Chapter 1

The foundations of quantum mechanics

Exercises

1.1 **(a)** $\int(f+g)\mathrm{d}x = \int f\mathrm{d}x + \int g\mathrm{d}x$; linear.

(b) $(f+g)^{1/2} \neq f^{1/2} + g^{1/2}$; nonlinear.

(c) $f(x+a) + g(x+a) = f(x+a) + g(x+a)$; linear.

(d) $f(-x) + g(-x) = f(-x) + g(-x)$; linear.

Exercise: Repeat the exercise for (a) differentiation, (b) exponentiation.

1.2 **(a)** $(\mathrm{d}/\mathrm{d}x)\mathrm{e}^{ax} = a\mathrm{e}^{ax}$; e^{ax} is an eigenfunction, eigenvalue a.

$(\mathrm{d}/\mathrm{d}x)\mathrm{e}^{ax^2} = 2ax\mathrm{e}^{ax^2} = 2a\{x\mathrm{e}^{ax^2}\}$; e^{ax^2} not an e.f.

$(\mathrm{d}/\mathrm{d}x)x = 1$; x not an e.f.

$(\mathrm{d}/\mathrm{d}x)x^2 = 2x$; x^2 not an e.f.

$(\mathrm{d}/\mathrm{d}x)(ax+b) = a$; $ax+b$ not an e.f.

$(\mathrm{d}/\mathrm{d}x)\sin x = \cos x$; $\sin x$ not an e.f.

(b) $(\mathrm{d}^2/\mathrm{d}x^2)\mathrm{e}^{ax} = a^2\mathrm{e}^{ax}$; e^{ax} is an eigenfunction, eigenvalue a^2.

$(\mathrm{d}^2/\mathrm{d}x^2)\mathrm{e}^{ax^2} = 2a\mathrm{e}^{ax^2} + 4a^2x^2\mathrm{e}^{ax^2}$; e^{ax^2} not an e.f.

$(\mathrm{d}^2/\mathrm{d}x^2)x = 0 = 0x$; x is an e.f.; e.v. is 0.

$(\mathrm{d}^2/\mathrm{d}x^2)x^2 = 2$; x^2 not an e.f.

$(\mathrm{d}^2/\mathrm{d}x^2)(ax+b) = 0 = 0(ax+b)$; $ax+b$ is an e.f.; e.v. is 0.

$(\mathrm{d}^2/\mathrm{d}x^2)\sin x = -\sin x$; $\sin x$ is an e.f.; e.v. is -1.

Exercise: Find the operator of which e^{ax^2} is an eigenfunction. Find the eigenfunction of the operator 'multiplication by x^2'.

1.3

$$\langle m|A + iB|n\rangle = \langle m|A|n\rangle + i\langle m|B|n\rangle$$

$$= \langle n|A|m\rangle^* + i\langle n|B|m\rangle^* \text{ [}A, B\text{ hermitian, eqn 1.26]}$$

$$= \{\langle n|A|m\rangle - i\langle n|B|m\rangle\}^* = \langle n|A - iB|m\rangle^*.$$

Hence, $A - iB$ is the hermitian conjugate of $A + iB$ (and $A + iB$ is not *self-conjugate*, another term for hermitian).

Exercise: Confirm that $x + (d/dx)$ and $x - (d/dx)$ are hermitian conjugates.

1.4 If the maximum uncertainty in the position x of the electron is Δx, the minimum uncertainty in the momentum p_x will be given by $\Delta x \Delta p_x = \frac{1}{2}\hbar$. Since the electron is confined to the linear box, $\Delta x = 0.10$ nm. Therefore

$$\Delta p_x = \frac{\hbar}{2\Delta x}$$

$$= \frac{1.055 \times 10^{-34} \text{ J s}}{2 \times 0.10 \times 10^{-9} \text{ m}}$$

$$= 5.3 \times 10^{-25} \text{ kg m s}^{-1}$$

(a) Since $p_x = m_e v$, the uncertainty in the velocity is

$$\Delta v = \Delta p_x / m_e$$

$$= (5.3 \times 10^{-25} \text{ kg m s}^{-1})/(9.109\,38 \times 10^{-31} \text{ kg})$$

$$= \underline{5.8 \times 10^5 \text{ m s}^{-1}}$$

(b) Since, $E_K = p_x^2/2m_e$

$$\Delta E_K = (\Delta p_x)^2/2m_e$$

$$= (5.3 \times 10^{-25}\ \text{kg m s}^{-1})^2/(2 \times 9.109\ 38 \times 10^{-31}\ \text{kg})$$

$$= \underline{1.5 \times 10^{-19}\ \text{J}}$$

Exercise: If the length of the box is doubled to 0.20 nm, what are the minimum uncertainties? If a proton is confined to a linear box of length 0.20 nm, what are the minimum uncertainties?

1.5

Use the integral

$$\int x^2 \sin^2 ax\text{d}x = \tfrac{1}{6}x^3 + (1/4a^3)\{\tfrac{1}{2}\sin(2ax) - ax\cos(2ax) - a^2x^2\sin(2ax)\}$$

$$\langle x^2\rangle_n = (2/L)\int_0^L x^2 \sin^2(n\pi x/L)\text{d}x = \underline{\tfrac{1}{3}L^2\{1-(3/2n^2\pi^2)\}}$$

$$\langle x^2\rangle_2 = \tfrac{1}{3}L^2\{1-(3/8\pi^2)\}$$

Since the particle is equally likely to be found in the right-hand side of the box (between $L/2$ and L) and in the left-hand side of the box (between 0 and $L/2$), the average value $<x> = L/2$ for all values of n. Therefore,

$$\Delta x_n = \{\langle x^2\rangle_n - \langle x\rangle_n^2\}^{1/2} = \{\tfrac{1}{3}L^2 - (1/2n^2\pi^2)L^2 - \tfrac{1}{4}L^2\}^{1/2}$$

$$= \underline{(L/2\sqrt{3})\{1-(6/n^2\pi^2)\}^{1/2}}$$

$$\Delta x_2 = \underline{(L/2\sqrt{3})\{1-(3/2\pi^2)\}^{1/2}}$$

As for the momentum, the intuitive solution is $\langle p\rangle_n = 0$ because the wavefunction is a standing wave. The elegant solution is

$$\langle p\rangle = \langle n|p|n\rangle = \langle n|p|n\rangle^* \text{ [hermiticity]} = \langle n|p^*|n\rangle = -\langle n|p|n\rangle [p^* = -p].$$

Therefore since $\langle p\rangle = -\langle p\rangle$, $\langle p\rangle = 0$.

The straightforward solution is:

$$\langle p\rangle_n = (\hbar/i)(2/L)\int_0^L \sin(n\pi x/L)(\mathrm{d}/\mathrm{d}x)\sin(n\pi x/L)\mathrm{d}x$$

$$= (2\hbar/\mathrm{i}L)(n\pi/L)\int_0^L \sin(n\pi x/L)\cos(n\pi x/L)\mathrm{d}x = 0$$

Also, note that

$$\langle p^2\rangle_n = 2mE_n = \underline{n^2h^2/4L^2}$$

Thus, $\Delta p_n = \{\langle p^2\rangle_n - \langle p\rangle_n^2\}^{1/2} = \langle p^2\rangle_n^{1/2} = \underline{nh/2L}$

Therefore:

$$\Delta x_n \Delta p_n = (L/2\sqrt{3})\{1-(6/n^2\pi^2)\}^{1/2}(nh/2L)$$

$$= (n/4\sqrt{3})\{1-(6/n^2\pi^2)\}^{1/2}h = \underline{(n\pi/\sqrt{3})\{1-(6/n^2\pi^2)\}^{1/2}(\hbar/2)}$$

$$\Delta x_2 \Delta p_2 = (2\pi/\sqrt{3})\{1-(3/2\pi^2)\}^{1/2}(\hbar/2) = \underline{3.3406(\hbar/2)} > \hbar/2$$

as required. As n increases, the uncertainty product $\Delta x_n \Delta p_n$ increases.

Exercise: Repeat the calculation for the mixed state $\psi_1 \cos\beta + \psi_2 \sin\beta$.

What value of β minimizes the uncertainty product?

1.6 To use the Born interpretation to find the probability, we need to first normalize the wavefunction, $\psi(x) = N\mathrm{e}^{-2x}$. Normalization requires that

$$\int_0^\infty \psi^*\psi\,\mathrm{d}x = \int_0^\infty N^2\mathrm{e}^{-4x}\mathrm{d}x = 1$$

which yields $N = 2$. The probability of finding the particle at a distance $x \geq 1$ is given by

$$\text{Probability} = \int_1^\infty (2e^{-2x})^2 dx$$

$$= \underline{e^{-4}}$$

Exercise: Suppose that the particle is now described by the unnormalized wavefunction $\psi(x) = e^{-3x}$. Between 0 and what other distance is the probability of finding the particle equal to $\frac{1}{2}$?

1.7 Use eqn 1.44. Since $l_z = (\hbar/i)(\partial/\partial\phi)$, $V(\phi) = V$, a constant, and $H = (1/2mr^2)\,l_z^2 + V$:

$$[H, l_z] = (1/2mr^2)[l_z^2, l_z] + [V, l_z] = 0 \ \{[V, l_z] \propto dV/d\phi = 0\},$$

Hence, $(d/dt)\langle l_z\rangle = 0$

Exercise: Find the equation of motion for the expectation value of l_z for a particle on a vertical ring in a uniform gravitational field. Examine the equations for small displacements from the lowest point.

1.8 The most probable location is given by the value of x corresponding to the maximum (or maxima) of $|\psi|^2$; write this location x_*. In the present case

$$|\psi|^2 = N^2x^2e^{-x^2/\Gamma^2}$$

$$(d/dx)|\psi|^2 = N^2\{2xe^{-x^2/\Gamma^2} - 2(x^3/\Gamma^2)e^{-x^2/\Gamma^2}\} = 0 \text{ at } x = x_*$$

Hence, $1 - x_*^2/\Gamma^2 = 0$, so $\underline{x_* = \pm\Gamma}$

Exercise: Evaluate N for the wavefunction. Consider then another excited state wave-function $\{2(x/\Gamma)^2 - 1\}e^{-x^2/2\Gamma^2}$, and locate x_*.

1.9 Base the answer on $|\psi|^2 = (b^3/\pi)e^{-2br}$. The probability densities are

(a) $|\psi(0)|^2 = b^3/\pi = 1/(53\text{ pm})^3\pi = \underline{2.1 \times 10^{-6}\text{ pm}^{-3}}$

(b) $|\psi(r = 1/b, \theta, \phi)|^2 = (b^3/\pi)e^{-2} = \underline{2.9 \times 10^{-7}\text{ pm}^{-3}}$

[The values of θ and ϕ do not matter because ψ is spherically symmetrical.] The probabilities are given by

$$P = \int_{\text{volume}} \psi^2 d\tau \approx |\psi|^2 \, \delta V$$

because $|\psi|^2$ is virtually constant over the small volume of integration $\delta V = 1$ pm^3.

Hence:

(a) $P = |\psi(0)|^2 \delta V = \underline{2.1 \times 10^{-6}}$;

(b) $P = |\psi(1/b, \theta, \phi)|^2 \delta V = \underline{2.9 \times 10^{-7}}$

Problems

1.1 **(a)**

$$\langle p_x \rangle \propto \langle \sin(\pi x/L) \left| \frac{\mathrm{d}}{\mathrm{d}x} \right| \sin(\pi x/L) \rangle$$

$$\propto \langle \sin(\pi x/L) | \cos(\pi x/L) \rangle = \underline{0}$$

(b)

$$\langle p_x^2 \rangle = 2m\langle T \rangle = 2mE \; [V = 0] \quad \text{[see eqn 2.30]}$$

$$= 2m\left(\frac{h^2}{8mL^2}\right) [\text{for } n = 1] = \underline{h^2/4L^2}$$

Alternatively, integrate explicitly.

Exercise: Evaluate **(a)** $\langle p_x^3 \rangle$, **(b)** $\langle p_x^4 \rangle$.

1.4 **(a)** $[A, B] = AB - BA = -(BA - AB) = -[B, A]$

(b) $[A^m, A^n] = A^m A^n - A^n A^m = A^{m+n} - A^{m+n} = 0$

(c)

$$[A^2, B] = AAB - BAA = ABA + (AAB - ABA) - ABA + (ABA - BAA)$$

$$= A[A, B] + [A, B]A$$

(d)

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]]$$

$$= (ABC - ACB - BCA + CBA) + (BCA - BAC - CAB + ACB)$$

$$+ (CAB - CBA - ABC + BAC) = 0$$

Exercise: Express $[A^2, B^2]$, $[A^3, B]$, and $[A, [B, [C, [D, E]]]]$ in terms of individual commutators.

1.7 Find a normalization constant N such that eqn 1.18 is satisfied.

$$\int |\psi|^2 \, d\tau = N^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} r^2 e^{-2br} dr$$

$$= N^2 \{2\pi\}\{2\} \int_0^{\infty} r^2 e^{-2br} dr = 4\pi N^2 \{2!/(2b)^3\}$$

$$= N^2 \pi / b^3.$$

Hence $\underline{N = (b^3/\pi)^{1/2} = 1.5 \times 10^{15} \text{ m}^{-3/2}}$

Consequently, $\psi = (b^3/\pi)^{1/2} e^{-br}$

Exercise: ψ depends on Z as e^{-Zbr}. Find N for general Z.

1.10 **(a)** $[1/x, p_x]$; use the position representation.

$$[1/x, p_x] = [x^{-1}, (\hbar/\mathrm{i})\mathrm{d}/\mathrm{d}x] = (\hbar/\mathrm{i})\{x^{-1}(\mathrm{d}/\mathrm{d}x) - (\mathrm{d}/\mathrm{d}x)x^{-1}\}$$

$$= (\hbar/\mathrm{i})\{x^{-1}(\mathrm{d}/\mathrm{d}x) - (\mathrm{d}x^{-1}/\mathrm{d}x) - x^{-1}(\mathrm{d}/\mathrm{d}x)\}$$

$$= -(\hbar/\mathrm{i})(\mathrm{d}x^{-1}/\mathrm{d}x) = \underline{(\hbar/\mathrm{i})x^{-2}}$$

(b)

$$[1/x, p_x^2] = [x^{-1}, -\hbar^2(\mathrm{d}^2/\mathrm{d}x^2)]$$

$$= -\hbar^2\{x^{-1}(\mathrm{d}^2/\mathrm{d}x^2) - (\mathrm{d}^2/\mathrm{d}x^2)x^{-1}\}$$

$$= -\hbar^2\{x^{-1}(\mathrm{d}^2/\mathrm{d}x^2) - (\mathrm{d}/\mathrm{d}x)[(\mathrm{d}x^{-1}/\mathrm{d}x) + x^{-1}(\mathrm{d}/\mathrm{d}x)]\}$$

$$= -\hbar^2\{x^{-1}(\mathrm{d}^2/\mathrm{d}x^2) - (\mathrm{d}/\mathrm{d}x)[-x^{-2} + x^{-1}(\mathrm{d}/\mathrm{d}x)]\}$$

$$= -\hbar^2\{x^{-1}(\mathrm{d}^2/\mathrm{d}x^2) + (\mathrm{d}x^{-2}/\mathrm{d}x) + x^{-2}(\mathrm{d}/\mathrm{d}x)$$

$$-(\mathrm{d}x^{-1}/\mathrm{d}x)(\mathrm{d}/\mathrm{d}x) - x^{-1}(\mathrm{d}^2/\mathrm{d}x^2)\}$$

$$= -\hbar^2\{-2x^{-3} + 2x^{-2}(\mathrm{d}/\mathrm{d}x)\}$$

$$= 2\hbar^2/x^3 - 2\hbar^2x^{-2}(\mathrm{i}/\hbar)p_x = \underline{(2\hbar/x^3)(\hbar - \mathrm{i}xp_x)}$$

(c)

$$[xp_y - yp_x, yp_z - zp_y]$$

$$= [xp_y, yp_z] - [xp_y, zp_y] - [yp_x, yp_z] + [yp_x, zp_y]$$

$$= x[p_y, y]p_z - 0 - 0 + p_x[y, p_y]z$$

$$= x(-\mathrm{i}\hbar)p_z + p_x(\mathrm{i}\hbar)z = \underline{\mathrm{i}\hbar(zp_x - xp_z)}$$

(d)

$[x^2(\partial^2/\partial y^2), y(\partial/\partial x)]$

$= x^2(\partial^2/\partial y^2)y(\partial/\partial x) - y(\partial/\partial x)x^2(\partial^2/\partial y^2)$

$= x^2(\partial/\partial x)(\partial^2/\partial y^2)y - (\partial/\partial x)x^2y(\partial^2/\partial y^2)$

$= x^2(\partial/\partial x)(\partial/\partial y)\{1 + y(\partial/\partial y)\} - \{2x + x^2(\partial/\partial x)\}y(\partial^2/\partial y^2)$

$= x^2(\partial/\partial x)\{2(\partial/\partial y) + y(\partial^2/\partial y^2)\} - 2xy(\partial^2/\partial y^2) - x^2(\partial/\partial x)y(\partial^2/\partial y^2)$

$= 2x^2(\partial/\partial x)(\partial/\partial y) - 2xy(\partial^2/\partial y^2)$

$= \underline{2x^2(\partial^2/\partial x\partial y) - 2xy(\partial^2/\partial y^2)}$

Exercise: Evaluate $[xy(\partial^2/\partial x\partial y), x^2(\partial^2/\partial y^2)]$.

1.13 Use the correspondence in Section 1.5.

(a)

$T = p^2/2m = \underline{-(\hbar^2/2m)(\mathrm{d}^2/\mathrm{d}x^2)}$ in one dimension.

$T = p^2/2m = -(\hbar^2/2m)\{(\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)\}$

$= \underline{-(\hbar^2/2m)\nabla^2}$ in three dimensions.

(b) $1/x \rightarrow$ <u>multiplication by $(1/x)$</u>

(c) $\mu = \sum_i Q_i \boldsymbol{r}_i \rightarrow$ <u>multiplication by $\sum_i Q_i \boldsymbol{r}_i$</u>

(d)

$l_z = xp_y - yp_x = \underline{(\hbar/\mathrm{i})\{x(\partial/\partial y) - y(\partial/\partial x)\}}$

$= \underline{(\hbar/\mathrm{i})(\partial/\partial\phi)}$ for $x = r\cos\phi$, $y = r\sin\phi$

(e) $\delta x^2 = x^2 - \langle x\rangle^2 \rightarrow$ multiplication by $\{x^2 - \langle x\rangle^2\}$

$\delta p^2 = p^2 - \langle p\rangle^2 \rightarrow \{-\hbar^2(\partial^2/\partial x^2) - \langle p\rangle^2\}$

Exercise: Devise operators for $1/r$, xp_x, and $\mathrm{e}^{\alpha x}$.

1.16 Take $H\Psi = \kappa(\partial^2\Psi/\partial t^2)$. Because H has the dimensions of energy, κ must have the dimensions of energy × time^2, or ML^2. Try $\Psi = \psi\theta$, with H an operator on x, not t. The equation separates into $H\psi = E\psi$, $d^2\theta/dt^2 = (E/\kappa)\theta$. The latter admits solutions of the form $\theta \propto \cos(E/\kappa)^{1/2}t$. Then

$$\int |\Psi|^2 d\tau \propto \int |\psi|^2 \, d\tau \cos^2(E/\kappa)^{1/2} t$$

which oscillates in time between 0 and 1; hence the total probability is not conserved.

1.19 **(a)**

$$e^A e^B = (1 + A + \tfrac{1}{2}A^2 + \ldots)(1 + B + \tfrac{1}{2}B^2 + \ldots)$$

$$= 1 + (A + B) + \tfrac{1}{2}(A^2 + 2AB + B^2) + \ldots$$

$$e^{A+B} = 1 + (A + B) + \tfrac{1}{2}(A + B)^2 + \ldots$$

$$= 1 + (A + B) + \tfrac{1}{2}(A^2 + AB + BA + B^2) + \ldots$$

Therefore, $e^A e^B = e^{A+B}$ only if $AB = BA$, which is so if $[A, B] = 0$.

(b) If $[A, [A, B]] = [B, [A, B]] = 0$, then

$$e^{A+B} = 1 + (A+B) + \tfrac{1}{2}(A^2 + AB + BA + B^2)$$

$$+ (1/3!)(A^3 + A^2B + ABA + BAA + BBA + BAB + ABB + B^3) + \ldots$$

$$= 1 + (A+B) + \tfrac{1}{2}(A^2 + 2AB + B^2) - \tfrac{1}{2}[A, B]$$

$$+ (1/3!)(A^3 + 3A^2B + 3AB^2 + B^3) - \tfrac{1}{2}(A+B)[A, B] + \ldots$$

$$= \underline{e^A e^B e^{-\frac{1}{2}[A,B]}}$$

Therefore, $e^A e^B = e^{A+B} e^f$ where $f = [A, B]/2$.

Exercise: Find expressions for cos A cos B and cos A sin B, where A and B are operators such that

$$[A, [A, B]] = [B, [A, B]] = 0$$

(Use $\cos A = \tfrac{1}{2}(e^{iA} + e^{-iA})$, etc.)

1.22 $(d/dt)\langle\Omega\rangle = (i/\hbar)\langle[H, \Omega]\rangle$ [eqn 1.44].

For a harmonic oscillator, $H = p_x^2/2m + \tfrac{1}{2}k_f x^2$, and

$$[H, x] = [p_x^2/2m, x] = -(i\hbar/m)p_x \qquad \text{[Problem 1.11]}$$

$$[H, p_x] = [\tfrac{1}{2}k_f x^2, p_x] = -i\hbar k_f x \qquad \text{[Problem 1.11]}$$

$$(d/dt)\langle x\rangle = \underline{(1/m)\langle p_x\rangle}\,;\ (d/dt)\langle p_x\rangle = \underline{-k_f\langle x\rangle}$$

Therefore

$$(d^2/dt^2)\langle x\rangle = (1/m)(d/dt)\langle p_x\rangle = -(k_f/m)\langle x\rangle$$

The solution of $(d^2/dt^2)\langle x\rangle = -(k_f/m)\langle x\rangle$ is

$$\langle x\rangle = A\cos\omega t + B\sin\omega t, \quad \omega^2 = k_f/m$$

$$\langle p\rangle = m(\mathrm{d}/\mathrm{d}t)\langle x\rangle = -Am\omega\sin\omega t + Bm\omega\cos\omega t$$

which is the classical trajectory.

Exercise: Find the equation of motion of the expectation values of x and p for a quartic oscillator ($V \propto x^4$).

1.25 $(-\hbar^2/2m)(\mathrm{d}^2\Psi/\mathrm{d}x^2) + V(t)\Psi = \mathrm{i}\hbar(\partial\Psi/\partial t)$.

(a) Try $\Psi = \psi(x)\theta(t)$, then

$$(-\hbar^2/2m)\psi''\theta + V(t)\psi\theta = \mathrm{i}\hbar\,\psi\;\mathrm{d}\theta/\mathrm{d}t$$

$$-(\hbar^2/2m)(\psi''/\psi) + V(t) - \mathrm{i}\hbar(\mathrm{d}\theta/\mathrm{d}t)(1/\theta) = 0$$

By the same argument as that in Section 1.14, $(-\hbar^2/2m)(\psi''/\psi) = \varepsilon$, a constant; hence

$$\psi'' = -(2m\varepsilon/\hbar^2)\psi \qquad (1.1)$$

$\mathrm{i}\hbar(\mathrm{d}\theta/\mathrm{d}t)(1/\theta) - V(t) = \varepsilon$, the same constant; hence

$$(\mathrm{d}/\mathrm{d}t)\ln\theta = \varepsilon + V(t)/\mathrm{i}\hbar \qquad (1.2)$$

(b) Equation (1) has the solution $\psi = A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx}$, $k = (2m\varepsilon/\hbar^2)^{1/2}$

Equation (2) has the solution $\ln\theta(t) = \ln\theta(0) - (\mathrm{i}/\hbar)\int_0^t\{\varepsilon + V(t)\}\,\mathrm{d}t$

Therefore, on absorbing $\ln\theta(0)$ into A and B,

$$\Psi = \psi(x)\exp\left\{-\mathrm{i}(\varepsilon/\hbar)t - (\mathrm{i}/\hbar)\int_0^t V(t)\mathrm{d}t\right\}$$

Let $V(t) = V \cos \omega t$, then $\int_0^t V(t)\mathrm{d}t = (V/\omega) \sin \omega t$, so

$$\Psi = \psi(x)\exp\{-\mathrm{i}(\varepsilon/\hbar)t - \mathrm{i}(V/\hbar\omega) \sin \omega t\}$$

$$= \psi(x)(\cos \phi - \mathrm{i} \sin \phi),\ \phi = \varepsilon t/\hbar + (V/\hbar\omega) \sin \omega t.$$

The behaviour of the real and imaginary parts of Ψ (essentially the functions $\cos(\tau + \sin \tau)$ and $\sin(\tau + \sin \tau)$) is shown in Fig. 1.1. The dotted line is $\cos(\tau + \sin \tau)$ and the full line is $\sin(\tau + \sin \tau)$.

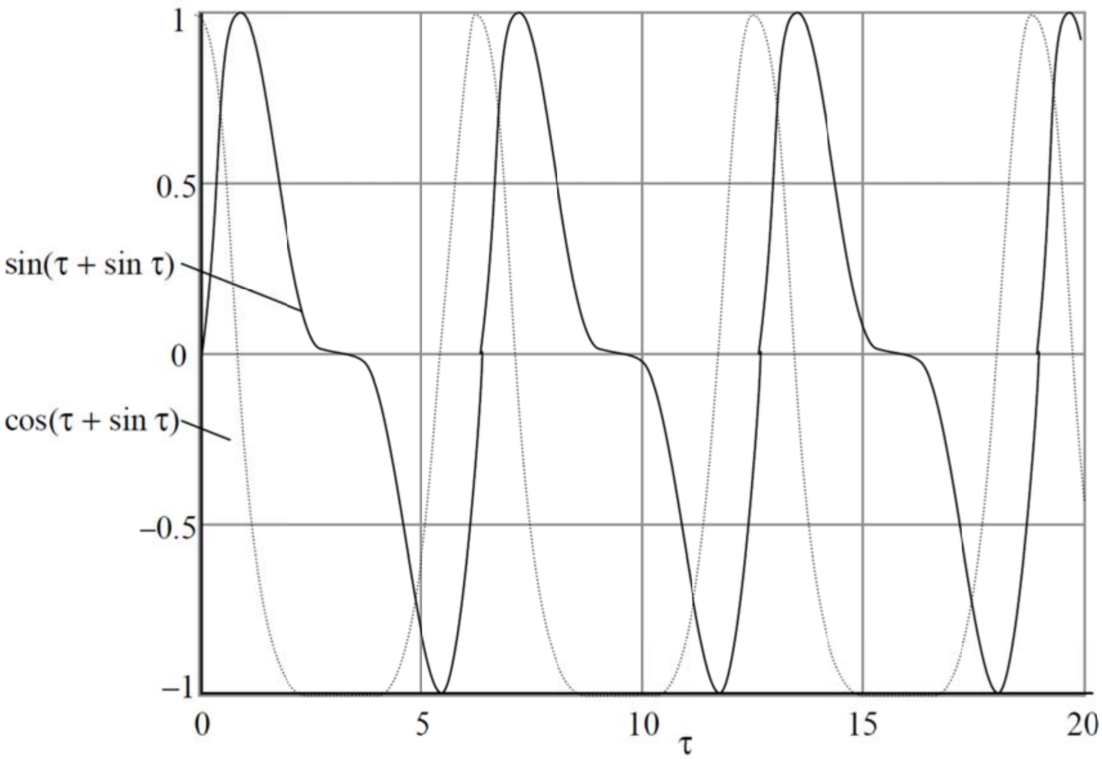

Figure 1.1: The real (dotted line) and imaginary (full line) components of Ψ.

(c) Note that $|\Psi|^2 = |\psi(x)|^2$, and so it is stationary.

Exercise: Consider the form of Ψ for an exponentially switched cosine potential energy, $V(t) = V(1 - e^{-t/T}) \cos \omega t$, for various switching rates.

1.28 From eqn 1.44,

$$\frac{d\langle x \rangle}{dt} = \frac{i}{\hbar}\langle [H, x] \rangle$$

The commutator has been evaluated in Problem 1.11(b):

$$\langle [H, x] \rangle = \frac{\hbar}{im} p_x$$

and therefore

$$\frac{d\langle x \rangle}{dt} = \frac{\langle p_x \rangle}{m}$$

which is eqn 1.47.