expansions for

$$e^{-x} = 1 - x + \frac{1}{2}x^2 + \cdots \text{ and } \frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 + \cdots.$$

Hence

$$\begin{aligned}\frac{e^{-x}}{1+x} &= \left(1 - x + \frac{1}{2}x^2 + \cdots\right)\left(1 - x + x^2 + \cdots\right). \\ &= 1 - 2x + \frac{5}{2}x^2 + \cdots\end{aligned}$$

(b) As in (a)

$$\begin{aligned}(1-x)^{\frac{1}{2}}e^x &= \left(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \cdots\right)\left(1 + x + \frac{1}{2}x^2 + \cdots\right) \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\end{aligned}$$

(c) This time square the series for $\ln(1-x)$:

$$\begin{aligned}\frac{1}{x^2}[\ln(1-x)]^2 &= \frac{1}{x^2}\left[-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots\right]^2 \\ &= \frac{1}{x^2}\left[x^2 + x^3 + \frac{11}{12}x^4 + \cdots\right] \\ &= 1 + x + \frac{11}{12}x^2 + \cdots\end{aligned}$$

5.7. Start with the Taylor series

$$1 + \ln(1+x) = 1 + x - \frac{1}{2}x^2 + \cdots.$$

Then, assuming that the expansion takes the form $b_0 + b_1x + b_2x^2 + \cdots$,

$$\frac{1}{[1+\ln(1+x)]} = \frac{1}{1 + x - \frac{1}{2}x^2 + \cdots} = b_0 + b_1x + b_2x^2 + \cdots.$$

We now equate coefficients of powers of x in the identity

$$\begin{aligned}1 &= \left(1 + x - \frac{1}{2}x^2 + \cdots\right)\left(b_0 + b_1x + b_2x^2 + \cdots\right), \\ &= b_0 + (b_1 + b_0)x + (b_2 - b_1 - \frac{1}{2}b_0)x^2 + \cdots\end{aligned}$$

Hence $b_0 = 1$, $b_1 = -b_0 = -1$, $b_2 = b_1 + \frac{1}{2}b_0 = \frac{3}{2}$ and the Taylor series starts with

$$\frac{1}{[1+\ln(1+x)]} = 1 - x + \frac{3}{2}x^2 + \cdots.$$

(b) Write $\tan x = \sin x/\cos x$ and use the series for $\sin x$ and $\cos x$. Thus

$$\begin{aligned}\tan x &= \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots}{1 - \frac{1}{2}x + \frac{1}{6}x^3 + \cdots} \\ &= b_1x + b_3x^3 + b_5x^5 + \cdots.\end{aligned}$$

Note that the series will contain only odd powers of x. By cross-multiplying

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots = \left(1 - \frac{1}{2}x + \frac{1}{6}x^3 + \cdots\right)\left(b_1 x + b_3 x^3 + b_5 x^5 + \cdots\right).$$

By matching the coefficients of $x, x^2, \ldots$ on either side we obtain

$$b_1 = 1, \quad b_3 = \frac{1}{3}, \quad b_5 = \frac{2}{15},$$

so that

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots.$$

(c) From (5.4b)

$$1 + \mathrm{e}^x = 2 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \cdots.$$

Assume that

$$\frac{1}{1+\mathrm{e}^x} = b_0 + b_1 x + b_2 x^2 + \cdots :$$

then

$$\begin{aligned} 1 &= \left(2 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \cdots\right)\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 \cdots\right) \\ &= 2b_0 + (b_0 + 2b_1)\,x + \left(\frac{1}{2}b_0 + b_1 + 2b_2\right) + \cdots \end{aligned}$$

Solving for b_0, b_1, b_2 and b_3,

$$\frac{1}{1+\mathrm{e}^x} = \frac{1}{2} - \frac{1}{4}x + \frac{1}{48}x^3 + \cdots.$$

(d) Use the definition

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^x},$$

where

$$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots, \quad \cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots.$$

Hence, if the required series is $b_1 x + b_3 x^3 + b_5 x^5 + \cdots$ (it must be an odd function), then

$$\begin{aligned} x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots &= \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right)\left(b_1 x + b_3 x^3 + b_5 x^5 + \cdots\right) \\ &= x + (b_1 x + (b_3 + \frac{1}{2}b_1)x^3 + (b_5 + \frac{1}{2}b_3 + \frac{1}{24})x^5 + \cdots \end{aligned}$$

Comparing powers of x, it follows that $b_1 = 1$, $b_3 = -\frac{1}{3}$, $b_5 = \frac{2}{15}$ so that

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots.$$

(e) Since $x/\sin x$ is an even function and $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, then

$$\begin{aligned} x &= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(b_0 + b_2 x^2 + b_4 x^4 + \cdots\right) \\ &= b_0 x + \left(b_2 - \frac{1}{6}b_0\right) + \left(b_4 - \frac{1}{6}b_2 + \frac{1}{120}b_0\right)x^5 + \cdots \end{aligned}$$

Hence $b_0 = 1$, $b_2 = \frac{1}{6}$ and $b_4 = \frac{7}{360}$ so that

$$\frac{x}{\sin x} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots.$$

5.8. The following series provide approximations for large values of x.
(a) Let $u = 1/x$. Then

$$\begin{aligned}\left(1-\frac{1}{x}\right)^{\frac{1}{2}} &= (1-u)^{\frac{1}{2}} = 1 - \frac{1}{2}u - \frac{1}{8}u^2 + \cdots \text{ (binomial series)} \\ &= 1 - \frac{1}{2x} - \frac{1}{8x^2} + \cdots\end{aligned}$$

which will be valid for $|u| < 1$, or equivalently, $|x| > 1$.
(b) Let $x > 0$ and $u = 1/x^{\frac{1}{2}} > 0$. Then

$$\begin{aligned}\ln\left(1+\frac{1}{x^{\frac{1}{2}}}\right) &= \ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + \cdots \\ &= \frac{1}{x^{\frac{1}{2}}} - \frac{1}{x} + \frac{1}{3x^{\frac{3}{2}}} + \cdots.\end{aligned}$$

This series will be valid for $0 < u \leq 1$, or $x \geq 1$.
(c) Let $x > 0$ and $u = 1/x$. Then

$$\begin{aligned}\frac{x^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} &= \frac{1}{(1+\frac{1}{x})^{\frac{1}{2}}} = (1+u)^{-1} \\ &= 1 - \frac{1}{2}u + \frac{3}{8}u^2 + \cdots \text{ (binomial expansion)} \\ &= 1 - \frac{1}{2x} + \frac{3}{8x^2} + \cdots\end{aligned}$$

The series is valid for $0 < u < 1$ or $x > 1$.
(d) Let $u = (1/x) + (1/x)^2$. Then, using (5.4e),

$$\begin{aligned}\ln(1+x+x^2) &= \ln(x^2) + \ln(1+u) = \ln(x^2) + u - \frac{1}{2}u^2 + \cdots \\ &= \ln(x^2) + \left(\frac{1}{x} + \frac{1}{x^2}\right) - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2}\right)^2 + \cdots \\ &= \ln(x^2) + \frac{1}{x} + \frac{1}{2x^2} + \cdots\end{aligned}$$

The series is valid for $-1 < (1/x)+(1/x^2) < 1$, that is, when $x < -\frac{1}{2}(\sqrt{5}-1)$ or when $x > \frac{1}{2}(1+\sqrt{5})$.
(e) Let $u = 1/x$. Then

$$\begin{aligned}\frac{1}{\sin(1/x)} &= \frac{1}{\sin u} = \frac{1}{u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \cdots} \\ &= \frac{1}{u} \cdot \frac{1}{1 - \frac{1}{3!}u^2 + \frac{1}{5!}u^4 \cdots} \\ &= \frac{1}{u}\left[1 + \left(\frac{1}{6}u^2 + \frac{1}{120}u^4\right) + \left(\frac{1}{6}u^2 + \frac{1}{120}u^4\right)^2 + \cdots\right] \\ &= \frac{1}{u}\left[1 + \frac{1}{6}u^2 + \frac{7}{360}u^4 + \cdots\right] \\ &= x + \frac{1}{6x} + \frac{7}{360x^3} + \cdots\end{aligned}$$

5.9. (a) Using the two-term Taylor expansion for $\sin x$,

$$\frac{1}{\sin x} \approx \frac{1}{x - \frac{1}{6}x^3} \approx \frac{1}{x}\frac{1}{1 - \frac{1}{6}x^2}$$

$$\begin{aligned} &\approx \frac{1}{x}\left(1+\frac{1}{6}x^2\right) \text{ (using (5.4f))} \\ &\approx \frac{1}{x}+\frac{1}{6}x, \end{aligned}$$

for small x. Note that for $x = 0.5$, the error is

$$\left|\sin x - \left(\frac{1}{x}+\frac{1}{6}x\right)\right| = 0.0024\ldots,$$

that is, about 0.1%.

(b) Write as

$$\begin{aligned} (1+x)^{\frac{1}{2}} &= x^{\frac{1}{2}}\left(1+\frac{1}{x}\right)^{\frac{1}{2}} \approx x^{\frac{1}{2}}\left(1+\frac{1}{2x}\right) \\ &= x^{\frac{1}{2}} + \frac{1}{2x^{\frac{1}{2}}}, \end{aligned}$$

for large x.

(c) Using the result from (b)

$$\begin{aligned} (2+x)^{\frac{1}{2}} - (1+x)^{\frac{1}{2}} &= 2^{\frac{1}{2}}(1+\frac{1}{2}x)^{\frac{1}{2}} - (1+x)^{\frac{1}{2}} \\ &\approx 2^{\frac{1}{2}}\left[\left(\frac{x}{2}\right)^{\frac{1}{2}} + 1/(2x)^{\frac{1}{2}}\right] - \left[x^{\frac{1}{2}} + \frac{1}{2x^{\frac{1}{2}}}\right] \\ &\approx \frac{1}{2x^{\frac{1}{2}}}, \end{aligned}$$

for large x.

(d) Using the three-term Taylor expansion for $\cos x$,

$$\begin{aligned} \frac{1}{(1-\cos x)^{\frac{1}{2}}} &\approx \left[\frac{x^2}{2} - \frac{x^4}{24}\right]^{-\frac{1}{2}} \\ &\approx \frac{2^{\frac{1}{2}}}{x}\left[1+\frac{1}{24}x^2\right] \text{ (binomial expansion)} \\ &\approx \frac{2^{\frac{1}{2}}}{x} + \frac{2^{\frac{1}{2}}x}{24}, \end{aligned}$$

for small x.

5.10. (a) Expanding about $x = 1$,

$$\ln x = \ln[1+(x-1)] = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots \text{ (using (5.4e)).}$$

The series is valid for $-1 < x-1 < 1$ or $0 < x < 2$.

(b) For an expansion about $x = \frac{1}{2}\pi$, write

$$\cos x = \cos[\frac{1}{2}\pi + (x - \frac{1}{2}\pi)] = -\sin(x-\frac{1}{2}\pi).$$

Now use the Taylor expansion for the sine:

$$\cos x = -\sin(x-\frac{1}{2}\pi) = -(x-\frac{1}{2}\pi) + \frac{1}{3!}(x-\frac{1}{2}\pi)^3 - \frac{1}{5!}(x-\frac{1}{2}\pi)^5 + \cdots$$

The series is valid for all $x - \frac{1}{2}\pi$, which means for all x.

(c) For a Taylor series centred at $x = 1$, write

$$(1+x)^{\frac{1}{2}} = [2+(x-1)]^{\frac{1}{2}} = 2^{\frac{1}{2}}[1+\frac{1}{2}(x-1)]^{\frac{1}{2}}.$$

Now expand the right-hand side using the binomial series (5.4f):

$$(1+x)^{\frac{1}{2}} = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) - \frac{1}{16\sqrt{2}}(x-1)^2 + \cdots.$$

The series is valid for $-1 < \frac{1}{2}(x-1) < 1$, that is for $-1 < x < 3$.

5.11. (a) Since $f(x)$ has a stationary point at $x = c$, $f'(c) = 0$ and its Taylor series about $x = c$ will be

$$f(x) = f(c) + \frac{1}{2!}f''(c)x^2 + \frac{1}{3!}f'''(c)x^3 + \cdots.$$

Approximately

$$f(x) \approx f(c) + \frac{1}{2}f''(c)x^2,$$

for $|x - c|$ small. Hence if $f''(c) > 0$, then $f(x) > f(c)$ close to $x = c$ excluding $x = c$. The conclusion is that $x = c$ is a minimum. Similarly if $f''(c) < 0$, then $x = c$ is a maximum.
(b) If $f''(c) = 0$, then we must look at the signs of higher derivatives. Suppose that $f^{(N)}(c) \neq 0$ is the first non-zero derivative (that is, $f^{(r)}(c) = 0$ for $r = 1, 2, \ldots, N-1$). Hence, approximately,

$$f(x) \approx f(c) + \frac{1}{N!}f^{N}(c)(x-c)^N.$$

If N is even and $f^{(N)}(c) > 0$ then $x = c$ is a minimum, whilst if $f^{(N)}(c) < 0$ then $x = c$ is a maximum. If N is odd then the stationary point will be a point of inflection.

5.12. (Compare eqn (2.15).) Put $(\mathrm{e}^x - 1)/x = f(x)$ for $x \neq 0$, and use the Taylor series (5.4b) for e^x:

$$f(x) = [(1 + x + \frac{1}{2!}x^2 + \cdots) - 1]/x = 1 + \frac{1}{2!}x + \cdots,$$

for $x \neq 0$. Therefore $\lim_{x\to 0} f(x) = 1$, which is the 'missing value' at $x = 0$.
(b) Put $(1 - \cos x)/x^2 = f(x)$ for $x \neq 0$. From (5.4d)

$$\begin{aligned} f(x) &= \frac{1}{x^2}[1 - (1 - \frac{1}{2!} + \frac{1}{4!}x^4 - \cdots)] \\ &= \frac{1}{x^2}(\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \cdots) \\ &= \frac{1}{2!} - \frac{1}{4!}x^2 + \cdots \text{ (for } x \neq 0) \end{aligned}$$

Therefore $\lim_{x\to 0} f(x) = \frac{1}{2}$.
(c) Put $[\ln(1 + x) - x]/\sin x = f(x)$, $x \neq 0$. From (5.4c,e),

$$\begin{aligned} f(x) &= \frac{(x - \frac{1}{2}x^2 + \cdots) - x}{x - \frac{1}{3!}x^3 + \cdots} = \frac{-\frac{1}{2}x^2 + \cdots}{x - \frac{1}{3!}x^3 + \cdots} \\ &= \frac{x(-\frac{1}{2} + \cdots}{1 - \frac{1}{3!}x^3 + \cdots} \text{ (for } x \neq 0). \end{aligned}$$

Therefore $\lim_{x\to 0} f(x) = 0$.
(Alternatively, rewrite $f(x)$ in the form

$$\frac{\ln(1+x) - x}{x}\frac{x}{\sin x}$$

and use the limits (2.13) and (2.14).)

(d) Put $\sin x/(1-\cos x) = f(x)$, $x \neq 0$:

$$\begin{aligned} f(x) &= \frac{x - \frac{1}{3!} + \cdots}{1-(1-\frac{1}{2!}x^2+\cdots)} = \frac{x(1-\frac{1}{3!}x^2+\cdots)}{\frac{1}{2}x^2(1-\frac{1}{4!}x^2+\cdots)} \\ &= \frac{1}{x}\frac{2(1-\cdots)}{(1-\cdots)}. \end{aligned}$$

This does not tend to a limit; it approaches ∞ as $x \to 0$. Therefore this function does not possess a fill-in value at $x = 0$ which would make it continuous.

5.13. (a)

$$\lim_{x\to 0}\frac{(1-x)^{12}-1}{(1-x)^{10}-1} = \lim_{x\to 0}\frac{-12x + \text{higher powers}}{-10x + \text{higher powers}} = \frac{12}{10},$$

where the binomial theorem (4.7) was used to expand the powers of $(1-x)$.

(b)

$$\begin{aligned} \lim_{x\to 0}\frac{\sin x - x}{\sin x - x\cos x} &= \lim_{x\to 0}\frac{x - \frac{1}{3!}x^3+\cdots) - x}{(x-\frac{1}{3!}x^3+\cdots) - x(1-\frac{1}{2!}x^2+\cdots)} \\ &= \lim_{x\to 0}\frac{-\frac{1}{3!} + \text{higher powers}}{\frac{1}{3} + \text{higher powers}} \\ &= \lim_{x\to 0}\frac{-\frac{1}{3!} + \text{higher powers}}{\frac{1}{3} + \text{higher powers}} \\ &= -\frac{1}{2}. \end{aligned}$$

(c) Put $x = \pi + u$. Then

$$\begin{aligned} \lim_{x\to\pi}\frac{\cos x + 1}{x - \pi} &= \lim_{u\to 0}\frac{\cos(\pi+u)+1}{u} \\ &= \lim_{u\to 0}\frac{\cos\pi\cos u - \sin\pi\sin u + 1}{u} \quad \text{(from (1.17a))} \\ &= \lim_{u\to 0}\frac{-\cos u + 1}{u} = \lim_{u\to 0}\frac{-(1-\frac{1}{2!}u^2+\cdots)+1}{u} \\ &= \lim_{u\to 0} u(\frac{1}{2}+\cdots) = 0. \end{aligned}$$

(d) Put $x = u + \frac{1}{2}\pi$: then

$$\begin{aligned} \lim_{x\to\frac{1}{2}\pi}\frac{\sin x - 1}{\cos 5x} &= \lim_{u\to 0}\frac{\sin(\frac{1}{2}\pi+u)-1}{\cos(\frac{5}{2}\pi+5u)} \\ &= \lim_{u\to 0}\frac{\sin\frac{1}{2}\pi\cos u + \cos\frac{1}{2}\pi\sin u - 1}{\cos\frac{5}{2}\pi\cos 5u - \sin\frac{5}{2}\pi\sin 5u} \quad \text{(from (1.17a))} \\ &= \lim_{u\to 0}\frac{\cos u - 1}{-\sin 5u} = \lim_{u\to 0}\frac{\frac{1}{2}u^2 - \cdots}{-5u+\cdots} \\ &= 0 \end{aligned}$$

5.14. Let

$$f(x) = \frac{e^x - 1}{e^x - 1 - x} = \frac{(e^x-1)/x}{[(\{e^x-1)/x\} - 1]}.$$

Since $\lim_{x\to 0}[(e^x-1)/x] = 1$ (see eqn (2.15) or Problem 5.12a), $f(x)$ approaches infinity as $x \to 0$

To determine the question of signs, return to the original form and take the following steps:
(i) $e^x - 1$ is negative when $x < 0$ and positive when $x > 0$.
(ii) $(d/dx)(e^x - 1 - x) = e^x$, which is greater than zero for all x. Therefore $e^x - 1 - x$ is a steadily increasing function for all x. Since also it is zero at $x = 0$, it must be negative when $x < 0$ and positive for $x > 0$.
(iii) From (i) and (ii), $f(x)$ is negative when $x < 0$ and positive when $x > 0$. Therefore $f(x) \to -\infty$ as $x \to 0$ from the left, and $f(x) \to +\infty$ as $x \to 0$ from the right.

5.15. (a)

$$\begin{aligned}\lim_{x\to 0}\left[\frac{\sin^3 3x}{1-\cos x}\right] &= \lim_{x\to 0}\frac{(3x+\cdots)^3}{1-(1-\frac{1}{2!}x^2+\cdots)}\\ &= \lim_{x\to 0}\frac{27x^3+\cdots}{\frac{1}{2}x^2+\cdots}\\ &= \lim_{x\to 0}\frac{27x(1+\cdots)}{\frac{1}{2}(1+\cdots)} = 0.\end{aligned}$$

(b) Let $[(e^x - 1)/x]^{\frac{1}{2}} = g(x)$. We know from eqn (2.15) that

$$\lim_{x\to 0}[g(x)]^2 = \lim_{x\to 0}\left[\frac{e^x-1}{x}\right] = 1.$$

But also

$$\lim_{x\to 0}[g(x)]^2 = \lim_{x\to 0}[g(x)g(x)] = \lim_{x\to 0} g(x)\lim_{x\to 0} g(x) = [\lim_{x\to 0} g(x)]^2.$$

Therefore

$$\lim_{x\to 0} g(x) = \sqrt{1} = 1,$$

the positive square root being taken because $g(x)$ is never negative.
(c)

$$\begin{aligned}\lim_{x\to 0}\left[\frac{(2+\tan x)\sin x}{x(3-\tan^2 x)}\right] &= \lim_{x\to 0}\left[\frac{2+\tan x}{3-\tan^2 x}\right]\lim_{x\to 0}\left[\frac{\sin x}{x}\right]\\ &= \frac{2+0}{3-0}\cdot 1 = \frac{2}{3} \text{ (where we refer to eqn (2.13))}\end{aligned}$$

5.16. Let $f(x) = 3x - \sin x$ and $g(x) = x$. Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \left.\frac{3-\cos x}{1}\right|_{x=0} = 2.$$

5.17. In the following, S represents the required sum.
(a)

$$\begin{aligned}S &= \sum_{n=0}^{\infty}\frac{1}{n!}x^n - \sum_{n=0}^{\infty}(-x)^n\\ &= e^x - (1 - x + x^2 - \cdots) = e^x - \frac{1}{1+x},\end{aligned}$$

the second term being a geometric series with common ratio $(-x)$: see (5.4a).
(b) $S = x^3 + \frac{1}{2}x^4 + \frac{1}{3}x^5 + \cdots = x^2(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots)$. From (5.4e),

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots .$$

Therefore $S = -x^2\ln(1-x)$.

(c) $S = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$. But from (5.4b),

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \text{ and } e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots.$$

Therefore $e^x + e^{-x} = 2S$, and

$$S = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

from (1.26).

5.18. In the following, S represents the required sum.
(a) From (5.4b)

$$e^2 = 1 + 2 + \tfrac{2^2}{2!} + \tfrac{2^3}{3!} + \cdots,$$

and

$$e^{-2} = 1 - 2 + \tfrac{2^2}{2!} - \tfrac{2^3}{3!} + \cdots,$$

so that

$$e^2 - e^{-2} = 2\left(2 + \tfrac{2^3}{3!} + \tfrac{2^5}{5!} + \cdots\right),$$

or

$$S = \tfrac{1}{2}(e^2 - e^{-2}).$$

(b) From (5.4b), $S = e^{\frac{1}{2}}$.
(c) S is geometric with common ratio $(-\frac{1}{4})$. Therefore, by (5.4a),

$$S = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5}.$$

Chapter 6: Complex numbers

6.1. (a) $x = -1 \pm i$; (b) $x = 3 \pm i$; (c) $x = i$ or $-3i$.

6.2. The equation is a quadratic equation in x^2. Hence $x^2 = -4$ or 1. Taking square roots

$$x = \pm 1 \text{ or } \pm 2i.$$

6.3. The standard form of a complex number is $a + ib$, where a and b are real numbers. Thus the answers are (a) $4 + 3i$; (b) $3 - 5i$; (c) $-11 + 15i$; (d) $9 + 3i$; (e) $\frac{1}{2} + \frac{1}{2}i$; (f) $1 + 6i$; (g) $-3 - 4i$: (h) $-\frac{78}{25} - \frac{96}{25}i$; (i) $-4 - 4gm$.

6.4. The boundary between real and complex roots in the (p, q) plane is the parabola $p^2 = 4q$: the roots are real if $p^2 \geq 4q$ and complex if $p^2 < 4q$. The roots are both real and negative in the quadrant $p > 0$, $q < 0$.

6.5. (a) $4 + i$; (b) $5 + 5i$; (c) $\frac{1}{5} - \frac{7}{5}i$; (d) $-\frac{13}{25} - \frac{9}{25}i$.

6.6. (a) $-4i$; (b) $-7 + 4i$; (c) $-\frac{1}{5} + \frac{8}{5}i$; (d) $-\frac{1}{5} - \frac{8}{5}i$.

6.7. (a) $1 - i$; (b) $2i$; (c) $-2i$; (d) $\frac{1}{2}(1 + gm)$; (e) i.

6.8. Numerically to 3 decimal places the answers are: (a) $4.482 + 2.218i$; (b) $16.233 - 0.167i$; (c) $-1.248 + 2.728i$; (d) 88.669; (e) $266.050 + 0.512i$.

6.9. (a) $|z_1| = 2\sqrt{2}$, $\operatorname{Arg} z_1 = \frac{3}{4}\pi$; (b) $|z_2| = 8$, $\operatorname{Arg} z_2 = -\frac{1}{3}\pi$; (c) $|z_3| = 5$, $\operatorname{Arg} z_3 = -\frac{1}{2}\pi$; (d) $|z_4| = 3$, $\operatorname{Arg} z_4 = \pi$; (e) $|z_5| = 5$, $\operatorname{Arg} z_5 = \arctan(\frac{4}{3})$.

6.10. The curves are: (a) the circle $x^2 + y^2 = 1$; (b) the straight line $y = 2$; (c) The circle $(x - a_1)^2 + (y - a_2)^2 = 1$ where $a_1 = \operatorname{Re}(a)$ and $a_2 = \operatorname{Im}(a)$; (d) the parabola $y^2 = 4x$; (e) the ellipse $3x^2 + 4y^2 = 12$ (need to square twice to remove the square roots); the complex formula expresses the well-known property of ellipses that the sum of the distances from any point on an ellipse to the foci is a constant; (f) the straight line $y = x$ for $x \geq 0$; (g) the archimedean spiral $r = \theta$.

6.11. (a) $\sqrt{2}\exp(\frac{3}{4}\mathrm{i}\pi)$; (b) $2\exp(\mathrm{i}\pi)$; (c) $3\exp(-\frac{1}{2}\mathrm{i}\pi)$;
(d) $14\exp(-\frac{1}{3}\mathrm{i}\pi)$; (e) $2\sqrt{2}\exp(\mathrm{i}\theta)$ where $\theta = \arctan[(\sqrt{3}-1)/(\sqrt{3}+1)]$;
(f) $\frac{\sqrt{2}}{1+\sqrt{3}}\exp(-\frac{1}{4}\mathrm{i}\pi)$; (g) $e^2\exp(\mathrm{i})$;
(h) $\sqrt{2}\exp(\mathrm{i}\theta)$ where $\theta = \arctan[(\cos 2 + \sin 2)/(\cos 2 - \sin 2)]$; (i) $512\exp(\mathrm{i}\pi)$;
(j) $\sqrt{2}\exp(\frac{3}{4}\mathrm{i}\pi)$.

6.12. Use the identity

$$e^{\mathrm{i}(\theta_1+\theta_2)} = \mathrm{e}^{\mathrm{i}\theta_1}e^{\mathrm{i}\theta_2}.$$

Hence

$$\begin{aligned}&\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)\\ &\quad= (\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2)\\ &\quad= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2).\end{aligned}$$

Equating real and imaginary parts it follows that

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,$$

and

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.$$

For the other identities use

$$e^{\mathrm{i}(\theta_1-\theta_2)} = \mathrm{e}^{\mathrm{i}\theta_1}\mathrm{e}^{-\mathrm{i}\theta_2}.$$

6.13.

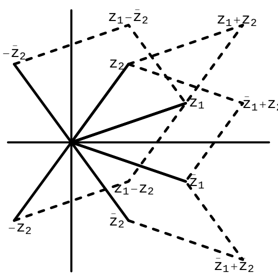

Figure 58: Problem: 6.13

6.14 For the general case with $f(\theta) = a\cos\theta + b\sin\theta$,

$$f'(\theta) = -a\sin\theta + b\cos\theta,$$

and

$$f''(\theta) = -a\cos\theta - b\sin\theta = -f(\theta).$$

The first case can be obtained by putting $a = 1$ and $b = \mathrm{i}$.

6.15. Using exponential forms for cos and sin, it follows that

$$\begin{aligned}\tan a &= \frac{\sin \mathrm{i}a}{\cos \mathrm{i}a} = \frac{2[\exp(a\mathrm{i}^2) - \exp(-a\mathrm{i}^2)]}{2\mathrm{i}[\exp(a\mathrm{i}^2) + \exp(-a\mathrm{i}^2)]}\\ &= \frac{1}{\mathrm{i}}\cdot\frac{\exp(-a) - \exp(a)}{\exp(-a) + \exp(a)} = \mathrm{i}\frac{\exp(a) - \exp(-a)}{\exp(a) + \exp(-a)}\\ &= \mathrm{i}\tanh a.\end{aligned}$$

6.16. (a) The equation $\cosh z = 1$ implies

$$\frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z}) = 1 \Rightarrow \mathrm{e}^{2z} - 2\mathrm{e}^z + 1 = 0 \Rightarrow (\mathrm{e}^z - 1)^2 = 0.$$

Hence $\mathrm{e}^z = 1$. If $z = a + b\mathrm{i}$ (a, b real), then $\mathrm{e}^a\mathrm{e}^{\mathrm{i}b} = 1 = \mathrm{e}^{2n\pi\mathrm{i}}$, $(n = 0, \pm 1, \pm 2, \ldots)$. Thus $a = 0$ and $b = 2n\pi$. The roots are given by $z = 2n\pi\mathrm{i}$, $(n = 0, \pm 1, \pm 2, \ldots)$.
(b) Similarly $\sinh z = 1$ implies

$$\mathrm{e}^{2z} - 2\mathrm{e}^z - 1 = 0.$$

Hence $\mathrm{e}^z = 1 \pm \surd 2$. If $z = a + b\mathrm{i}$, then

$$\mathrm{e}^{a+b\mathrm{i}} = \mathrm{e}^a(\cos b + \mathrm{i}\sin b) = 1 \pm \surd 2.$$

It follows that $\sin b = 0$ so that $b = n\pi$, $n = 0, \pm 1, \pm 2, \ldots)$. Hence

$$e^a \cos b = \mathrm{e}^a \cos(n\pi) = (-1)^n \mathrm{e}^a = 1 \pm \surd 2,$$

so that

$$a = \ln[\surd 2 - 1],\ (n \text{ odd})\ a = \ln[\surd 2 + 1]\ (n \text{ even})\ .$$

The complex roots are

$$z = \ln[\surd 2 - 1] + \mathrm{i}n\pi,\ (n \text{ odd})\ z = \ln[\surd 2 + 1] + \mathrm{i}n\pi\ (n \text{ even}).$$

(c) $\mathrm{e}^z = -1 = \mathrm{e}^{2n+1}\pi\mathrm{i}$, $(n = \ldots - 2, -1, 0, 1, 2, \ldots)$. It follows that the roots are

$$z = (2n + 1)\pi\mathrm{i} \quad (n = \ldots - 2, -1, 0, 1, 2, \ldots).$$

(d) $\cos z = \surd 2$ implies that

$$\frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) = \surd 2 \Rightarrow \mathrm{e}^{2\mathrm{i}z} - 2\surd 2\mathrm{e}^{\mathrm{i}z} + 1 = 0.$$

Hence

$$\mathrm{e}^{\mathrm{i}z} = \surd 2 \pm 1 = (\surd 2 \pm 1)e^{2n\pi\mathrm{i}}, \quad (n = 0, \pm 1, \pm 2, \ldots).$$

If $z = a + \mathrm{i}b$, then

$$\mathrm{e}^{-b} = (\surd 2 \pm 1),\ \text{and } a = 2n\pi.$$

Hence the roots are

$$z = 2n\pi - \mathrm{i}\ln[\surd 2 \pm 1].$$

6.17. (a) $\mathrm{Log}(1 + \mathrm{i}\surd 3) = \ln(\surd(1 + 3)) + \mathrm{i}\mathrm{Arg}\,(1 + \mathrm{i}\surd 3) = \ln 2 + \frac{1}{3}\mathrm{i}\pi$.
(b) We can write $\log z = \mathrm{Log}|z| + \mathrm{i}(\mathrm{Arg}\, z + 2k\pi)$ where k is an integer. Hence if $\log z = \pi\mathrm{i}$, then $|z| = 1$ and $Argz + 2k\pi = \pi$ so that $k = 0$ and $z = \pi\mathrm{i}$ is the only solution.
(c) $\mathrm{Log}(\mathrm{ei}) = \ln(\mathrm{e}) + \frac{1}{2}\pi\mathrm{i} = 1 + \frac{1}{2}\pi\mathrm{i}$.
(d) $\mathrm{e}^{\log z} = \mathrm{e}^{\ln r + \mathrm{i}\theta + 2k\pi\mathrm{i}} = \mathrm{e}^{\ln r}\mathrm{e}^{\mathrm{i}\theta} = r\mathrm{e}^{\mathrm{i}\theta} = z$. Therefore $\log z$ defines the *set of functions* inverse to e^z, as is suggested by the notation.

6.18. (a) $2^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\ln 2} = \cos(\ln 2) + \mathrm{i}\sin(\ln 2)$.
(b)

$$\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\ln\mathrm{i}} = \exp[\mathrm{i}\ln(\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}})] = \mathrm{e}^{-\frac{1}{2}\pi}.$$

This number is real: hence $\mathrm{Arg}(\mathrm{i}^{\mathrm{i}}) = 0$.
(c) The equation becomes

$$z^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\log z}\mathrm{e}^{\mathrm{i}[\mathrm{Log}|z| + \mathrm{i}(\mathrm{Arg}\, z + 2k\pi 0]} = \mathrm{e}^{-(\mathrm{Arg}\, z + 2k\pi)}\mathrm{e}^{\mathrm{Log}|z|} = -1 = \mathrm{e}^{(2n+1)\pi\mathrm{i}},$$

where n and k are integers. Hence

$$\mathrm{Log}\,|z| = (2n + 1)\pi, \quad \mathrm{Arg}\, z = 0, \quad k = 0.$$

Therefore, the solutions are given by $z = \mathrm{e}^{(2n+1)\pi}$, where n is any integer.

6.19. Write the equation as

$$z^5 = -1 = \mathrm{e}^{(2n+1)\pi \mathrm{i}}, \quad n, \text{ any integer.}$$

The solutions are given by

$$z = \exp\left[\frac{1}{5}(2n+1)\pi\mathrm{i}\right], \quad (n = 1, 2, 3, 4, 5).$$

Other values of n merely repeat these solutions. On the Argand diagram the solutions all lie on the unit circle centred at the origin at polar angles $\frac{1}{5}\pi$, $\frac{3}{5}\pi$, π, $\frac{7}{5}\pi$, $\frac{9}{5}\pi$.

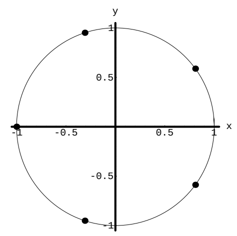

Figure 59: Problem: 6.19

6.20. Denote the complex number by z in each case.
(a) $z = 2\mathrm{e}^{3+2\mathrm{i}} = 2\mathrm{e}^3[\cos 2 + \mathrm{i}\sin 2]$. Hence

$$|z| = 2\mathrm{e}^3, \quad \operatorname{Arg} z = 2, \quad \operatorname{Re} z = 2\mathrm{e}^3\cos 2, \quad \operatorname{Im} z = 2\mathrm{e}^3 \sin 2.$$

(b) $z = 4\mathrm{e}^{\mathrm{i}} = 4[\cos 1 + \mathrm{i}\sin 1]$. Hence

$$|z| = 4, \quad \operatorname{Arg} z = 1, \quad \operatorname{Re} z = 4\cos 1, \quad \operatorname{Im} z = 4\sin 1.$$

(c) $z = 5\exp[\cos(\frac{1}{4}\pi) + \mathrm{i}\sin(\frac{1}{4}\pi)] = 5\exp(1/\sqrt{2})[\cos(1/\sqrt{2}) + \mathrm{i}\sin(1/\sqrt{2})]$. Hence

$$|z| = 5\exp(1/\sqrt{2}), \quad \operatorname{Arg} z = 1/\sqrt{2},$$

$$\operatorname{Re} z = 5\exp(1/\sqrt{2})\cos(1/\sqrt{2}, \quad \operatorname{Im} z = 5\exp(1/\sqrt{2}\sin(1/\sqrt{2}).$$

(d) $z = \mathrm{e}^{1+\mathrm{i}} = \mathrm{e}(\cos 1 + \mathrm{i}\sin 1)$. Hence

$$|z| = \mathrm{e}, \quad \operatorname{Arg} z = 1, \quad \operatorname{Re} z = \mathrm{e}\cos 1, \quad \operatorname{Im} z = \mathrm{e}\sin 1.$$

6.21. Let $z = c\mathrm{e}^{\alpha+\mathrm{i}\beta} = c\mathrm{e}^{\alpha}[\cos\beta + \mathrm{i}\sin\beta]$. Comparing with

$$x = 0.04\mathrm{e}^{-0.01t}\sin 12t,$$

we can identify

$$c = 0.04, \quad \alpha = -0.01t, \quad \beta = 12t + \frac{1}{2}\pi.$$

6.22. We have to express the sine as a cosine. Hence

$$\begin{aligned} i(t) &= c\mathrm{e}^{-0.05t}\sin(0.4t + 0.5) = c\mathrm{e}^{-0.05t}\cos(0.4t + 0.5 - \frac{1}{2}\pi), \\ &= \operatorname{Re}[c\mathrm{e}^{-0.05t}\mathrm{e}^{\mathrm{i}(0.4t+0.5-\frac{1}{2}\pi)}], \\ &= \operatorname{Re}[c\mathrm{e}^{-0.05t+\mathrm{i}(0.4t+0.5-\frac{1}{2}\pi)}] \end{aligned}$$

6.23. (a) $z^2 = (x + \mathrm{i}y)^2 = (x^2 - y^2) + \mathrm{i}2xy$. Hence

$$\mathrm{Re}(z^2) = x^2 - y^2, \quad \mathrm{Im}(z^2) = 2xy.$$

(b) First

$$z^3 = (x + \mathrm{i}y)(x^2 - y^2 + \mathrm{i}2xy) = (x^3 - 3xy^2) + \mathrm{i}(3x^2y - y^3).$$

Thus

$$z + 2z^2 + 3z^3 = (x + 2x^2 - 2y^2 + 3x^3 - 9xy^2) + \mathrm{i}(y + 4xy + 9x^2y - 3y^3).$$

Hence

$$\mathrm{Re}(z + 2z^2 + 3z^3) = x + 2x^2 - 2y^2 + 3x^3 - 9xy^2,$$
$$\mathrm{Im}(z + 2z^2 + 3z^3) = y + 4xy + 9x^2y - 3y^3.$$

(c)

$$\begin{aligned}
\sin z &= \sin(x + \mathrm{i}y) = \frac{1}{2\mathrm{i}}[\mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)} - \mathrm{e}^{-\mathrm{i}(x+\mathrm{i}y)}] \\
&= \frac{1}{2\mathrm{i}}[\mathrm{e}^{-y}\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{y}\mathrm{e}^{-\mathrm{i}x}] \\
&= \frac{1}{2\mathrm{i}}[\mathrm{e}^{-y}(\cos x + \mathrm{i}\sin x) - \mathrm{e}^{y}(\cos x - \mathrm{i}\sin x)] \\
&= \mathrm{e}^{-y}\sin x.
\end{aligned}$$

Hence

$$\mathrm{Re}(\sin z) = \mathrm{e}^{-y}\sin x, \quad \mathrm{Im}(\sin z) = 0.$$

(d)

$$\begin{aligned}
\cos z &= \frac{1}{2}[\mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)} + \mathrm{e}^{-\mathrm{i}(x+\mathrm{i}y)}] = \frac{1}{2}[\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{-y} + \mathrm{e}^{-\mathrm{i}x}\mathrm{e}^{y}] \\
&= \cos x \cosh y - \mathrm{i}\sin x \sinh y.
\end{aligned}$$

Hence

$$\mathrm{Re}(\cos z) = \cos x \cosh y, \quad \mathrm{Im}(\cos z) = -\sin x \sinh y.$$

(e) Using (d)

$$\begin{aligned}
e^z \cos z &= \frac{1}{2}\mathrm{e}^{x+\mathrm{i}y}[\cos x \cosh y - \mathrm{i}\sin x \sinh y] \\
&= \frac{1}{2}\mathrm{e}^{x}(\cos y + \mathrm{i}\sin y)[\cos x \cosh y - \mathrm{i}\sin x \sinh y] \\
&= \frac{1}{2}\mathrm{e}^{x}[(\cos y \cos x \cosh y + \sin y \sin x \sinh y) + \\
&\qquad (\sin y \cos x \cosh y - \cos y \sin x \sinh y)].
\end{aligned}$$

(f) $\exp(z^2) = \exp(x^2 - y^2)\exp(2xy\mathrm{i}) = \exp(x^2 - y^2)[\cos 2xy + \mathrm{i}\sin 2xy]$. Hence

$$\mathrm{Re}[\exp(z^2)] = \exp(x^2 - y^2 \cos 2xy, \quad \mathrm{Im}[\exp(z^2)] = \exp(x^2 - y^2 \sin 2xy.$$

6.24. $w = u + \mathrm{i}v = f(z) = z^2 = (x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i}$. Hence, equating real and imaginary parts

$$u = x^2 - y^2, \quad v = 2xy.$$

The hyperbolas map into the straight lines $u = 1$ and $v = 2$ respectively in the w plane.

6.25. Substituting for z it follows that

$$\begin{aligned}
w &= z + \frac{c}{z} = x + \mathrm{i}y + \frac{c(x - \mathrm{i}y)}{x^2 + y^2} \\
&= \left(x + \frac{cx}{x^2 + y^2}\right) + \mathrm{i}\left(y - \frac{cy}{x^2 + y^2}\right).
\end{aligned}$$

For the circle $|z| = 1$, $x^2 + y^2 = 1$, so that

$$w = x(1 + c) + \mathrm{i}y((1 - c).$$

Hence $u = x(1 + c)$ and $v = y(1 - c)$. Thus, on the circle $|z| = 1$

$$x^2 + y^2 = 1 = \frac{u^2}{(1+c)^2} + \frac{v^2}{(1-c)^2},$$

which is the equation of an ellipse.

6.26. The derivation of the formula for $\cos^6\theta$ is given in Example 6.20. For $\sin^6\theta$ use the identity

$$\sin n\theta = \frac{1}{2\mathrm{i}}\left(z^n - \frac{1}{z^n}\right),$$

where $z = \cos\theta + \mathrm{i}\sin\theta$. Then

$$\begin{aligned}
(2\sin\theta)^6 &= -\left(z - \frac{1}{z}\right)^6 \\
&= -\left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) - 15\left(z^2 + \frac{1}{z^2}\right) + 20 \\
&= 2\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 20\right).
\end{aligned}$$

Finally

$$\sin^6\theta = \tfrac{1}{32}(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 20).$$

6.27. The displacement is given by

$$x = \mathrm{Re}\, z = \mathrm{e}^{-0.2t}\cos 0.5t.$$

Hence $x = 0$ where $\cos 0.5t = 0$. The required zeros of x are given by

$$0.5t = \frac{1}{2}(2n+1)\pi, \qquad \text{for integer } n\ .$$

Hence $t = (2n+1)\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$.

The velocity is given by

$$\frac{dx}{dt} = \frac{d}{dx}\left[\mathrm{e}^{-0.2t}\cos 0.5t\right] = -\mathrm{e}^{-0.2t}[0.2\cos 0.5t + 0.5\sin 0.5t],$$

or, alternatively, by

$$\begin{aligned}
\mathrm{Re}\frac{dz}{dt} &= \mathrm{Re}\left[\frac{d}{dt}e^{(-0.2+0.5\mathrm{i})t}\right] \\
&= \mathrm{Re}\left[(-0.2 + 0.5\mathrm{i})e^{-0.2t}(\cos 0.5t + \mathrm{i}\sin 0.5t)\right] \\
&= -\mathrm{e}^{-0.2t}[0.2\cos 0.5t + 0.5\sin 0.5t].
\end{aligned}$$

6.28. If $z = 2 + \mathrm{i}$ is a solution then so is its conjugate $2 - \mathrm{i}$ since the coefficients of the polynomial are real. Therefore $(z - 2 - \mathrm{i})(z - 2 + \mathrm{i}) = z^2 - 4z + 5$ is a factor. Hence

$$z^4 - 2z^3 - z^2 + 2z + 10 = (z^2 - 4z + 5)(z^2 + 2z + 2).$$

The other solutions are given by $z^2 + 2z + 2 = 0$, that is, $z = -1 \pm \mathrm{i}$.

6.29. (a)

$$\begin{aligned}
S &= 1 - \sin\theta + \frac{1}{2!}\sin 2\theta - \frac{1}{3!}\sin\theta + \cdots \\
&= \mathrm{Im}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\mathrm{e}^{n\mathrm{i}\theta}}{n!}\right] = \mathrm{Im}\left[\sum_{n=0}^{\infty}\frac{z^n}{n!}\right],
\end{aligned}$$

where $z = -\mathrm{e}^{\mathrm{i}\theta}$. The infinite series is the Taylor series for the exponential function e^z (see Section 5.4). Hence

$$\begin{aligned} S &= 1 + \mathrm{Im}[\mathrm{e}^z] = 1 + \mathrm{Im}[\exp(-\mathrm{e}^{\mathrm{i}\theta})] \\ &= 1 + \mathrm{Im}[\exp(-\cos\theta - \mathrm{i}\sin\theta)] \\ &= 1 - \mathrm{e}^{-\cos\theta}\sin(\sin\theta). \end{aligned}$$

(b) In this case

$$\begin{aligned} T &= 1 + 2\cos\theta + \frac{2^2}{2!}\cos 2\theta + \frac{2^3}{3!}\cos 3\theta + \cdots \\ &= \sum_{n=0}^{\infty}\frac{2^n}{n!}\cos n\theta = \mathrm{Re}\left[\sum_{n=0}^{\infty}\frac{2^n}{n!}\mathrm{e}^{n\mathrm{i}\theta}\right] \end{aligned}$$

Using the Taylor series for an exponential function;

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{2^n}{n!}\mathrm{e}^{n\mathrm{i}\theta} &= \sum_{n=0}^{\infty}\frac{(2\mathrm{e}^{\mathrm{i}\theta})^n}{n!} = \exp[2\mathrm{e}^{\mathrm{i}\theta}] = \exp[2\cos\theta + 2\mathrm{i}\sin\theta] \\ &= \mathrm{e}^{2\cos\theta}[\cos(2\sin\theta) + \mathrm{i}\sin(2\sin\theta)] \end{aligned}$$

Hence

$$T = \mathrm{e}^{2\cos\theta}\cos(2\sin\theta).$$