Chapter 2

Linear motion and the harmonic oscillator

Exercises

2.1 For the energy in **(a)** use $E = e\mathbb{V}$.

(a) $k = (2m_e e\mathbb{V}/\hbar^2)^{1/2} = (5.123 \times 10^9\ \text{m}^{-1}) \times (\mathbb{V}/\text{Volt})^{1/2}$.

(i) $\mathbb{V} = 1.0$ V; $k = 5.123 \times 10^9\ \text{m}^{-1} = 5.123\ \text{nm}^{-1}$;

$\underline{\psi(x) = A\exp\{5.123\,\text{i}(x/\text{nm})\}}$, $A = 1/L^{1/2}$, $L \to \infty$.

(ii) $\mathbb{V} = 10$ kV; $k = 5.123 \times 10^{11}\ \text{m}^{-1} = 0.5123\ \text{pm}^{-1}$;

$\underline{\psi(x) = A\exp\{0.5123\,\text{i}(x/\text{pm})\}}$

(b) Because $p = (1.0\ \text{g}) \times (10\ \text{m s}^{-1}) = 1.0 \times 10^{-2}\ \text{kg m s}^{-1}$,

$k = p/\hbar$[eqn 2.7] $= 9.48 \times 10^{31}\ \text{m}^{-1}$; hence

$$\underline{\psi(x) = A\exp\{9.48\text{i} \times 10^{31}(x/\text{m})\}}$$

Exercise: What value of $\mathbb{V}$ is needed to accelerate an electron so that its wavelength is equal to its Compton wavelength?

2.2 In each case $\underline{|\psi(x)|^2 = A^2}$, a constant ($A^2 = 1/L$; $L \to \infty$)

2.3 Substituting eqn 2.5 for ψ in eqn 2.4 yields:

$$-\frac{\hbar^2}{2m}\frac{\text{d}^2}{\text{d}x^2}\left(A\text{e}^{\text{i}kx} + B\text{e}^{-\text{i}kx}\right) = \frac{\hbar^2 k^2}{2m}\left(A\text{e}^{\text{i}kx} + B\text{e}^{-\text{i}kx}\right)$$

confirming that the wavefunction is an eigenfunction with eigenvalue $\hbar^2k^2/2m$. The relation between k and E given in eqn 2.5 then follows. Similar substitution of eqn 2.6 for ψ in eqn 2.4 yields:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(C\cos kx + D\sin kx) = \frac{\hbar^2 k^2}{2m}(C\cos kx + D\sin kx)$$

and the wavefunction is seen to satisfy eqn 2.4.

2.4 The flux density for the wavefunction A sinkx is, using eqn 2.11,

$$J_x\,(A\sin kx) = \frac{1}{2m}\left(\frac{\hbar k}{\mathrm{i}}A^*\sin kx A\cos kx + \frac{\hbar k}{-\mathrm{i}}A\sin kx A^*\cos kx\right) = 0$$

2.5 Use the expression as given in the *brief illustration* in Section 2.7 for the penetration depth $1/\kappa$:

$$\frac{1}{\kappa} = \frac{\hbar}{\{2m(V-E)\}^{1/2}}$$
$$= \frac{1.055\times10^{-34}\ \mathrm{Js}}{\{2\times 9.109\times10^{-31}\ \mathrm{kg}\times(2.0\ \mathrm{eV} - E)\times 1.602\times10^{-19}\ \mathrm{J/eV}\}^{1/2}}$$
$$= 4.0\times10^{-10}\ \mathrm{m}$$

Solving for the kinetic energy yields E = 1.76 eV.

2.6 The transmission probability is given in eqn 2.26. Using the Worksheet entitled Equation 2.26 on the text's website and setting

$m = m_e$ (so that m/m_e=1)

$E = V_0$ = 2.0 eV (so that $E/E_h = V_0/E_h = 0.073499$)

$\beta = 1.0\times10^{10}\ \mathrm{m}^{-1}$ (so that $\beta/(1/a_0) = 0.529177$)

yields $T = 6.361\times10^{-1}$.

2.7 $\psi_4 = (2/L)^{1/2}\sin(4\pi x/L) = 0$ when $x = 0, \frac{1}{4}L, \frac{1}{2}L, \frac{3}{4}L, L,$ of which the central three are nodes.

Exercise: Repeat the question for n = 6.

2.8 To show that the n = 1 and n = 2 wavefunctions for a particle in a box are orthogonal, we must evaluate the integral

$$\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)\mathrm{d}x = 0$$

The integral can be evaluated using mathematical software or standard integration tables and does indeed vanish.

2.9 The wavefunction for a particle in a geometrically square two-dimensional box of length L is given by (see eqn 2.35)

$$\psi_{n_1 n_2}(x,y) = \frac{2}{L}\sin\left(\frac{n_1 \pi x}{L}\right)\sin\left(\frac{n_2 \pi y}{L}\right)$$

(a) Nodes correspond to points where the wavefunction passes through zero. For (i) $n_1 = 2$, $n_2 = 1$, this occurs at points (x,y) such that $\underline{x = L/2}$. For (ii) $n_1 = 3$, $n_2 = 2$, this occurs at points (x,y) such that $\underline{x = L/3 \text{ or } 2L/3}$ and at points (x,y) such that $\underline{y = L/2}$.

(b) Maxima in the probability densities occur where ψ^2 is maximized. For (i) $n_1 = 2$, $n_2 = 1$, this occurs at points (x,y) such that $\underline{x = L/4 \text{ or } 3L/4}$ and $\underline{y = L/2}$. For (ii) $n_1 = 3$, $n_2 = 2$, this occurs at points (x,y) such that $\underline{x = L/6, L/2 \text{ or } 5L/6}$ and $\underline{y = L/4 \text{ or } 3L/4}$.

2.10 The energy of a particle in a three-dimensional cubic box is given by:

$$E_{n_1 n_2 n_3} = \frac{h^2}{8m}\left(\frac{n_1^2}{L^2} + \frac{n_2^2}{L^2} + \frac{n_3^2}{L^2}\right) \qquad n_1 = 1,2,\ldots \quad n_2 = 1,2,\ldots \quad n_3 = 1,2,\ldots$$

The lowest energy level corresponds to ($n_1 = n_2 = n_3 = 1$) and equals $3h^2/8mL^2$. Three times this energy, that is $9h^2/8mL^2$, can be achieved with the following sets of quantum numbers:

($n_1 = 2$, $n_2 = 2$, $n_3 = 1$)
($n_1 = 2$, $n_2 = 1$, $n_3 = 2$)
($n_1 = 1$, $n_2 = 2$, $n_3 = 2$)

Therefore the degeneracy of the energy level is 3.

2.11 The harmonic oscillator wavefunction is given in eqn 2.41. Nodes correspond to those points x such that the Hermite polynomial $H_v(\alpha x)$ vanishes. Using Table 2.1, we seek values of $z = \alpha x$ such that $4z^2 - 2 = 0$. This equation is satisfied by

$$z = \pm\sqrt{\frac{1}{2}}$$

and therefore $x = \alpha/\sqrt{2}$ and $x = -\alpha/\sqrt{2}$.

2.12 The energy levels of the harmonic oscillator are given by 2.40. The separation between neighboring vibrational energy levels v and $v+1$ is given by

$$\Delta E = \hbar\omega = \hbar\sqrt{\frac{k_f}{m}} = 1.055 \times 10^{-34}\text{Js} \times \sqrt{\frac{275\ \text{Nm}^{-1}}{1.33 \times 10^{-25}\text{kg}}} = 4.797 \times 10^{-21}\text{J}$$

Equating this with the photon energy hc/λ yields a wavelength of 4.14×10^{-5} m and a corresponding wavenumber of $1/(4.14 \times 10^{-3}\ \text{cm}) = 241\ \text{cm}^{-1}$.

Problems

2.1 See Fig. 2.1.

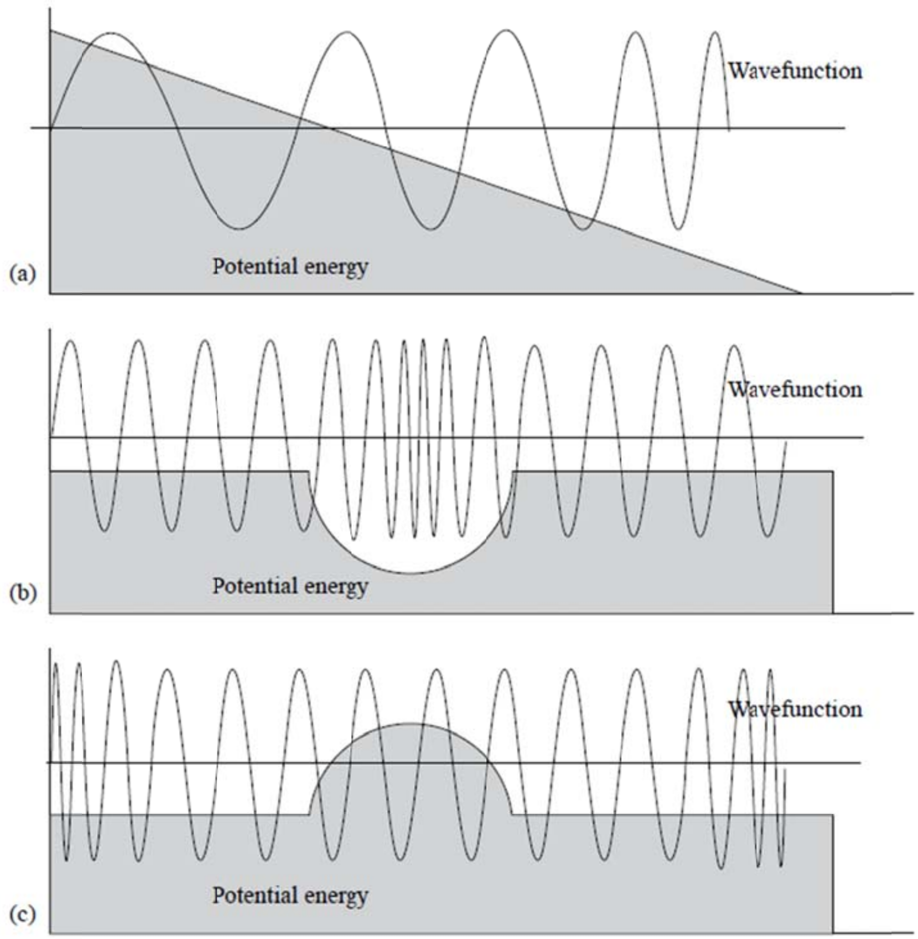

Figure 2.1: The wavefunction in the presence of various potentials.

Exercise: Sketch the general form of the wavefunction for a potential with two parabolic wells separated and surrounded by regions of constant potential.

2.4 From eqns 2.12 and 2.13,

$$\Psi(x,\, t) = \int g(k)\Psi_k(x,t)\mathrm{d}k = AB\int_{k-\frac{1}{2}\Delta k}^{k+\frac{1}{2}\Delta k}\exp\{\mathrm{i}kx - \mathrm{i}k^2\hbar t/2m\}\mathrm{d}k$$

$$\Psi(x,\, 0) = AB\int_{k-\frac{1}{2}\Delta k}^{k+\frac{1}{2}\Delta k}\exp\{\mathrm{i}kx\}\mathrm{d}k$$

$$= (AB/\mathrm{i}x)\left\{\mathrm{e}^{\mathrm{i}(k+\frac{1}{2}\Delta k)x} - \mathrm{e}^{\mathrm{i}(k-\frac{1}{2}\Delta k)x}\right\}$$

$$= (AB\mathrm{e}^{\mathrm{i}kx}/\mathrm{i}x)\left\{\mathrm{e}^{\frac{1}{2}\mathrm{i}\Delta kx} - \mathrm{e}^{-\frac{1}{2}\mathrm{i}\Delta kx}\right\} = 2AB(\mathrm{e}^{\mathrm{i}kx}\sin\tfrac{1}{2}\Delta kx)/x$$

$$|\Psi(x,\, 0)|^2 = 4A^2B^2\{\sin(\tfrac{1}{2}\Delta kx)/x\}^2$$

For normalization (to unity), write $AB = N$; then

$$\int|\Psi|^2\,\mathrm{d}\tau = 4N^2\int_{-\infty}^{\infty}\{\sin(\tfrac{1}{2}\Delta kx)/x\}^2\mathrm{d}x = 2N^2\Delta k\int_{-\infty}^{\infty}(\sin z/z)^2\mathrm{d}z\ \ [z = \tfrac{1}{2}\Delta kx]$$

$$= 2N^2\Delta k\pi = 1;\ \text{hence}\ N = (2\Delta k\pi)^{-1/2}$$

Therefore, $\Psi(x,0) = (2/\Delta k\pi)^{1/2}(\mathrm{e}^{\mathrm{i}kx}\sin\tfrac{1}{2}\Delta kx)/x$

$$|\Psi(x,\, 0)|^2 = (2/\Delta k\pi)(\sin\ \tfrac{1}{2}\Delta kx/x)^2$$

$$|\Psi(0,\, 0)|^2 = (2/\Delta k\pi)\lim_{x\to 0}(\sin\ \tfrac{1}{2}\Delta kx/x)^2 = (2/\Delta k\pi)(\tfrac{1}{2}\Delta kx/x)^2$$

$$= (2/\Delta k\pi)(\Delta k/2)^2 = \Delta k/(2\pi)$$

We seek the value of x for which $|\Psi(x,\, 0)|^2/|\Psi(0,\, 0)|^2 = \tfrac{1}{2}$; that is

$$\frac{\{\sin(\tfrac{1}{2}\Delta kx)/x\}^2}{(\tfrac{1}{2}\Delta k)^2} = \tfrac{1}{2}$$

or

$$\frac{\sin\frac{1}{2}\Delta kx}{\frac{1}{2}\Delta kx} = \frac{1}{\sqrt{2}}$$

which is satisfied by $\frac{1}{2}\Delta kx = \pm 1.392$ [solve numerically]. Hence the probability density falls to one half its value at $x = 0$ when $x = \pm 2.784/\Delta k$. From the uncertainty principle $\Delta p_x \Delta x \geq \frac{1}{2}\hbar$, so $\Delta k \Delta x \geq \frac{1}{2}$, and hence $\Delta x \geq 0.5/\Delta k$ which is in accord with $\Delta x \approx 2 \times 2.784/\Delta k$.

Exercise: Examine the properties of a Gaussian wavepacket in the same way.

2.7 Consider the zones set out in Fig. 2.5; impose the condition of continuity of ψ and ψ' at each interface.

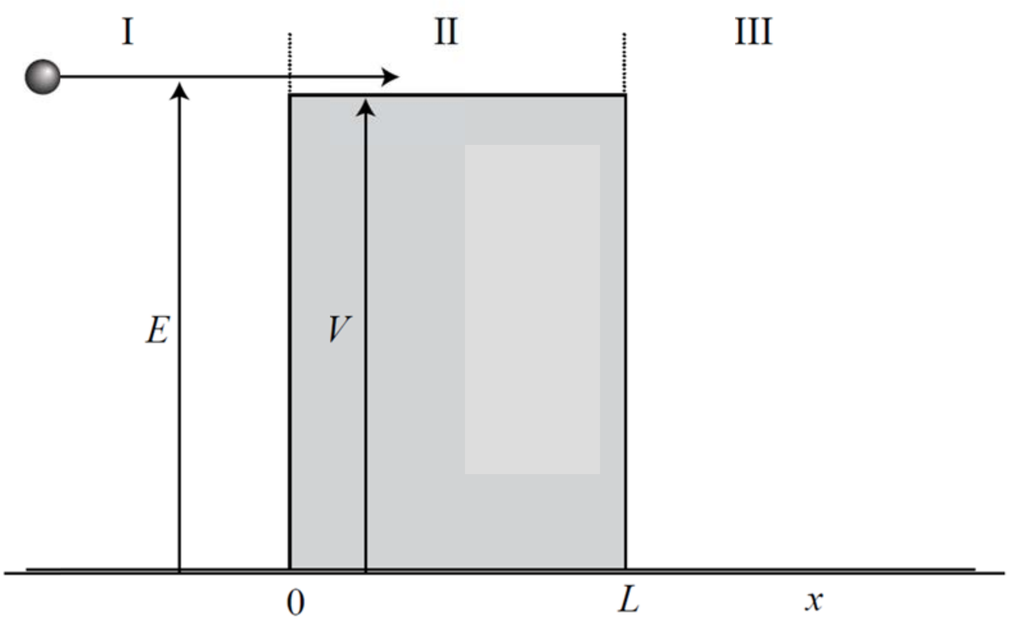

Fig 2.5 The zones of potential energy used in Problem 2.7.

$$\left.\begin{aligned}\psi_{\rm I} &= A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}, k^2 = 2mE/\hbar^2 \\ \psi_{\rm II} &= A'\mathrm{e}^{ik'x} + B'\mathrm{e}^{-ik'x}, k'^2 = 2m(E-V)/\hbar^2\end{aligned}\right\} \gamma = k/k'$$

$\psi_{\rm III} = A''\mathrm{e}^{ikx}$ [no particles incident from right]

(1) $A + B = A' + B'$, [from $\psi_{\rm I}(0) = \psi_{\rm II}(0)$]

(2) $A'{\rm e}^{{\rm i}k'L} + B'{\rm e}^{-{\rm i}k'L} = A''{\rm e}^{{\rm i}kL}$, [from $\psi_{\rm II}(L) = \psi_{\rm III}(L)$]

(3) $kA - kB = k'A' - k'B'$, [from $\psi'_{\rm I}(0) = \psi'_{\rm II}(0)$]

(4) $k'A'{\rm e}^{{\rm i}k'L} - k'B'{\rm e}^{-{\rm i}k'L} = kA''{\rm e}^{{\rm i}kL}$ [from $\psi'_{\rm II}(L) = \psi'_{\rm III}(L)$]

From (1) and (3):

$$A' = \tfrac{1}{2}(1 + \gamma)A + \tfrac{1}{2}(1 - \gamma)B;\ B' = \tfrac{1}{2}(1 - \gamma)A + \tfrac{1}{2}(1 + \gamma)B$$

From (2) and (4)

$$A'' = A'{\rm e}^{{\rm i}(k'-k)L} + B'{\rm e}^{-{\rm i}(k'+k)L}$$

$$\gamma A'' = A'{\rm e}^{{\rm i}(k'-k)L} - B'{\rm e}^{-{\rm i}(k'+k)L}$$

so

$$\tfrac{1}{2}(1 + \gamma)A'' = A'{\rm e}^{{\rm i}(k'-k)L},\ \tfrac{1}{2}(1 - \gamma)A'' = B'{\rm e}^{-{\rm i}(k'+k)L}$$

Then

$$A''{\rm e}^{{\rm i}kL}\left\{(1+\gamma)^2{\rm e}^{-{\rm i}k'L} - (1-\gamma)^2{\rm e}^{{\rm i}k'L}\right\} = 4\gamma A$$

$$A''/A = 2\gamma\,{\rm e}^{-{\rm i}kL}/\{2\gamma\cos k'L - {\rm i}(1 + \gamma^2)\sin k'L\}$$

The transmission coefficient (or tunnelling probability) is

$$P = |A''|^2/|A|^2 = |A''/A|^2$$

$$= \underline{4\gamma^2/\{4\gamma^2 + (1 - \gamma^2)^2\sin^2 k'L\}},\ \gamma^2 = E/(E - V)$$

Exercise: Find the transmission coefficient for a particle incident on a rectangular dip in the potential energy.

2.10 Use the normalized wavefunctions in eqn 2.31:

$$\psi_n = (2/L)^{1/2} \sin(n\pi x/L)\text{; also use}$$

$$\int \sin^2 ax\mathrm{d}x = \tfrac{1}{2}x - (1/4a)\sin 2ax$$

(a) $P_n = \int_0^{\frac{1}{2}L} \psi_n^2 \mathrm{d}x = (2/L)\int_0^{\frac{1}{2}L} \sin^2(n\pi x/L)\mathrm{d}x = \tfrac{1}{2}$ for all n

(b) $P_n = \int_0^{\frac{1}{4}L} \psi_n^2 \mathrm{d}x = (2/L)\int_0^{\frac{1}{4}L} \sin^2(n\pi x/L)\mathrm{d}x = \underline{\tfrac{1}{4}\{1-(2/\pi n)\sin(\tfrac{1}{2}n\pi)\}}$

$P_1 = \tfrac{1}{4}\{1-(2/\pi)\} = \underline{0.090\ 85}$

(c)

$$P_n = \int_{\frac{1}{2}L-\delta x}^{\frac{1}{2}L+\delta x} \psi_n^2 \mathrm{d}x = (2/L)\int_{\frac{1}{2}L-\delta x}^{\frac{1}{2}L+\delta x} \sin^2(n\pi x/L)\mathrm{d}x$$

$$= (2/L)\{\delta x - (L/2\pi n)\cos(n\pi)\sin(2n\pi\delta x/L)\}$$

$$= \underline{(2/L)\{\delta x - (-1)^n(L/2\pi n)\sin(2n\pi\delta x/L)\}}$$

$P_1 = (2/L)\{\delta x + (L/2\pi)\sin(2\pi\delta x/L)\} \approx \underline{4\delta x/L}$ when $\delta x/L << 1$

Note that

$$\lim_{n\to\infty} P_n = \text{(a) } \tfrac{1}{2}\text{, (b) } \tfrac{1}{4}\text{, (c) } 2\delta x/L$$

the last corresponding to a uniform distribution (the classical limit).

Exercise: Find P_n (and P_1) for the particle being in a short region of length δx centred on the general point x.

2.13 Use the wavefunction $\psi_n = (2/L)^{1/2} \sin(n\pi x/L)$ and the integral

$$\int x \sin^2 ax\mathrm{d}x = (1/4a^2)\{a^2x^2 - ax \sin(2ax) - \tfrac{1}{2}\cos(2ax)\}$$

$$\langle x \rangle_n = \int_0^L x\psi_n^2 \mathrm{d}x = (2/L)\int_0^L x \sin^2(n\pi x/L)\mathrm{d}x$$

$$= (L/2n^2\pi^2)\{n^2\pi^2 - n\pi \sin(2n\pi) - \tfrac{1}{2}[\cos(2n\pi) - 1]\} = \underline{\tfrac{1}{2}L}$$

The result is also obvious, by symmetry.

Exercise: Evaluate $\langle x \rangle$ when the particle is in the normalized mixed state $\psi_1 \cos\beta +$ $\psi_2 \sin\beta$. Account for its dependence on the parameter β.

2.16 Refer to Fig. 2.8. Consider the case $E < V$.

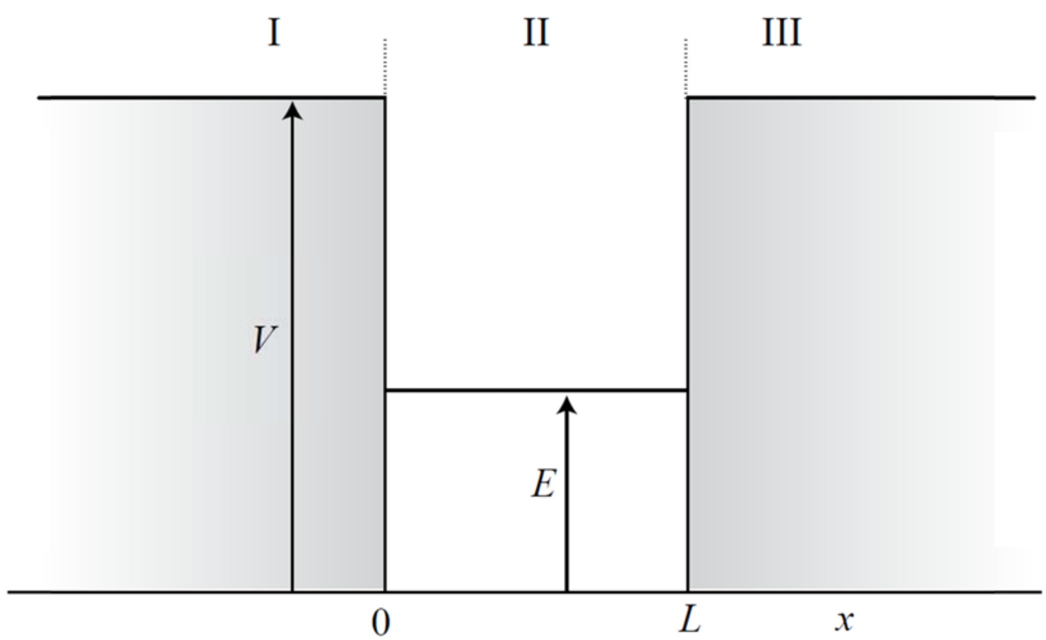

Figure 2.8: The zones of potential energy used in Problem 2.16.

$$\psi_{\mathrm{I}} = Ae^{-\kappa x} + Be^{\kappa x}, \qquad \kappa^2 = 2m(V-E)/\hbar^2$$

$$\psi_{\mathrm{II}} = A'e^{ikx} + B'e^{-ikx}, \qquad k^2 = 2mE/\hbar^2$$

$$\psi_{\mathrm{III}} = A''e^{-\kappa x} + B''e^{\kappa x}, \qquad \kappa^2 = 2m(V-E)/\hbar^2$$

Because $\psi < \infty$ everywhere, $A = 0$, $B'' = 0$ [consider $x \to -\infty$ and $x \to \infty$ respectively].

At the interfaces of the zones:

$$\psi_{\mathrm{I}}'(0)/\psi_{\mathrm{I}}(0) = -\kappa(A - B)/(A + B) = \kappa \qquad [A = 0]$$

$$\psi_{\mathrm{II}}'(0)/\psi_{\mathrm{II}}(0) = \mathrm{i}k(A' - B')/(A' + B')$$

$$\psi_{\mathrm{II}}'(L)/\psi_{\mathrm{II}}(L) = \mathrm{i}k(A'\mathrm{e}^{\mathrm{i}kL} - B'\mathrm{e}^{-\mathrm{i}kL})/(A'\mathrm{e}^{\mathrm{i}kL} + B'\mathrm{e}^{-\mathrm{i}kL})$$

$$\psi_{\mathrm{III}}'(L)/\psi_{\mathrm{III}}(L) = -\kappa(A''\mathrm{e}^{-\kappa L} - B''\mathrm{e}^{\kappa L})/(A''\mathrm{e}^{-\kappa L} + B''\mathrm{e}^{\kappa L}) = -\kappa \quad [B'' = 0]$$

Because ψ'/ψ is continuous at each boundary,

$$(A' - B')/(A' + B') = \kappa/\mathrm{i}k = -\mathrm{i}\kappa/k = -\mathrm{i}\gamma \quad [\gamma = \kappa/k]$$

$$(A'\mathrm{e}^{\mathrm{i}kL} - B'\mathrm{e}^{-\mathrm{i}kL})/(A'\mathrm{e}^{\mathrm{i}kL} + B'\mathrm{e}^{-\mathrm{i}kL}) = -\kappa/\mathrm{i}k = \mathrm{i}\kappa/k = \mathrm{i}\gamma$$

This pair of equations solves to

$$(1 + \mathrm{i}\gamma)A' = (1 - \mathrm{i}\gamma)B', \quad (1 - \mathrm{i}\gamma)A'\mathrm{e}^{\mathrm{i}kL} = (1 + \mathrm{i}\gamma)B'\mathrm{e}^{-\mathrm{i}kL}$$

It follows that

$$(1 - \gamma^2)\sin kL - 2\gamma\cos kL = 0, \text{ or } \tan kL = 2\gamma/(1 - \gamma^2)$$

Then, since

$$\tan kL = 2\tan(\tfrac{1}{2}kL)/[1 - \tan^2(\tfrac{1}{2}kL)], \quad \tan(\tfrac{1}{2}kL) = \gamma$$

Consequently,

$$\cos(\tfrac{1}{2}kL) = 1/(1 + \gamma^2)^{1/2} = \hbar k/(2mV)^{1/2}$$

Therefore,

$$kL = 2\arccos\{\hbar k/(2mV)^{1/2}\} + n\pi, \quad n = 0, 1, \ldots$$

But $\arccos z = \tfrac{1}{2}\pi - \arcsin z$, so

$$kL + 2\arcsin\{\hbar k/(2mV)^{1/2}\} = n\pi, \quad n = 1, 2, \ldots$$

Solve this equation for k by plotting $y = kL$ and

$$y = n\pi - 2\arcsin(\hbar^2k^2/2mV)^{1/2} \quad \text{for } n = 1, 2, \ldots$$

and finding the values of k at which the two lines coincide, and then form $E_n = \hbar^2k^2/2m$ for each value of n. This procedure is illustrated in Fig. 2.9 for the special case $V = 225\hbar^2/2mL^2$, so, with $kL = z$, $y = z$ and $y = n\pi - 2\arcsin z/15$, $E_n = z_n^2(\hbar^2/2mL^2)$ with z_n the intersection value of n. (Because $E < V$, $z < 15$.) We find $z = 2.9, 5.9, 8.8, 11.7$ for $n = 1, 2, 3, 4$; hence $E/(\hbar^2/2mL^2) = 8.4, 35, 77, 137$ for $n = 1, 2, 3, 4$.

When V is large in the sense $2mV >> \hbar^2k^2$, $\arcsin(\hbar^2k^2/2mV)^{1/2} \approx 0$. Hence the equation to solve is $kL \approx n\pi$. Consequently $E_n \approx n^2h^2/8mL^2$ in accord with the infinitely deep square-well solutions.

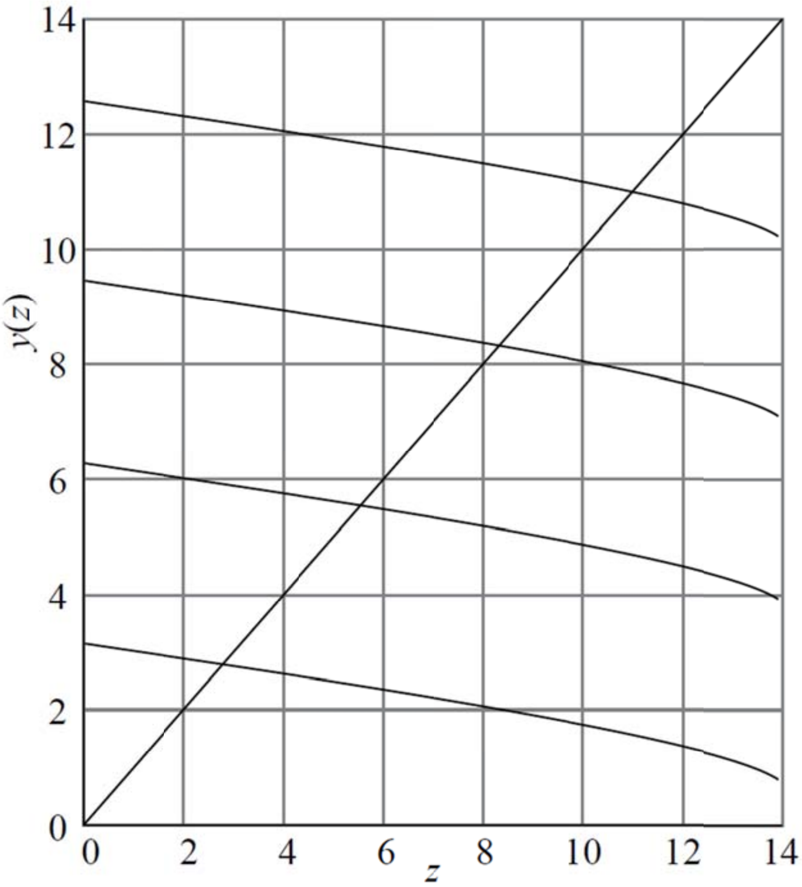

Figure 2.9: The determination of allowed energies.

Exercise: First consider the special case $V = 6\hbar^2/2mL^2$, and find the allowed solutions. Then repeat the calculation for an unsymmetrical well in which the potential energy rises to V on the left and to $4V$ on the right.

2.19 **(a)** $E_{n_1n_2} = \dfrac{h^2}{8m}\left(\dfrac{n_1^2}{L_1^2} + \dfrac{n_2^2}{L_2^2}\right) = \dfrac{h^2}{8mL_2^2}\left(\dfrac{n_1^2}{\lambda^2} + n_2^2\right)$

where $\lambda = L_1/L_2$. Therefore, if λ is an integer, the states (n_1, n_2) and $(\lambda n_2, n_1/\lambda)$ are degenerate.

(b) The states related by the relation in **(a)** are doubly degenerate.

2.22 The Schrödinger equation is

$$-(\hbar^2/2m)(\mathrm{d}^2\psi/\mathrm{d}x^2) + \tfrac{1}{2}k_\mathrm{f}x^2\psi = E\psi$$

Substitute $y = (m\omega/\hbar)^{1/2}x$ with $\omega^2 = k_\mathrm{f}/m$; then $\psi'' - y^2\psi = -\lambda\psi$, with $\lambda = E/\tfrac{1}{2}\hbar\omega$ and

$\psi'' = \mathrm{d}^2\psi/\mathrm{d}y^2$.

Substitute eqn 2.41: $\psi = N_vH_v\mathrm{e}^{-y^2/2}$:

$$(\mathrm{d}^2/\mathrm{d}y^2)(H_v\mathrm{e}^{-y^2/2}) - y^2H_v\mathrm{e}^{-y^2/2} = -\lambda H_v\mathrm{e}^{-y^2/2}$$

Use

$$(\mathrm{d}^2/\mathrm{d}y^2)(H_v\mathrm{e}^{-y^2/2}) = (H_v'' - 2yH_v' - H_v + y^2H_v)\mathrm{e}^{-y^2/2}$$

$$= (2yH_v' - 2vH_v - 2yH_v' - H_v + y^2H_v)e^{-y^2/2} \quad \text{[given]}$$

$$= \{y^2H_v - (2v+1)H_v\}e^{-y^2/2}$$

Then

$$\{y^2H_v - (2v+1)H_v - y^2H_v\}\mathrm{e}^{-y^2/2} = -\lambda H_v\mathrm{e}^{-y^2/2}$$

so $\lambda = 2v + 1$, or $E = \tfrac{1}{2}(2v+1)\hbar\omega = (v + \tfrac{1}{2})\hbar\omega$, as required.

2.25 **(a)**

$$\langle v+1|x|v\rangle = N_{v+1}N_v\alpha^{-2}\int_{-\infty}^{\infty} H_{v+1}(y)yH_v(y)\mathrm{e}^{-y^2}\mathrm{d}y \quad [y=\alpha x]$$

$$= \alpha^{-2}N_{v+1}N_v\int_{-\infty}^{\infty} H_{v+1}\{\tfrac{1}{2}H_{v+1}+vH_{v-1}\}e^{-y^2}\mathrm{d}y \quad \text{[Table 2.1]}$$

$$= \tfrac{1}{2}\alpha^{-2}N_{v+1}N_v\int_{-\infty}^{\infty} H_{v+1}^2\mathrm{e}^{-y^2}\mathrm{d}y \quad \text{[orthogonality]}$$

$$= \tfrac{1}{2}\alpha^{-2}N_{v+1}N_v\pi^{1/2}2^{v+1}(v+1)!$$

$$= \frac{\pi^{1/2}2^{v+1}(v+1)!}{2\pi^{1/2}\alpha\{2^v2^{v+1}v!(v+1)!\}^{1/2}} = \underline{\frac{1}{\sqrt{2}\alpha}(v+1)^{1/2}}$$

(b)

$$\langle v+2|x^2|v\rangle = N_{v+2}N_v\alpha^{-3}\int_{-\infty}^{\infty} H_{v+2}y^2H_v\mathrm{e}^{-y^2}\mathrm{d}y$$

$$= \alpha^{-3}N_{v+2}N_v\int_{-\infty}^{\infty} H_{v+2}y\{\tfrac{1}{2}H_{v+1}+vH_{v-1}\}\mathrm{e}^{-y^2}\mathrm{d}y$$

$$= \alpha^{-3}N_{v+2}N_v\int_{-\infty}^{\infty} H_{v+2}\{\tfrac{1}{4}H_{v+2}+\tfrac{1}{2}vH_v+\tfrac{1}{2}vH_v+v^2H_{v-2}\}\mathrm{e}^{-y^2}\mathrm{d}y$$

$$= \tfrac{1}{4}\alpha^{-3}N_{v+2}N_v\int_{-\infty}^{\infty} H_{v+2}^2\mathrm{e}^{-y^2}\mathrm{d}y \quad \text{[orthogonality]}$$

$$= \tfrac{1}{4}\alpha^{-3}N_{v+2}N_v\pi^{1/2}2^{v+2}(v+2)! \quad \text{[Table 2.1]}$$

$$= \frac{\alpha^{-3}\pi^{1/2}2^{v+2}(v+2)!}{4\{2^{v+2}2^v(v+2)!v!\alpha^{-2}\pi\}^{1/2}} = \underline{\tfrac{1}{2}\alpha^{-2}\{(v+2)(v+1)\}^{1/2}}$$

Exercise: Evaluate $\langle v+3|x^3|v\rangle$ in the same way.

2.28 According to classical mechanics, the turning point x_{tp} occurs when all the energy of the oscillator is potential energy and its kinetic energy is zero. This equality occurs when

$$E = \tfrac{1}{2}k_{\text{f}}x_{\text{tp}}^2 \ \text{ or } \ x_{\text{tp}} = \left(\frac{2\text{E}}{k_{\text{f}}}\right)^{1/2}$$

Since we are only considering the stretching of the harmonic oscillator beyond the classical turning point, we only choose the positive square root for x_{tp}. The probability P of finding the ground-state harmonic oscillator stretched beyond a displacement x_{tp} is given by:

$$P = \int_{x_{tp}}^{\infty} \psi_0^2 \, dx$$

Using eqn 2.41 and the Hermite polynomial H_0 in Table 2.1, we obtain:

$$P = \frac{\alpha}{\pi^{1/2}} \int_{x_{tp}}^{\infty} e^{-\alpha^2 x^2} \, dx$$

The turning point can be expressed in terms of α, using (i) the definition of α in eqn 2.41 and (ii) the ground-state energy $E = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar(k_f/m)^{1/2}$. This results in $x_{tp} = 1/\alpha$. Now introduce the variable $y = \alpha x$ so that $dy = \alpha dx$, $y^2 = \alpha^2 x^2$ and $y_{tp} = \alpha x_{tp} = 1$. The above integral then becomes, in terms of the variable y:

$$P = \frac{1}{\pi^{1/2}} \int_{1}^{\infty} e^{-y^2} \, dy$$

The above integral is related to the error function given in the Problem, and using the value of erf 1 given:

$$P = \frac{1}{\pi^{1/2}} \int_1^{\infty} e^{-y^2} \, dy = \frac{1}{2}(1 - \text{erf } 1) = \frac{1}{2}(1 - 0.8427)$$

The probability is 7.865×10^{-2}.

2.31 The wavefunction $\psi(x)$ is given as a sum of normalized particle-in-a-box eigenfunctions $\psi_n(x)$. Therefore, according to quantum mechanical postulate 3′, a single measurement of the energy yields a single outcome which is one of the eigenvalues E_n (associated with the eigenfunction ψ_n appearing in the expansion of ψ). The probability of obtaining E_n is $|c_n|^2$ where c_n is the coefficient of ψ_n in the expansion.

(a) When the energy of the particle is measured, possible outcomes are

$$E_1 = \frac{h^2}{8mL^2} \quad E_3 = \frac{9h^2}{8mL^2} \quad E_5 = \frac{25h^2}{8mL^2}$$

(b) The probability of obtaining each result is

$$|c_1|^2 = (1/3)^2 = 1/9 \quad \text{for } E_1$$

$$|c_3|^2 = |(\mathrm{i}/3)|^2 = 1/9 \quad \text{for } E_3$$

$$|c_5|^2 = [-(7/9)^{1/2}]^2 = 7/9 \quad \text{for } E_5$$

(c) The expectation value is the weighted sum of the possible eigenvalues:

$$\tfrac{1}{9}E_1 + \tfrac{1}{9}E_3 + \tfrac{7}{9}E_5 = \frac{185h^2}{72mL^2}$$

Exercise: If the linear momentum of the particle described above were measured, what would we expect to find?