26

Vectors

Answers to additional problems

26.1 Inserting terms, $w = \int_0^2 (2t\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}) \cdot (5\,\hat{\mathbf{i}} - t\,\hat{\mathbf{j}})\,\mathrm{d}t = \int_0^2 6t\,\mathrm{d}t = [3t^2]_0^2 = 12\,\mathrm{J}$

Initially, we could have used a slightly different notation,

$$w = \int_0^2 \begin{pmatrix} 2t \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ -t \end{pmatrix} \mathrm{d}t = \int_0^2 (10t - 4t)\,\mathrm{d}t$$

The rest of the solution remains unchanged.

26.2 The cross product of this vector is given from eqn. (26.10).
By definition, $\mathbf{H} \times \mathbf{H} = (H_y H_z - H_z H_y)\,\hat{\mathbf{i}} - (H_x H_z - H_z H_x)\,\hat{\mathbf{j}} + (H_x H_y - H_y H_x)\hat{\mathbf{k}}$
Since $H_y H_z = H_z H_y$, $H_x H_z = H_z H_x$, and $H_x H_y = H_y H_x$, the sum $= 0\,\hat{\mathbf{i}} - 0\,\hat{\mathbf{j}} - 0\,\hat{\mathbf{k}}$.
so there is no vector product of a vector with itself.

Alternatively, we could say, $\mathbf{H} \times \mathbf{H} = |\,\mathbf{H}\,| \times |\,\mathbf{H}\,| \sin\theta$. As the two vectors are parallel, $\theta = 0$; and $\sin 0 = 0$. Therefore, $\mathbf{H} \times \mathbf{H} = 0$.

Remember, $\hat{\mathbf{i}} \bullet \hat{\mathbf{i}} = \hat{\mathbf{j}} \bullet \hat{\mathbf{j}} = 0$.

26.3

Strategy

1. Force $\mathbf{F} = ma\,\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit vector pointing vertically upwards.
2. Calculate the displacement vector $\mathbf{d}$ as a simple vector subtraction.
3. Work w is the dot product, $\mathbf{F} \bullet \mathbf{d}$

Solution

1. $\mathbf{F} = ma\,\hat{\mathbf{k}} = 10 \times -9.8\,\hat{\mathbf{k}} = -98\,\hat{\mathbf{k}}\mathrm{N}$
2. The displacement vector $\mathbf{d} = (5-2)\,\hat{\mathbf{i}} + (4-3)\,\hat{\mathbf{j}} + (7-3)\,\hat{\mathbf{k}} = 3\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
3. Work $= \mathbf{F} \bullet \mathbf{d} = (-98\,\hat{\mathbf{k}}\mathrm{J}) \bullet (3\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

so work $= (0 \times 3) + (0 \times 1) + (-98 \times 4) = -392\,\mathrm{J}$.

In an alternative notation, $\mathbf{F} \bullet \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ -98 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$.

Notice the negative sign verifies that the work is done against gravity because we're lifting the evaporator. Therefore, it requires 392 J of work to move the rotary evaporator.

26.4 We start by remembering from p. 476 that we obtain the determinant of a matrix by reducing it.

$$\text{so}\quad \mathbf{curl\,A} = \hat{\mathbf{i}}\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_y & A_z \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ A_x & A_z \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ A_x & A_y \end{vmatrix}$$

We then obtain the determinant of each minor,

$$\mathbf{curl\,A} = \hat{\mathbf{i}}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) - \hat{\mathbf{j}}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) + \hat{\mathbf{k}}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$$

Tidying up then yields,

$$\textbf{curl A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{k}}$$

which is eqn. (26.12) if each of the coefficients A_i relate to the vector **A**.

26.5 By definition, power = $\mathbf{F} \cdot \mathbf{v}$

so power $= (5t^2\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}) \cdot (t\hat{\mathbf{i}} - 2t^2\hat{\mathbf{j}}) = \begin{pmatrix} 5t^2 \\ 2t \end{pmatrix} \cdot \begin{pmatrix} t \\ -2t^2 \end{pmatrix} \begin{pmatrix} 5t^3 \\ -4t^3 \end{pmatrix}$

so $\mathbf{F} \cdot \mathbf{v} = 5t^3 - 4t^3 = t^3$.

Alternatively, we could multiply the brackets,

We also call the torque a **moment of force**. A torque produces (or tends to produce) torsion or rotation. Torque is to angular motion what a force is to linear motion.

power $= (5t^2\hat{\mathbf{i}}) \cdot (t\,\hat{\mathbf{i}}) + (5t^2\hat{\mathbf{i}}) \cdot (-2t^2\hat{\mathbf{j}}) + (2t\,\hat{\mathbf{j}}) \cdot (t\,\hat{\mathbf{i}}) + (2t\,\hat{\mathbf{j}}) \cdot (-2t^2\hat{\mathbf{j}})$

power $= 5t^3\,\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} - 10t^4\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + 2t^2\hat{\mathbf{j}} \cdot \hat{\mathbf{i}} - 4t^3\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}$

by definition, however, $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{i}} = 0$, so power $= 5t^3 - 4t^3 = t^3$.
Notice how this result, being a scalar product, has magnitude but no direction.

26.6 From eqn. (26.11), $\mathbf{r} \times \mathbf{F} = (1 \times 4 - 3 \times 2)\,\hat{\mathbf{i}} - (2 \times 4 - 3 \times 3)\,\hat{\mathbf{j}} + (2 \times 2 - 1 \times 3)\,\hat{\mathbf{k}}$

so $\mathbf{r} \times \mathbf{F} = -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$.

In matrix notation, we write, $\mathbf{r} \times \mathbf{F} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$

26.7 From eqn. (26.1), the horizontal component of the vector = 40.2 cos 40° = 30.8 m s^{-1}

From eqn. (26.2), the vertical component of the vector = 40.2 sin 40° = 25.8 m s^{-1}

26.8 **Strategy**

We are being asked to find a directional derivative, so we,

1. Calculate ∇f.
2. Find the derivative in the direction of $\hat{\mathbf{r}}$ using the equation, $\nabla f \cdot \hat{\mathbf{d}}$.
3. Substitute in for (x,y,z).

Solution

1. The three differentials are,

$$\left(\frac{\partial f}{\partial x}\right) = \frac{-Ax}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}$$

$$\left(\frac{\partial f}{\partial y}\right) = \frac{-Ay}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}$$

$$\left(\frac{\partial f}{\partial z}\right) = \frac{-Az}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}$$

$$\nabla f = \frac{-A}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)$$

2. The gradient of f in the direction of $\hat{\mathbf{d}}$ is given by,

$$\nabla f \cdot \hat{\mathbf{d}} = \frac{-A}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right) \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) / \sqrt{3}$$

$$= \frac{-A}{\sqrt{3}(x^2+y^2+z^2)^{\frac{3}{2}}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{-A(x+y+z)}{\sqrt{3}(x^2+y^2+z^2)^{\frac{3}{2}}}$$

3. At the point, (1,2,2),

$$\nabla f \bullet \hat{\mathbf{d}} = \frac{-A(1+2+2)}{\sqrt{3}(1^2+2^2+2^2)^{\frac{3}{2}}} = \frac{-5A}{27\sqrt{3}}$$

(We should note that if we define $\hat{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and let $|\,\mathbf{r}\,| = r$, then the gradient can be written as, $\nabla f = \frac{-A\mathbf{r}}{r^3} = \frac{-A\hat{\mathbf{r}}}{r^2}$, which is a gravitational vector field.)

In fact, we can derive the gradient much easily using the form of the operator ∇ in spherical polar coordinates, $f(r,\theta,\phi) = \frac{A}{r}$

so the gradient of this scalar function is,

$$\nabla f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} = \frac{-A\mathbf{e}_r}{r^2} = \frac{-A\hat{\mathbf{r}}}{r^2}.$$

26.9 We know from eqn. (26.9) that $\mathbf{l} = \mathbf{r} \times \mathbf{p} = rp\sin\theta\,\hat{\mathbf{n}}$.

Since the motion is circular, the position and momentum vectors are at right angles, therefore $\sin\theta = 1$.

Thus, $|\,\mathbf{l}\,| = l = rp$

One form of the kinetic energy is, $E = \frac{p^2}{2\mu}$

Substituting in for the momentum gives, $E = \frac{l^2}{2\mu r^2} = \frac{l^2}{2I}$

26.10 Strategy

1. Calculate $\mathbf{b} \times \mathbf{c}$.

Note that the area of the base of the parallelepiped is $|\,\mathbf{b} \times \mathbf{c}\,|$.

2. Calculate the dot product $\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}$

Solution

1. $$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4\times 1 - 1\times 2 \\ -(2\times 1 - 1\times 3) \\ 2\times 2 - (-4)\times 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ 16 \end{pmatrix}$$

Alternatively, we may write this answer as,

$\mathbf{b} \times \mathbf{c} = (2\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (3\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}) = (-6\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + 16\hat{\mathbf{k}})$

2. $$\mathbf{a} \bullet \mathbf{b} \times \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -6 \\ 1 \\ 16 \end{pmatrix} = (1\times -6) + (1\times 1) + (3\times 16) = 43$$

Alternatively, we may write this as,

$\mathbf{a} \bullet \mathbf{b} \times \mathbf{c} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \bullet (-6\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + 16\hat{\mathbf{k}}) = (1 \times -6) + (1 \times 1) + (3 \times 16) = 43$

The volume of the parallelepiped is 43 volume units.