Chapter 16

RBC and New Keynesian models — Web Appendix

Acknowledgement[1]

16.1 The equations in a simple RBC model

Production The model begins with the production function. This takes the form of the Cobb-Douglas production function:

$$y_t = B_t K_t^{\alpha} N_t^{1-\alpha} \tag{16.1}$$

where y is output, K is capital, and N is hours of labour (rather than employment as we usually define it). α is capital's share of income and $1-\alpha$ is labour's. We assume that next period's capital is equal to the existing capital stock (adjusted for depreciation, δ) plus any new investment:

$$K_{t+1} = I_t + (1-\delta)K_t.$$

The first step in setting out the model is to take the first order conditions for profit maximization by differentiating the profit function with respect to labour and with respect to capital and setting each equal to zero. For simplicity, we set the price level equal to 1:

$$\begin{aligned}
\text{Max profits} &= y_t - w_t N_t - r_t K_t \\
\frac{\partial \text{profits}}{\partial N} &= (1-\alpha)B_t K_t^{\alpha} N_t^{1-\alpha-1} - w_t = 0 \\
&\rightarrow w_t = (1-\alpha)B_t K_t^{\alpha} N_t^{-\alpha} = MPL \qquad (16.2) \\
\frac{\partial \text{profits}}{\partial K} &= \alpha B_t K_t^{\alpha-1} N_t^{1-\alpha} - r_t = 0 \\
&\rightarrow r_t = \alpha B_t K_t^{\alpha-1} N_t^{1-\alpha} = MPK. \qquad (16.3)
\end{aligned}$$

[1] Bob Rowthorn made a major contribution to this appendix.

Profit maximization produces the familiar results that under perfect competition, the wage is equal to the marginal product of labour and the real interest rate is equal to the marginal product of capital. Note that the production decision does not involve any expectations because all the variables are for the current period.

Consumption Consumption behaviour follows from the permanent income model of Chapter 1. The difference is that we now include leisure as well as consumption in the utility function. The agent maximizes utility over an infinite horizon subject to the intertemporal budget constraint

$$\text{Max} U_t = \underbrace{E_t \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u\left(C_t, l_t\right)}_{\text{discounted PV of utility}}$$

$$\text{s.t.} \quad \underbrace{\sum_{t=0}^{\infty} \frac{1}{(1+r_t)^t} C_t}_{\text{PV of lifetime consumption}} \leq \underbrace{a_0 + \sum_{t=0}^{\infty} \frac{1}{(1+r_t)^t} (w_t(1-l_t))}_{\text{PV of lifetime wealth = initial assets + income from working}} .$$

Note that there are minor presentational variations from other chapters. Firstly, we use B to refer to technology in the production function and we use a to refer to the individual's assets. Second, because of our focus here on the leisure/labour choice, we use l to refer to the the fraction of time devoted to leisure. This means that $0 < l < 1$. It is also the case that the proportion of time devoted to labour, N, varies between zero and one and that $1 = l_t + N_t$. Third, instead of assuming r constant as we did in Chapter 1, we allow it to vary.

We assume a specific form for the consumption function. We use a constant relative risk aversion (CRRA) utility function as this simplifies the analysis by making the labour supply curve vertical. The utility function can be written as,

$$\text{Max} U_t = \underbrace{E_t \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t}}_{\text{discounted PV of utility}} \left(\ln C_t + \theta \frac{l_t^{1-\eta}}{1-\eta} \right).$$

Intertemporal optimization The RBC model has two intertemporal optimization conditions. One is the intertemporal consumption optimization condition (the consumption Euler equation from Chapter 1) and the second is the leisure Euler equation. We firstly derive the consumption Euler equation. This involves equating the marginal rate of substitution (MRS) and the marginal rate of transformation (MRT). The MRS is the ratio of marginal utilities of consumption in periods t and $t+1$, taking into account the fact that utility in period $t+1$ must be discounted by the subjective discount factor, ρ. The MRT is the rate at which savings in period t can be transformed into income in period $t+1$, which rests on the interest rate, r_t.

$$\underbrace{\frac{\frac{1}{C_t}}{E_t \frac{1/C_{t+1}}{1+\rho}}}_{MRS} = \underbrace{E_t(1+r_t)}_{MRT} \tag{16.4}$$

$$\rightarrow \frac{1}{C_t} = E_t \frac{(1+r_t)}{(1+\rho)} \frac{1}{C_{t+1}} \tag{16.5}$$

$$\rightarrow C_t = E_t \frac{(1+\rho)}{(1+r_t)} C_{t+1} \quad \text{(consumption Euler equation)}$$

This produces a consumption Euler equation the same as that derived in Chapter 1. The equation shows that consumption today is a function of expected consumption tomorrow. The preference for consumption today or tomorrow depends on the relative size of r_t and ρ. In the special case where $r_t = \rho$ consumption is perfectly smooth and the agent consumes the same in every period.

We can now move on to derive the leisure Euler equation. This again involves equating the MRS and MRT, but this time for leisure instead of consumption:

$$\underbrace{\frac{\theta l_t^{-\eta}}{E_t \frac{\theta l_{t+1}^{-\eta}}{1+\rho}}}_{MRS} = \underbrace{E_t \frac{(1+r_t) w_t}{w_{t+1}}}_{MRT} \tag{16.6}$$

$$\rightarrow \theta l_t^{-\eta} = E_t \frac{(1+r_t)}{(1+\rho)} \frac{w_t}{w_{t+1}} \theta l_{t+1}^{-\eta} \tag{16.7}$$

$$\rightarrow l_t^{\eta} = E_t \frac{(1+\rho)}{(1+r_t)} \frac{w_{t+1}}{w_t} l_{t+1}^{\eta} \quad \text{(leisure Euler equation)}$$

In the leisure Euler equation, the left hand side relates to the leisure choice. And this time, the agent make the hours decision controlling as well for the rate at which utility from working today is transformed into utility from working tomorrow ($\frac{w_t}{w_{t+1}}$).

The technology shock In the RBC model, the equilibrium is disturbed by a (temporary and persistent) shock to B_t in the production function. B_t is referred to as total factor productivity or the Solow residual. It is easy to see what it means by rearranging the production function to get:

$$y_t = B_t K_t^{\alpha} N_t^{1-\alpha}$$
$$B_t = \frac{y_t}{K_t^{\alpha} N_t^{1-\alpha}}.$$

Whereas labour productivity is output per worker or per hour and capital productivity is output per unit of fixed capital. B_t is a measure of productivity where both capital and labour inputs are taken into account. If B_t shifts then the output produced by given inputs changes and it is in this sense that it is a measure of technology.

In the Ramsey growth model, B is growing at a constant exponential rate: this is the economy's steady state rate of growth of per capita output. In the RBC model, the trend growth of B is removed and we examine the consequences of a shock to B_t. The technology shock is modelled in the following way:

$$B_t = \xi B_{t-1} + \varepsilon_t, \qquad \text{(shock to technology)}$$

where $0 < \xi < 1$ captures the persistence of the technology shock. This is an ad hoc assumption about the shock — it does not have any microeconomic foundations. It says that the technology shock dies away gradually rather than disappearing the following period, which would be the case if $\xi = 0$. On the other hand, the shock does not last forever, because $\xi < 1$. ε_t is the random shock.

The technology shock and the business cycle

We now repeat the description of the propagation and amplification mechanisms from Section 16.2.3 of Chapter 16, discussing how the underlying equations are affected at each step. A positive shock to ε_t means that B_t goes up.

1. Output goes up because of the technology shock as shown by the production function (Equation 16.1).

2. From the first order conditions of the profit function (Equations 16.2 and 16.3), we know that both the real wage and the real interest rate rise in response to a higher B_t.

3. The rise in the interest rate calls for a response via the consumption Euler equation. The higher return from saving leads to a cut in current consumption. Saving and therefore investment go up.

4. The leisure Euler equation has also been disturbed by the increase in the wage and in the interest rate. The agent re-optimizes by reducing leisure and 'making hay while the sun shines' taking advantage of the temporarily higher wage.

5. The outcome of higher saving is a larger capital stock next period. From the production function, the labour demand curve shifts outward extending the upswing initiated by the technology shock.

6. As the technology shock peters out, the economy gradually returns to the steady state growth path.

16.2 The derivation of the New Keynesian Phillips curve

The first stage in deriving the New Keynesian Phillips curve is to formally set out a model of Calvo pricing. If we first assume that firms can adjust their price in every period then what price would they set? Following Calvo's logic, the optimal price, p_t^{**}, depends on the output gap, x_t, and the general price level, p_t:

$$p_t^{**} = p_t + \alpha x_t. \tag{16.8}$$

In Equation 16.8, the prices are expressed in natural logs and $x_t = \log y_t - \log y_{e_t}$. This expression shows that the optimal price depends positively on the output gap — i.e. firms would like to set a higher price when output is above equilibrium. Firms choose their price in each period to maximise their profits. We assume a quadratic profit function to capture the idea that profits are higher the closer is $p_t^*$to p_t^{**}(as well as simplifying the mathematics):

$$F(p_t^*, p_t, x_t) = A - (p_t^* - p_t^{**})^2 = A - (p_t^* - p_t - \alpha x_t)^2. \tag{16.9}$$

The next stage in deriving the New Keynesian Phillips curve is to introduce price stickiness. As discussed in Section 16.3.1 of Chapter 16, under Calvo pricing, firms can only change their prices in a given period if they receive a 'green light'. Each period, the probability that a firm will receive the green light is δ. This means that when firms do get to change their price in this period, they also take into account the fact they might not be able to change their prices in future periods. In fact, there is a $(1 - \delta\ )^i$ chance that a firm will not be able to change its price for at least the next i periods. In this Calvo pricing model, the present value of the future stream of profits from choosing price p_t^* is therefore given by:

$$V_t = F(p_t^*, p_t, x_t) + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i F(p_t^*, p_{t+i}, x_{t+i}), \tag{16.10}$$

where ψ is the discount factor. Equation 16.10 shows that the firm must set the current price taking into account their optimal price for this period and the expected optimal prices for future periods. Firms are fully forward-looking in this model and have to form expectations about the future price level (p_{t+i}) and the future output gap (x_{t+i}).

We now need to find the price (p_t^*) that maximises the present value of the future stream of profits (V_t). We can do this by substituting Equation 16.9 into Equation 16.10, differentiating it with respect to p_t^* and setting it equal to zero:

$$V_t = A - (p_t^* - p_t - \alpha x_t)^2 + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i [A - (p_t^* - p_{t+i} - \alpha x_{t+i})]^2 \tag{16.11}$$

$$\frac{\partial V_t}{\partial p_t^*} = \frac{\partial}{\partial p_t^*} \left[A - (p_t^* - p_t - \alpha x_t)^2 + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i [A - (p_t^* - p_{t+i} - \alpha x_{t+i})]^2 \right] \tag{16.12}$$

$$0 = (p_t^* - p_t - \alpha x_t) + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i (p_t^* - p_{t+i} - \alpha x_{t+i}). \tag{16.13}$$

The next step is to rearrange equation 16.13 to get the equation in terms of p_t^*:[2]

$$p_t^* \left(1 + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i\right) = p_t + \alpha x_t + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i (p_{t+i} + \alpha x_{t+i}) \tag{16.14}$$

$$p_t^* = (1 - (1-\delta)\psi) \left[p_t + \alpha x_t + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i (p_{t+i} + \alpha x_{t+i})\right]. \tag{16.15}$$

By the same logic as used up to this point, we can use Equation 16.15 to find an expression for the optimal price for firms in period $t+1$. The maximisation problem is exactly the same in period $t+1$, but just moved forward one period:

$$p_{t+1}^* = (1 - (1-\delta)\psi) \left[p_{t+1} + \alpha x_{t+1} + \sum_{i=1}^{\infty} [(1-\delta)\psi]^i (p_{t+i+1} + \alpha x_{t+i+1})\right] \tag{16.16}$$

$$p_{t+1}^* = (1 - (1-\delta)\psi) \left[\sum_{i=1}^{\infty} [(1-\delta)\psi]^{i-1} (p_{t+i} + \alpha x_{t+i})\right]. \tag{16.17}$$

We now multiply both sides of 16.17 through by $(1-\delta)\psi$ and then using Equations 16.15 and 16.18 we can find an expression for p_t^* in terms of p_{t+1}^*:

$$(1-\delta)\psi p_{t+1}^* = (1 - (1-\delta)\psi) \left[\sum_{i=1}^{\infty} [(1-\delta)\psi]^i (p_{t+i} + \alpha x_{t+i})\right]. \tag{16.18}$$

$$p_t^* = [1 - (1-\delta)\psi](p_t + \alpha x_t) + (1-\delta)\psi p_{t+1}^* \tag{16.19}$$

Phillips curves are typically expressed in terms of inflation and not prices, so we need to convert Equation 16.19 into terms of inflation. This requires the following definitions:

$$\begin{aligned} \pi_t^* &= p_t^* - p_{t-1} \\ \pi_{t+1}^* &= p_{t+1}^* - p_t \\ \pi_t &= p_t - p_{t-1} \\ \pi_{t+1} &= p_{t+1} - p_t. \end{aligned}$$

These definitions can then be substituted into Equation 16.19 to become:

$$\pi_t^* = \pi_t + [1 - (1-\delta)\psi]\alpha x_t + (1-\delta)\psi \pi_{t+1}^*. \tag{16.20}$$

[2] To get from Equation 16.14 to Equation 16.15 requires using the formula for the sum of a geometric series. In this case the geometric series starts from period 1 and not period 0 as normal. The rule therefore becomes: $z + z^2 + z^3 ... + z^\infty = \frac{1}{1-z} - 1$.

If we assume that firms actually set the optimal prices in each period, then all firms who get the green light to change prices in a given period will set the optimal price. The remaining firms are not able to change their prices, which means we can express inflation in periods t and $t+1$ as:

$$\begin{aligned}\pi_t &= \delta\pi_t^* \\ \pi_{t+1} &= \delta\pi_{t+1}^*\end{aligned}$$

The next stage of the derivation involve substituting these expressions back into Equation 16.20 and simplifying:

$$\begin{aligned}\frac{\pi_t}{\delta} &= \pi_t + [1-(1-\delta)\psi]\alpha x_t + (1-\delta)\psi\frac{\pi_{t+1}}{\delta} \\ \pi_t(1-\delta) &= \delta[1-(1-\delta)\psi]\alpha x_t + (1-\delta)\psi\pi_{t+1} \\ \pi_t &= \psi\pi_{t+1} + \frac{\delta[1-(1-\delta)\psi]}{1-\delta}\alpha x_t \qquad (16.21)\end{aligned}$$

The last stage of the derivation is to replace π_{t+1} by its expectation. This reflects the fact that π_{t+1} is not known with certainty in period t and firms have to form expectations about the future whilst making decisions in the present. This yields the New Keynesian Phillips curve that is discussed in Section 16.3.1 of Chapter 16:

$$\pi_t = \psi E_t\pi_{t+1} + \frac{\delta[1-(1-\delta)\psi]}{1-\delta}\alpha x_t.$$

16.3 The behaviour of inflation in the New Keynesian model

The New Keynesian Phillips curve is of the form

$$\pi_t = \psi E_t\pi_{t+1} + \beta x_t,$$

where x_t is the output gap and β is equal to $\frac{\delta[1-(1-\delta)\psi]}{1-\delta}a$. We can represent this $NKPC$ in its alternative form by carrying out repeated substitution (following the same method as shown in Section 16.3.1 of Chapter 16). This yields a $NKPC$ of the form

$$\pi_t = \psi^k E_t\pi_{t+k} + \beta\sum_{i=0}^{i=k-1}\psi^i E_t x_{t+i}.$$

Note that this solution is based on the rule of iterated expectations, which implies that $E_tE_{t+1} = E_t$ etc. This form of the $NKPC$ also assumes a finite amount of periods (up to period $t+k$), rather than the infinite time horizon used in Chapter 16. This allows us to more easily model the reaction of inflation to shocks lasting a finite amount of time (e.g. temporary inflation or demand shocks).

Suppose that an economy initially has zero inflation and the output gap is zero. Then $\pi_t = 0$ and $\pi_{t-1} = 0$. Suppose also that there is a temporary positive shock to output, which starts in period t and ends at period $t+T$. This positive output gap is the same magnitude in each future period (until it disappears), such that $x_t = x_{t+1} = x_{t+2} = x_{t+3}... = x_{t+T-1} = \Delta > 0$ and that $x_{t+i} = 0$ for $i \geq T$. Finally, suppose that expectations with regard to future output gaps are fulfilled, so that $E_t x_{t+i} = x_{t+i}$ for all $i \geq 0$. And assume that $k > T$.

$$\begin{aligned}
\pi_t &= \psi^k E_t \pi_{t+k} + \beta \sum_{i=0}^{i=T-1} \psi^i E_t x_{t+i} \\
&= \psi^k E_t \pi_{t+k} + \beta \sum_{i=0}^{i=T-1} \psi^i x_{t+i} \\
&= \psi^k E_t \pi_{t+k} + \beta \sum_{i=0}^{i=T-1} \psi^i \Delta \\
&= \psi^k E_t \pi_{t+k} + \beta \Delta \left(\frac{1-\psi^T}{1-\psi} \right).
\end{aligned}$$

The last line of the derivation shown above uses the formula for the sum of a geometric series.[3] If $0 < \psi < 1$ and $E_t \pi_{t+k}$ is bounded for all t and k, then letting $k \to \infty$ yields

$$\pi_t = \beta \left(\frac{1-\psi^T}{1-\psi} \right) \Delta. \tag{16.22}$$

By the same logic,

$$\begin{aligned}
\pi_{t+m} &= \beta \left(\frac{1-\psi^{T-m}}{1-\psi} \right) \Delta \text{ for } T > m \geq 0 \qquad (16.23) \\
\pi_{t+m} &= 0 \text{ for } m \geq T.
\end{aligned}$$

Thus, inflation jumps and then gradually falls back again to zero. We can show this using Equations 16.22 and 16.23: as $0 < \psi < 1$ we know that $\psi^T < \psi^{T-m}$ for $T > m \geq 0$, which means that $\left(\frac{1-\psi^T}{1-\psi}\right) > \left(\frac{1-\psi^{T-m}}{1-\psi}\right)$ and that inflation is higher in period t than it is in period $t+m$. We can also see from Equation 16.23 that the nearer we are to time T (i.e. the higher is m), the smaller $\left(\frac{1-\psi^{T-m}}{1-\psi}\right)$

[3] We have a geometric series of the form: $1 + \psi + \psi^2 + ... + \psi^{T-1}$. We can solve this series by using the formula for the sum of the first T terms of a geometric series, which states that: $\sum_{i=0}^{i=T-1} \psi^i = \frac{1-\psi^T}{1-\psi}$.

becomes, which means that inflation gradually falls between period t and period $t+T$.

Let us now consider the case where $\psi = 1$. If $\psi = 1$, then

$$\begin{aligned}
\pi_t &= \psi^k E_t \pi_{t+k} + \beta \sum_{i=0}^{i=T-1} \psi^i x_{t+i} \text{ for } k \geq T \\
&= E_t \pi_{t+k} + \beta \sum_{i=0}^{i=T-1} \Delta \text{ for } k \geq T \\
&= E_t \pi_{t+k} + \beta T \Delta \text{ for } k \geq T.
\end{aligned}$$

Likewise,

$$\pi_{t+m} = E_t \pi_{t+k} + \beta (T - m) \Delta \text{ for } m < T \text{ and } k \geq T.$$

Since this holds for all $k \geq T$, it follows that $E_t \pi_{t+k} = E_t \pi_{t+i}$ for all $k, i \geq T$.

Suppose that $E_t \pi_{t+k} = \pi^*$ for all $k \geq T$. Suppose also, that all price expectations are fulfilled. Then the actual path of inflation is as follows

$$\begin{aligned}
\pi_{t+m} &= \pi^* + \beta (T - m) \Delta \text{ for } T > m \geq 0 \\
\pi_{t+m} &= \pi^* \text{ for } m \geq T.
\end{aligned}$$

Thus, if $\psi = 1$, the New Keynesian Phillips curve does not determine a unique inflation path. What happens depends on long-term expectations. When the positive shocks to output begin inflation jumps upwards and then gradually falls, but how high it initially jumps and where it eventually stabilizes depends on long-term expectations. There is an infinite number of possible inflation trajectories along which expectations are fulfilled. To determine the unique path we require that $\psi < 1$. In this case, we also require that future inflation expectations are bounded.

Why is there an initial jump in inflation?

The initial jump in inflation occurs because all future output gap shocks are taken into account when firms are making pricing decisions in the current period (i.e. period t). In future periods (i.e. $t+1$ onwards), there are fewer shocks to take into account in the summation and hence inflation falls from its period t level.

16.4 Inflation trajectories in the 3-equation and New Keynesian models

The subsection compares the behaviour of inflation implied by the following adaptive expectations (i.e. backwards-looking) Phillips curve

$$\pi_t = \pi_{t-1} + \alpha x_t \tag{16.24}$$

and the following forward-looking New Keynesian Phillips curve

$$\pi_t = E_t\pi_{t+1} + \alpha x_t, \tag{16.25}$$

where x_t is the output gap. To make the $NKPC$ as simple as possible, we have assumed that $\psi = 1$ (i.e. there is no discounting) and δ is such that the term $\frac{\delta[1-(1-\delta)\psi]}{1-\delta}$ is equal to 1 and hence can be removed from the equation.[4]

We can now use Equations 16.24 and 16.25 to trace the path of inflation following a temporary positive demand shock using both the adaptive expectations and NK Phillips curves. This provides the mathematical underpinnings to the impulse response functions shown in Fig. 16.6 in Chapter 16.

We start with the adaptive (i.e. backward-looking) expectations case. Suppose that $\pi_{t-1} = 0$ and that the economy is subject to a finite series of identical positive shocks; $x_t, x_{t+1}, x_{t+2....}x_{t+T-1} = \Delta > 0$ and that $x_{t+k} = 0$ for $k \geq T$ — i.e. there is a positive output gap that starts in period t and ends in period $t+T$. As in the previous section of the appendix, the time horizon of this problem runs from period t to period $t+k$. This is in contrast to the infinite time horizon shown in the main body of the chapter. Under these assumptions, the adaptive expectations Phillips curve (Equation 16.24) generates the following sequence:

$$\begin{aligned}
\pi_{t-1} &= 0 \\
\pi_t &= \alpha x_t = \alpha\Delta \\
\pi_{t+1} &= \alpha x_t + \alpha x_{t+1} = 2\alpha\Delta \\
\pi_{t+2} &= \alpha x_t + \alpha x_{t+1} + \alpha x_{t+2} = 3\alpha\Delta \\
&\ldots \\
\pi_{t+T-2} &= \alpha x_t + \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} = (T-1)\alpha\Delta \\
\pi_{t+T-1} &= \alpha x_t + \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} + \alpha x_{t+T-1} = T\alpha\Delta \\
\pi_{t+T} &= \alpha x_t + \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} + \alpha x_{t+T-1} = T\alpha\Delta \\
&\ldots \\
\pi_{t+T+i} &= \alpha x_t + \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} + \alpha x_{t+T-1} = T\alpha\Delta.
\end{aligned}$$

We can see from the equations above that the inflation rate builds up and then stabilizes at the rate $\pi = T\alpha\Delta$. This is because in each period inflation is equal to lagged inflation plus the output gap. This means that inflation will rise for as long as the positive output gap persists. Once the output gap returns to normal in period $t+T$, then inflation stops rising.

We now move onto the New Keynesian Phillips curve. To solve this forward-looking case, we require some assumption about expectations. Suppose that, from time t onwards all future shocks are foreseen and that the long-term expectation is that inflation will stabilize at rate π^*. In this case, $E_{t+k}\pi_{t+k+1} = \pi_{t+k+1}$ for all $k \geq 0$. The solution to the New Keynesian Phillips curve (Equation 16.25) is then:

[4]The value of δ which satisfies this condition is 0.618033988.

$$
\begin{aligned}
\pi_{t-1} &= 0 \\
\pi_t &= \pi^* + \alpha x_t + \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} + \alpha x_{t+T-1} = \pi^* + T\alpha\Delta \\
\pi_{t+1} &= \pi^* + \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} + \alpha x_{t+T-1} = \pi^* + (T-1)\alpha\Delta \\
&\dots \\
\pi_{t+T-2} &= \pi^* + \alpha x_{t+T-2} + \alpha x_{t+T-1} = \pi^* + 2\alpha\Delta \\
\pi_{t+T-1} &= \pi^* + \alpha x_{t+T-1} = \pi^* + \alpha\Delta \\
\pi_{t+T} &= \pi^* + 0 = \pi^* \\
&\dots \\
\pi_{t+T+i} &= \pi^* + 0 = \pi^*.
\end{aligned}
$$

We can see from the equations above that in the forward-looking case, inflation jumps upwards in period t and then slowly falls back to its long-run expected level over the course of the output gap. This is because inflation in each period depends on the entire path of expected future output gaps. As time goes on, the time the output gap is expected to persist for is diminished, causing firms to want to adjust their prices by less and inflation to slowly fall back towards its long-run expected level. Note that the solution is not unique. It depends on the value of π^*. If we assume that the long-term expectation is that inflation will eventually disappear (i.e. $\pi^* = 0$), the solution to the New Keynesian curve is as follows:

$$
\begin{aligned}
\pi_{t-1} &= 0 \\
\pi_t &= \alpha x_t + \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} + \alpha x_{t+T-1} = \pi^* + T\alpha\Delta \\
\pi_{t+1} &= \alpha x_{t+1} + \alpha x_{t+2} + .. + \alpha x_{t+T-2} + \alpha x_{t+T-1} = \pi^* + (T-1)\alpha\Delta \\
&\dots \\
\pi_{t+T-2} &= \alpha x_{t+T-2} + \alpha x_{t+T-1} = \pi^* + 2\alpha\Delta \\
\pi_{t+T-1} &= \alpha x_{t+T-1} = \pi^* + \alpha\Delta \\
\pi_{t+T} &= 0 \\
&\dots \\
\pi_{t+T+i} &= 0.
\end{aligned}
$$

We can see from the equations in this subsection that the trajectory of inflation differs in the adaptive expectations and New Keynesian cases. How can we explain this in simple terms? In the adaptive expectations (i.e. backward-looking) solution, the sequence of disturbances that are taken into account becomes longer over time. In the New Keynesian (i.e. forward-looking) solution, this sequence gets shorter over time. The disturbances are all positive (or zero from $t + T$ onwards), which means that inflation rises over time (up to $t + T$) in the backward-looking case, but falls over time (up to $t + T$) in the forward-looking case.