Chapter 4

Angular momentum

Exercises

4.1 Since $l_y = zp_x - xp_z$ and $l_z = xp_y - yp_x$, one commutator is $[l_y, l_z] = [zp_x - xp_z, xp_y - yp_x]$.

Since in any representation, the operators x, y, z all commute and similarly the operators p_x, p_y, p_z all commute, this commutator is

$$[l_y, l_z] = (zp_xxp_y - xp_yzp_x) - (xxp_zp_y - p_zp_yxx) - (zyp_xp_x - p_xp_xzy) + (xyp_zp_x - p_zp_xxy)$$

Since, again in any representation, y commutes with p_x and p_z, x commutes with p_y and p_z, and z commutes with p_x and p_y, the above expression simplifies to

$$[l_y, l_z] = -zp_y[x,p_x] - 0 - 0 + yp_z[x,p_x]$$

In both the (a) position representation and (b) the momentum representation, $[x,p_x] = i\hbar$ so

$$[l_y, l_z] = i\hbar(-zp_y + yp_z) = \underline{i\hbar l_x}$$

Similarly,

$$[l_z, l_x] = [xp_y - yp_x, yp_z - zp_y]$$

$$= (xp_yyp_z - yp_zxp_y) - (yyp_xp_z - p_xp_zyy) - (xzp_yp_y - p_yp_yxz) + (yzp_xp_y - p_xp_yyz)$$

$$= -xp_z[y,p_y] - 0 - 0 + zp_x[y,p_y]$$

$$= i\hbar(-xp_z + zp_x) = \underline{i\hbar l_y}$$

4.2 $[l_z, l_-] = [l_z, l_x - il_y] = [l_z, l_x] - i[l_z, l_y] = i\hbar l_y - i(-i\hbar l_x) = \hbar(il_y - l_x) = -\hbar l_-$

$$[l_+, l_-] = [l_x + il_y, l_x - il_y] = [l_x, l_x] - i[l_x, l_y] + i[l_y, l_x] - [l_y, l_y]$$

$$= -i(i\hbar l_z) + i(-i\hbar l_z) = 2\hbar l_z$$

4.3 Using eqns 4.23 and 4.29:

(a)

$$l_+|3,3\rangle = (3\times 4 - 3\times 4)^{\frac{1}{2}}\hbar|3,4\rangle = 0$$

$$l_-|3,3\rangle = (3\times 4 - 3\times 2)^{\frac{1}{2}}\hbar|3,2\rangle = \sqrt{6}\,\hbar|3,2\rangle$$

(b)

$$l_+|3,-3\rangle = (3\times 4 - -3\times -2)^{\frac{1}{2}}\hbar|3,-2\rangle = \sqrt{6}\hbar|3,-2\rangle$$

$$l_-|3,-3\rangle = (3\times 4 - -3\times -4)^{\frac{1}{2}}\hbar|3,-4\rangle = 0$$

4.4 Using eqns 4.20 and 4.21:

(a)

$$j^2|3,2\rangle = (3\times 4)\hbar^2|3,2\rangle = 12\hbar^2|3,2\rangle$$

$$j_z|3,2\rangle = 2\hbar|3,2\rangle$$

(b)

$$j^2|1,-1\rangle = (1\times 2)\hbar^2|1,-1\rangle = 2\hbar^2|1,-1\rangle$$

$$j_z|1,-1\rangle = -\hbar|1,-1\rangle$$

4.5 **(a)** For $l = 4$, permitted values of m_l are 0, ±1, ±2, ±3, ±4.

(b) For $l = 5/2$, permitted values of m_l are ±1/2, ±3/2, ±5/2.

4.6 Two operators A and B are each other's Hermitian conjugate if

$$\langle a|A|b\rangle = \langle b|B|a\rangle^*$$

To confirm that s_+ and s_- are each other's Hermitian conjugate, we note the following (using eqn 4.34 and the orthonormality of the states α and β):

$$\langle\alpha|s_+|\alpha\rangle = \langle\beta|s_+|\alpha\rangle = \langle\beta|s_+|\beta\rangle = \langle\alpha|s_-|\alpha\rangle = \langle\beta|s_-|\beta\rangle = \langle\alpha|s_-|\beta\rangle = 0$$

The only non-zero matrix elements are

$$\langle\alpha|s_+|\beta\rangle = \hbar$$

$$\langle\beta|s_-|\alpha\rangle = \hbar$$

Therefore, since $\hbar$ is real,

$$\langle\alpha|s_+|\beta\rangle = \langle\beta|s_-|\alpha\rangle^*$$

and s_+ and s_- are each other's Hermitian conjugate.

4.7 **(a)** The matrix element is

$$\langle 1,0|j_z|1,0\rangle = 0\times\hbar\langle 1,0|1,0\rangle = 0$$

(b) The matrix element is

$$\langle 1,1|j_+|1,0\rangle = \left\langle 1,1\middle|\sqrt{1\times 2-0\times 1}\hbar\middle|1,1\right\rangle = \sqrt{2}\hbar$$

4.8 An electron has a spin angular momentum quantum number s = ½ and a quantum number for the z-component of $m_s = +1/2$ (α state) or $m_s = -1/2$ (β state). In general, the magnitude of the angular momentum is given by $\{s(s+1)\}^{1/2}\hbar$ and the z-component is $m_s\hbar$. Therefore, for both **(a)** the α state and **(b)** the β state, the magnitude of the spin angular momentum is

$$\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\,\hbar = \frac{\sqrt{3}}{2}\hbar = 9.133\times 10^{-35}\,\mathrm{Js}$$

and for the z-component: **(a)** $\hbar/2 = 5.273\times 10^{-35}$ Js, **(b)** $-\hbar/2 = -5.273\times 10^{-35}$ Js.

4.9 The three components of $\boldsymbol{j}$ are $j_x = j_{1x} + j_{2x}$; $j_y = j_{1y} + j_{2y}$; $j_z = j_{1z} + j_{2z}$. We have shown in eqn 4.38 one of the commutation relations of eqn 4.7, namely $[j_x, j_y] = \mathrm{i}\hbar j_z$. The other two commutation relations are confirmed as follows:

$$[j_y, j_z] = [j_{1y} + j_{2y}, j_{1z} + j_{2z}] = [j_{1y}, j_{1z}] + [j_{1y}, j_{2z}] + [j_{2y}, j_{1z}] + [j_{2y}, j_{2z}]$$
$$= \mathrm{i}\hbar j_{1x} + 0 + 0 + \mathrm{i}\hbar j_{2x} = \mathrm{i}\hbar j_z$$

$$[j_z, j_x] = [j_{1z} + j_{2z}, j_{1x} + j_{2x}] = [j_{1z}, j_{1x}] + [j_{1z}, j_{2x}] + [j_{2z}, j_{1x}] + [j_{2z}, j_{2x}]$$

$$= i\hbar j_{1y} + 0 + 0 + i\hbar j_{2y} = i\hbar j_y$$

4.10 A system with two sources of angular momentum, $j_1 = 1$ and $j_2 = 3/2$ can give rise to total angular momenta, using eqn 4.42, $j = 5/2, 3/2, 1/2$. States can be specified as either

$$|j_1 m_{j1}; j_2 m_{j2}\rangle \text{ or } |j_1 j_2; j m_j\rangle$$

$|j_1 m_{j1}; j_2 m_{j2}>$ states are |1,1; 3/2,3/2>, |1,0; 3/2,3/2>, |1,−1; 3/2,3/2>,

|1,1; 3/2,1/2>, |1,0; 3/2,1/2>, |1,−1; 3/2,1/2>,

|1,1; 3/2,−1/2>, |1,0; 3/2,−1/2>, |1,−1; 3/2,−1/2>,

|1,1; 3/2,−3/2>, |1,0; 3/2,−3/2>, |1,−1; 3/2,−3/2>

$|j_1 j_2; j m_j>$ states are |1, 3/2; 5/2, 5/2>, |1, 3/2; 5/2, 3/2>, |1, 3/2; 5/2, 1/2>,

|1, 3/2; 5/2, −1/2>, |1, 3/2; 5/2, −3/2>, |1, 3/2; 5/2, −5/2>,

|1, 3/2; 3/2, 3/2>, |1, 3/2; 3/2, 1/2>, |1, 3/2; 3/2, −1/2>,

|1, 3/2; 3/2, −3/2>, |1, 3/2; 1/2, 1/2>, |1, 3/2; 1/2, −1/2>

4.11 For a p-electron, $l = 1$ and $s = 1/2$. We can construct the state $|j, m_j>$ from the uncoupled states $|l, m_l; s, m_s>$ using the vector coupling coefficients of *Resource section* 2:

$$|1/2, 1/2> = (2/3)^{1/2}|1,1;1/2,-1/2> - (1/3)^{1/2}|1,0;1/2,+1/2>$$

4.12 Couple the three spin angular momenta $s_1 = ½$, $s_2 = ½$, $s_3 = ½$. Coupling of s_1 and s_2 yields angular momenta of 1 and 0. Now couple each of these values with s_3. Permitted values from coupling s_3 and 1 are 3/2 and 1/2. Permitted values from coupling s_3 and 0 are 1/2. So the net result is 3/2, 1/2 (twice).

Problems

4.1 **(a)**

$$[l_x, l_y] = (\hbar/\mathrm{i})^2[y(\partial/\partial z) - z(\partial/\partial y), z(\partial/\partial x) - x(\partial/\partial z)] \quad \text{[eqn 4.5]}$$

$$= (\hbar/\mathrm{i})^2\{[y(\partial/\partial z), z(\partial/\partial x)] + [z(\partial/\partial y), x(\partial/\partial z)]\}$$

$$= (\hbar/\mathrm{i})^2\{y[(\partial/\partial z), z](\partial/\partial x) + x[z, (\partial/\partial z)](\partial/\partial y)\}$$

$$[(\partial/\partial z), z] = (\partial/\partial z)z - z(\partial/\partial z) = 1 + z(\partial/\partial z) - z(\partial/\partial z) = 1$$

Therefore,

$$[l_x, l_y] = (\hbar/\mathrm{i})^2\{y(\partial/\partial x) - x(\partial/\partial y)\} = -(\hbar/\mathrm{i})l_z \quad \text{[eqn 4.6]}$$

$$= \underline{\mathrm{i}\hbar l_z}$$

(b)

$$[l_x, l_y] = [yp_z - zp_y, zp_x - xp_z]$$

$$= y[p_z, z]p_x + p_y[z, p_z]x$$

$$= y\left[p_z, -\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial p_z}\right]p_x + p_y\left[-\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial p_z}, p_z\right]x$$

$$= \left(-\frac{\hbar}{\mathrm{i}}\right)\left\{y\left[p_z, \frac{\partial}{\partial p_z}\right]p_x - p_y\left[p_z, \frac{\partial}{\partial p_z}\right]x\right\}$$

$$\left[p_z, \frac{\partial}{\partial p_z}\right] = p_z\frac{\partial}{\partial p_z} - \frac{\partial}{\partial p_z}p_z$$

$$= p_z\frac{\partial}{\partial p_z} - 1 - p_z\frac{\partial}{\partial p_z}$$

$$= -1$$

Therefore,

$$[l_x, l_y] = \left(-\frac{\hbar}{\mathrm{i}}\right)\{-yp_x + p_y x\}$$

$$= \left(-\frac{\hbar}{\mathrm{i}}\right) l_z = \mathrm{i}\hbar l_z$$

Exercise: Evaluate $[l_y, l_x]$ in the position representation.

4.4

$$\boldsymbol{l} \times \boldsymbol{l} = \begin{bmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ l_x & l_y & l_z \\ l_x & l_y & l_z \end{bmatrix}$$

$$= \hat{\boldsymbol{i}}\ (l_y l_z - l_z l_y) - \hat{\boldsymbol{j}}\ (l_x l_z - l_z l_x) + \hat{\boldsymbol{k}}\ (l_x l_y - l_y l_x)$$

$$\mathrm{i}\hbar\boldsymbol{l} = \hat{\boldsymbol{i}}\ (\mathrm{i}\hbar l_x) + \hat{\boldsymbol{j}}\ (\mathrm{i}\hbar l_y) + \hat{\boldsymbol{k}}\ (\mathrm{i}\hbar l_z)$$

Hence, equating both sides term by term reproduces the commutation rules, eqn 4.7.

Exercise: Show that if $\boldsymbol{l}_1 \times \boldsymbol{l}_1 = \mathrm{i}\hbar\boldsymbol{l}_1$ and $\boldsymbol{l}_2 \times \boldsymbol{l}_2 = \mathrm{i}\hbar\boldsymbol{l}_2$, then $\boldsymbol{l} \times \boldsymbol{l} = \mathrm{i}\hbar\boldsymbol{l}$, where $\boldsymbol{l} = \boldsymbol{l}_1 + \boldsymbol{l}_2$,

but only if $[l_{1q}, l_{2q'}] = 0$ for all q, q'.

4.7 (a)

$$[s_x, s_y] = \left(\frac{1}{2}\hbar\right)^2 [\sigma_x, \sigma_y]$$

$$= \left(\frac{1}{2}\hbar\right)^2 \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

$$= \left(\frac{1}{2}\hbar\right)^2 \left\{\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} - \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\right\} = \frac{1}{2}\mathrm{i}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underline{\mathrm{i}\hbar s_z}$$

(b)

$$s^2 = s_x^2 + s_y^2 + s_z^2 = \left(\frac{1}{2}\hbar\right)^2 \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \right\}$$

$$= \left(\frac{1}{2}\hbar\right)^2 \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of s^2 are therefore $\frac{3}{4}\hbar^2$; hence, identifying these with $s(s + 1)\hbar^2$ identified $s = \frac{1}{2}$.

Exercise: Confirm that the following matrices constitute a representation of an angular momentum with $l = 1$.

$$l_x = (1/\sqrt{2})\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ l_y = (\mathrm{i}/\sqrt{2})\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ l_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

4.10 Suppose $[l_x, l_y] = -\mathrm{i}\hbar l_z$, $l_\pm = l_x \pm \mathrm{i}l_y$; then

$$[l_+, l_z] = \hbar l_+,\ [l_-, l_z] = -\hbar l_-, \text{ and } [l^+, l^-] = -2\hbar l_z$$

Then, following the development that led to eqn 4.17,

$$l_z l_+|l, m\rangle = \{l_+ l_z + [l_z, l_+]\}|l, m\rangle = \{l_+ l_z - \hbar l_+\}|l, m\rangle$$

$$= \{l_+ m\hbar - \hbar l_+\}|l, m\rangle = (m - 1)\hbar l_+|l, m\rangle$$

Consequently, $l_+|l, m\rangle \propto |l, m - 1\rangle$ and $l_-|l, m\rangle \propto |l, m + 1\rangle$; therefore l_+ is a *lowering* operator and l_- is a *raising* operator.

Exercise: Find a matrix representation of these l_x and l_y ‘angular momenta’ corresponding to $L = 1$ (draw on the matrices in the *Exercise* to Problem 4.7).

4.13 In each case $l = 1$ and $\mathrm{p}_x = (\mathrm{p}_- - p_+)/\sqrt{2}$, $\mathrm{p}_y = (\mathrm{p}_- + \mathrm{p}_+)(\mathrm{i}/\sqrt{2})$. Then $\mathrm{p}_+ \to |1, 1\rangle$, $\mathrm{p}_- \to |1, -1\rangle$ and $\mathrm{p}_z \to |1, 0\rangle$ in the notation $|l, m_l\rangle$. The l label will be omitted henceforth.

(a)

$$\langle p_x|l_z|p_y\rangle = (i/2)\{\langle -1| - \langle 1|\}l_z\{|-1\rangle + |1\rangle\}$$

$$= (i/2)\{\langle -1|l_z|-1\rangle - \langle 1|l_z|1\rangle\} = (i/2)\{-\hbar - \hbar\}$$

$$= \underline{-i\hbar}$$

(b) $$\langle p_x|l_+|p_y\rangle = (i/2)\{\langle -1| - \langle 1|\}l_+\{|-1\rangle + |1\rangle\} = 0$$

(c)

$$\langle p_z|l_y|p_x\rangle = (1/2i)(1/\sqrt{2})\langle 0|(l_+ - l_-)\{|-1\rangle - |1\rangle\}$$

$$= (1/2i\sqrt{2})\{\langle 0|l_+|-1\rangle + \langle 0|l_-|1\rangle\}$$

$$= (1/2i\sqrt{2})\{\hbar\sqrt{2} + \hbar\sqrt{2}\} = \underline{-i\hbar}$$

(d)

$$\langle p_z|l_x|p_y\rangle = (1/2)(i/\sqrt{2})\langle 0|(l_+ + l_-)\{|-1\rangle + |1\rangle\}$$

$$= (i/2\sqrt{2})\{\langle 0|l_+|-1\rangle + |\langle 0|l_-|1\rangle\}$$

$$= (i/2\sqrt{2})\{\hbar\sqrt{2} + \hbar\sqrt{2}\} = \underline{i\hbar}$$

(e)

$$\langle p_z|l_x|p_x\rangle = (1/2)(1/\sqrt{2})\langle 0|(l_+ + l_-)\{|-1\rangle - |1\rangle\}$$

$$= (1/2\sqrt{2})\{\langle 0|l_+|-1\rangle - \langle 0|l_-|1\rangle\} = \underline{0}$$

Exercise: Evaluate $\langle p_y|l_-|p_z\rangle$, $\langle p_x|l_y|p_z\rangle$, $\langle p_x|l_+l_-|p_z\rangle$, and $\langle d_{xy}|l_x|d_{xz}\rangle$.

4.16

$$[l_x, l_y] = \hbar^2[\sin\phi(\partial/\partial\theta) + \cot\theta\cos\phi(\partial/\partial\phi), \cos\phi(\partial/\partial\theta) - \cot\theta\sin\phi(\partial/\partial\phi)]$$

$$= \hbar^2\{-[\sin\phi(\partial/\partial\theta), \cot\theta\sin\phi(\partial/\partial\phi)]$$

$+ [\cot\theta\cos\phi(\partial/\partial\phi), \cos\phi(\partial/\partial\theta)] - [\cot\theta\cos\phi(\partial/\partial\phi), \cot\theta\sin\phi\,(\partial/\partial\phi)]$

$+ [\sin\phi(\partial/\partial\theta), \cos\phi\,(\partial/\partial\theta)]\}$

$[\sin\phi(\partial/\partial\theta), \cos\phi(\partial/\partial\theta)] = 0$

$[\sin\phi(\partial/\partial\theta), \cot\theta\sin\phi(\partial/\partial\phi)]$

$= \sin\phi(\partial/\partial\theta)\cot\theta\sin\phi(\partial/\partial\phi) - \cot\theta\sin\phi(\partial/\partial\phi)\sin\phi(\partial/\partial\theta)$

$= \sin^2\phi(\partial\cot\theta/\partial\theta)(\partial/\partial\phi) + \sin^2\phi\cot\theta(\partial^2/\partial\theta\partial\phi)$

$- \cot\theta\sin\phi(\partial\sin\phi/\partial\phi)(\partial/\partial\theta) - \cot\theta\sin^2\phi(\partial^2/\partial\theta\partial\phi)$

$= -\sin^2\phi\,\mathrm{cosec}^2\theta(\partial/\partial\phi) - \cot\theta\sin\phi\cos\phi(\partial/\partial\theta)$

$[\cot\theta\cos\phi(\partial/\partial\phi), \cos\phi(\partial/\partial\theta)]$

$= \cot\theta\cos\phi(\partial/\partial\phi)\cos\phi(\partial/\partial\theta) - \cos\phi(\partial/\partial\theta)\cot\theta\cos\phi(\partial/\partial\phi)$

$= \cot\theta\cos\phi(\partial\cos\phi/\partial\phi)(\partial/\partial\theta) + \cot\theta\cos^2\phi(\partial^2/\partial\phi\partial\theta)$

$- \cos^2\phi(\partial\cot\theta/\partial\theta)(\partial/\partial\phi) - \cos^2\phi\cot\theta(\partial^2/\partial\theta\partial\phi)$

$= -\cot\theta\cos\phi\sin\phi(\partial/\partial\theta) + \cos^2\phi\,\mathrm{cosec}^2\theta(\partial/\partial\phi)$

$[\cot\theta\cos\phi(\partial/\partial\phi), \cot\theta\sin\phi(\partial/\partial\phi)]$

$= \cot^2\theta[\cos\phi(\partial/\partial\phi), \sin\phi(\partial/\partial\phi)]$

$= \cot^2\theta\{\cos\phi(\partial/\partial\phi)\sin\phi(\partial/\partial\phi) - \sin\phi(\partial/\partial\phi)\cos\phi(\partial/\partial\phi)\}$

$= \cot^2\theta\{\cos^2\phi(\partial/\partial\phi) + \sin^2\phi(\partial/\partial\phi)\} = \cot^2\theta(\partial/\partial\phi)$

$[l_x, l_y] = \hbar^2\{\sin^2\phi\,\mathrm{cosec}^2\theta(\partial/\partial\phi) + \cot\theta\sin\phi\cos\phi(\partial/\partial\theta)$

$+ \cos^2\phi\,\mathrm{cosec}^2\theta(\partial/\partial\phi) - \cot\theta\cos\phi\sin\phi(\partial/\partial\theta) - \cot^2\theta(\partial/\partial\phi)\}$

$= \hbar^2\{(\sin^2\phi + \cos^2\phi)\mathrm{cosec}^2\theta(\partial/\partial\phi) - \cot^2\theta(\partial/\partial\phi)\}$

$= \hbar^2\{\mathrm{cosec}^2\theta(\partial/\partial\phi) - \cot^2\theta(\partial/\partial\phi)\} = \hbar^2(\partial/\partial\phi) = \underline{i\hbar l_z}$

as required.

$$[l_+, l_-] = \left[\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), -\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\right]$$

$$= -\hbar^2\left[e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\right]$$

$$= -\hbar^2\left\{\left[e^{i\phi}\frac{\partial}{\partial\theta}, e^{-i\phi}\frac{\partial}{\partial\theta}\right] + \left[e^{i\phi} i\cot\theta\frac{\partial}{\partial\phi}, e^{-i\phi}\frac{\partial}{\partial\theta}\right]\right.$$

$$\left. - \left[e^{i\phi}\frac{\partial}{\partial\theta}, e^{-i\phi} i\cot\theta\frac{\partial}{\partial\phi}\right] - \left[e^{i\phi} i\cot\theta\frac{\partial}{\partial\phi}, e^{-i\phi} i\cot\theta\frac{\partial}{\partial\phi}\right]\right\}$$

$$= -\hbar^2\left\{ie^{i\phi}\cot\theta\frac{\partial}{\partial\phi}e^{-i\phi}\frac{\partial}{\partial\theta} - ie^{-i\phi}\frac{\partial}{\partial\theta}e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\right.$$

$$- ie^{i\phi}\frac{\partial}{\partial\theta}e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi} + ie^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}e^{i\phi}\frac{\partial}{\partial\theta}$$

$$\left. + e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi} - e^{-i\phi}\cot\theta\frac{\partial}{\partial\phi}e^{i\phi}\cot\theta\frac{\partial}{\partial\phi}\right\}$$

$$= -\hbar^2\left\{\cot\theta\frac{\partial}{\partial\theta} + i\,\mathrm{cosec}^2\theta\frac{\partial}{\partial\phi} + i\,\mathrm{cosec}^2\theta\frac{\partial}{\partial\phi}\right.$$

$$\left. - \cot\theta\frac{\partial}{\partial\theta} - 2i\cot^2\theta\frac{\partial}{\partial\phi}\right\}$$

$$= -2i\hbar^2\{\mathrm{cosec}^2\theta - \cot^2\theta\}\frac{\partial}{\partial\phi}$$

$$= -2i\hbar^2\frac{\partial}{\partial\phi}$$

$$= 2\hbar l_z$$

Exercise: Evaluate $[l_z, l_x]$ in this representation.

4.19

$$\boldsymbol{j}_1 \times \boldsymbol{j}_2 = \begin{bmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ j_{1x} & j_{1y} & j_{1z} \\ j_{2x} & j_{2y} & j_{2z} \end{bmatrix}$$

$$= \hat{\boldsymbol{i}}\ (j_{1y}j_{2z} - j_{1z}j_{2y}) - \hat{\boldsymbol{j}}\ (j_{1x}j_{2z} - j_{1z}j_{2x}) + \hat{\boldsymbol{k}}\ (j_{1x}j_{2y} - j_{1y}j_{2x})$$

$$= \hat{\boldsymbol{i}}\ (j_{2z}j_{1y} - j_{2y}j_{1z}) - \hat{\boldsymbol{j}}\ (j_{2z}j_{1x} - j_{2x}j_{1z}) + \hat{\boldsymbol{k}}\ (j_{2y}j_{1x} - j_{2x}j_{1y})$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -j_{2x} & -j_{2y} & -j_{2z} \\ j_{1x} & j_{1y} & j_{1z} \end{vmatrix} = -\boldsymbol{j}_2 \times \boldsymbol{j}_1$$

$$\boldsymbol{j} \times \boldsymbol{j} = (\boldsymbol{j}_1 + \boldsymbol{j}_2) \times (\boldsymbol{j}_1 + \boldsymbol{j}_2)$$

$$= \boldsymbol{j}_1 \times \boldsymbol{j}_1 + \boldsymbol{j}_2 \times \boldsymbol{j}_2 + \boldsymbol{j}_1 \times \boldsymbol{j}_2 + \boldsymbol{j}_2 \times \boldsymbol{j}_1 = \boldsymbol{j}_1 \times \boldsymbol{j}_1 + \boldsymbol{j}_2 \times \boldsymbol{j}_2$$

$$= \mathrm{i}\hbar\boldsymbol{j}_1 + \mathrm{i}\hbar\boldsymbol{j}_2 = \mathrm{i}\hbar\boldsymbol{j}$$

Exercise: Under what circumstances do $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ satisfy the vector relations set out in eqn 4.9?

4.22 $j_1 = 1, j_2 = \frac{1}{2}$ gives the states $j = \frac{3}{2}, \frac{1}{2}$. The state $|j, m_j\rangle = |\frac{3}{2}, +\frac{3}{2}\rangle$ is

$$|j_1, m_{j1}; j_2, m_{j2}\rangle = |1, +1;\ \tfrac{1}{2}, +\tfrac{1}{2}\rangle.$$

Generate $|\frac{3}{2}, +\frac{1}{2}\rangle$ using $j_-|\frac{3}{2}, \frac{3}{2}\rangle = \hbar\sqrt{3}\,|\frac{3}{2}, +\frac{1}{2}\rangle$ and

$$j_-|\tfrac{3}{2}, \tfrac{3}{2}\rangle = (j_{1-} + j_{2-})\,|1, +1; \tfrac{1}{2}, +\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{2}\,|1, 0; \tfrac{1}{2}, +\tfrac{1}{2}\rangle + \hbar\,|1, +1; \tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

Therefore,

$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \sqrt{(2/3)}\,|1,0;\tfrac{1}{2},+\tfrac{1}{2}\rangle + \sqrt{(1/3)}\,|1,+1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Next, generate $|\tfrac{3}{2},-\tfrac{1}{2}\rangle$ by using

$$j_-\,|\tfrac{3}{2},+\tfrac{1}{2}\rangle = 2\hbar\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

and

$$\begin{aligned}
j_-\,|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= (j_{1-}+j_{2-})\left\{\sqrt{(2/3)}\,|1,0;\tfrac{1}{2},+\tfrac{1}{2}\rangle + \sqrt{(1/3)}\,|1,+1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right\} \\
&= \sqrt{(2/3)}\sqrt{2}\hbar\,|1,-1;\tfrac{1}{2},+\tfrac{1}{2}\rangle + \sqrt{(2/3)}\hbar\,|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle \\
&\quad + \sqrt{(1/3)}\sqrt{2}\hbar\,|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle + 0 \\
&= (2/\sqrt{3})\hbar\,|1,-1;\tfrac{1}{2},+\tfrac{1}{2}\rangle + (2\sqrt{2}/\sqrt{3})\hbar\,|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle
\end{aligned}$$

Therefore,

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{(1/3)}\,|1,-1;\tfrac{1}{2},+\tfrac{1}{2}\rangle + \sqrt{(2/3)}\,|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

We could generate $|\tfrac{3}{2},-\tfrac{3}{2}\rangle$ using $j_-\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle$ or, more simply, by noting that there is only one way to achieve this state since $m_j = m_{j_1} + m_{j_2}$. Therefore,

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |1,-1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

The state $|\tfrac{1}{2},+\tfrac{1}{2}\rangle$ is orthogonal to $|\tfrac{3}{2},+\tfrac{1}{2}\rangle$ so we require

$$\begin{aligned}
\langle\tfrac{1}{2},+\tfrac{1}{2}\,|\,\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \{a\langle 1,0;\tfrac{1}{2},+\tfrac{1}{2}\,| + b\langle 1,+1;\tfrac{1}{2},-\tfrac{1}{2}\,|\} \\
&\quad \times\left\{\sqrt{(2/3)}\,|1,0;\tfrac{1}{2},+\tfrac{1}{2}\rangle + \sqrt{(1/3)}\,|1,+1;\tfrac{1}{2},-\tfrac{1}{2}\rangle\right\} \\
&= \sqrt{(2/3)}a + \sqrt{(1/3)}b = 0
\end{aligned}$$

Therefore $a = -\sqrt{(1/3)}, b = \sqrt{(2/3)}$ and

$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = -\sqrt{(1/3)}\,|1,0;\tfrac{1}{2},+\tfrac{1}{2}\rangle + \sqrt{(2/3)}\,|1,+1;\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

The remaining $|\tfrac{1}{2},-\tfrac{1}{2}\rangle$ may be generated using $j_-\,|\tfrac{1}{2},+\tfrac{1}{2}\rangle$ and yields

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{(1/3)}\,|1,0;\tfrac{1}{2},-\tfrac{1}{2}\rangle + -\sqrt{(2/3)}\,|1,-1;\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

For the matrix elements, write

$$|\,j,m_j\rangle = \sum_{m_1m_2} c(j_1m_1j_2m_2\,|\,jm_j)\,|\,j_1m_1j_2m_2\rangle$$

Then

$$\langle j',m_j'\,|\,j_{1z}\,|\,j,m_j\rangle = \sum_{m_1m_2}\sum_{m_1'm_2'} c(j_1m_1'j_2m_2'\,|\,j'm_j')c(j_1m_1j_2m_2\,|\,jm_j)\langle j_1m_1'j_2m_2'\,|\,j_{1z}\,|\,j_1m_1j_2m_2\rangle$$

$$= \sum_{m_1m_2}\sum_{m_1'm_2'} c(j_1m_1'j_2m_2'\,|\,j'm_j')c(j_1m_1j_2m_2\,|\,jm_j)m_1\hbar$$

$$= \delta_{m_1m_1'}\delta_{m_2m_2'}\hbar\sum_{m_1m_2} c(j_1m_1j_2m_2\,|\,j'm_j')c(j_1m_1j_2m_2\,|\,jm_j)m_1$$

$$= \hbar\sum_{m_1m_2} c(j_1m_1j_2m_2\,|j'm_j)c(j_1m_1j_2m_2\,|\,jm_j)m_1$$

Individual numerical values may now be obtained by substituting the coefficients.

Exercise: Repeat the procedure for $j_1 = 2, j_2 = \frac{1}{2}$.

4.25 Refer to Fig. 4.3.

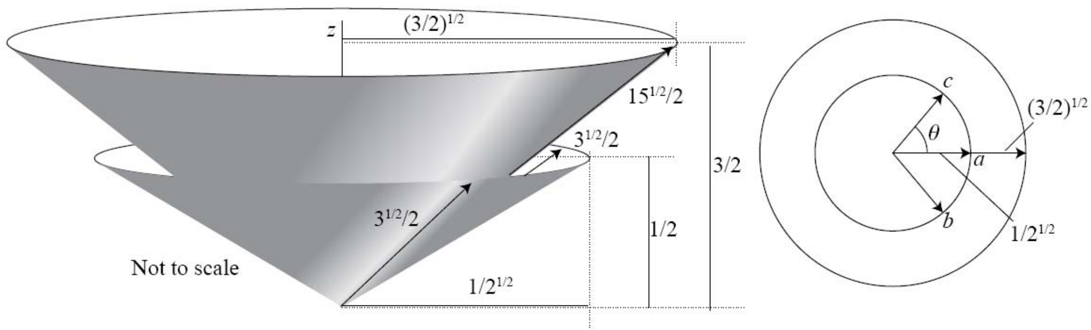

Figure 4.3: The construction used in Problem 4.25.

We shall interpret the question as requiring the angle between the spins projected onto the *xy*-plane. The projection of a spin-$\frac{1}{2}$ vector (of length $\frac{1}{2}\sqrt{3}$) onto the *xy*-plane, given its projection of $\frac{1}{2}$ onto the *z*-axis, is $(\frac{3}{4}-\frac{1}{4})^{1/2}=1/\sqrt{2}$. Similarly, the projection of a spin-$\frac{3}{2}$ vector (of length $\frac{1}{2}\sqrt{15}$), with $m_z=+\frac{3}{2}$, is $(\frac{15}{4}-\frac{9}{4})^{1/2}=\sqrt{\frac{3}{2}}$. Therefore, the projection of the resultant of *b* and *c* must account for $\sqrt{3/2}-\sqrt{1/2}$. Consequently,

$$2\times\frac{1}{\sqrt{2}}\cos\tfrac{1}{2}\theta=(3/2)^{1/2}-(1/2)^{1/2}$$

or

$$\cos\tfrac{1}{2}\theta=\frac{\sqrt{3}-1}{2},\ \theta=2\arccos\left(\frac{\sqrt{3}-1}{2}\right)$$

The angle between *b* and *c* is therefore <u>137.06°</u>, and that between *a* and *c* (and *a* and *b*) is one-half this angle, or <u>68.529°</u>.

For the second part, we shall calculate the actual inter-spin angle, θ, by noting that by symmetry the angles between a, b, and c are all the same. Hence

$$(\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3)\cdot(\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3) = s_1^2 + s_2^2 + s_3^2 + 2\mathbf{s}_1\cdot\mathbf{s}_3 + 2\mathbf{s}_2\cdot\mathbf{s}_3 + 2\mathbf{s}_1\cdot\mathbf{s}_2$$

$$= 3\times\tfrac{3}{4} + 3\times\tfrac{3}{2}\cos\theta = \tfrac{9}{4} + \tfrac{9}{2}\cos\theta$$

$$= S^2 = \tfrac{15}{4}$$

Therefore,

$$\tfrac{9}{2}\cos\theta = \tfrac{3}{2}, \text{so } \theta = \arccos(\tfrac{1}{3}),\ \text{ or } \underline{70.53°}$$

Exercise: Show, by the second method, that for n spins, the angle between vectors is 70.53° in the state with maximum S and M_s for all n.

4.28

$$\langle j_1 j_2; jm_j | j_1 j_2; jm_j\rangle$$

$$= \sum_{m'_{j1}}\sum_{m'_{j2}}\sum_{m_{j1}}\sum_{m_{j2}} C^*_{m'_{j1}m'_{j2}} C_{m_{j1}m_{j2}} \langle j_1 m'_{j1} j_2 m'_{j2} \mid j_1 m_{j1} j_2 m_{j2}\rangle$$

$$= \sum_{m'_{j1}}\sum_{m'_{j2}}\sum_{m_{j1}}\sum_{m_{j2}} C^*_{m'_{j1}m'_{j2}} C_{m_{j1}m_{j2}} \delta_{m'_{j1}m_{j1}} \delta_{m'_{j2}m_{j2}}$$

$$= \sum_{m_{j1}}\sum_{m_{j2}} C^*_{m_{j1}m_{j2}} C_{m_{j1}m_{j2}} = \sum_{m_{j1},m_{j2}} |C_{m_{j1}m_{j2}}|^2$$

But $\langle j_1 j_2; jm_j | j_1 j_2; jm_j\rangle = 1$, which completes the proof.

Exercise: Find a general expression for $\langle j_1 j_2; jm_j | j_{1z} | j_1 j_2; jm_j\rangle$ and evaluate it for $\langle G, M_L | l_{1z} | G, M_L\rangle$; see Problem 4.27.