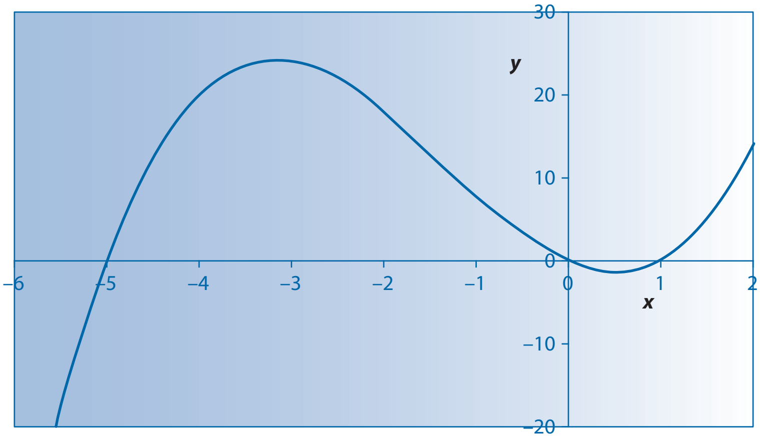
Differentiation V

Higher-order differentials and turning points

17

Answers to additional problems

17.1 The graph looks like this.



The highest power in the equation is 3 (there is an x^3 term) so the maximum number of turning points in the graph is (3 – 1), so 2.

Within the accuracy of the graph (above), the turning points are,

- A **maximum** at (–3, 24) and
- A **minimum** at (0.5,–1.38).

The graph cuts the x-axis at $x = -5$, 0, and 1.

17.2 Firstly, each term includes x, so we can factorize as, $y = x(x^2 + 4x - 5)$.

Secondly, we factorize the bracket using the methods learnt in Chapter 7 to give, $y = x(x - 1)(x + 5)$.

Accordingly, the graph will cut the axis when,

$$x = 0$$

$$(x - 1) = 0 \quad \text{so } x = +1$$

$$(x + 5) = 0 \quad \text{so } x = -5$$

These results agree with the values of x obtained using a graphical method in Additional Problem 17.1.

17.3 ***Strategy***

1. Differentiate in the usual way.
2. Equate the derivative to zero to obtain the coordinates of the turning points.
3. Take the second derivative and substitute for the values of x obtained in part 2.

Solution

1. First differential $y = x^3 + 4x^2 - 5x$ so $\frac{dy}{dx} = 3x^2 + 8x - 5$

   The turning points occur when the derivative is 0 so when $3x^2 + 8x - 5 = 0$.

2. The turning points occur when $(3x^2 + 8x - 5) = 0$. Using the quadratic formula (eqn. 7.6) yields, $x = \frac{-8 \pm \sqrt{8^2 - 4 \times 3 \times (-5)}}{2 \times 3} = \frac{-4 \pm \sqrt{31}}{3} = 0.523$ or -3.19.

   If $x = 0.523$, $y = -1.38$
   If $x = -3.19$, $y = 24.2$.

3. Second differential $\frac{d^2y}{dx^2} = 6x + 8$

   Substituting for $x = -3.19$ yields a negative result, so the first turning point (–3.19, 24.2) is a maximum.

   Substituting for $x = 0.523$ yields a positive result, so the second turning point (0.523, –1.38) is a minimum.

17.4

1. First derivative $\frac{dy}{dx} = 4x^3 - 3x^2 + 12 \times \frac{1}{x} + -2x^{-3}$ so $\frac{dy}{dx} = 4x^3 - 3x^2 + \frac{12}{x} - \frac{2}{x^3}$
2. Second derivative $\frac{d^2y}{dx^2} = 12x^2 - 6x - 12x^{-2} + 6x^{-4}$ so $\frac{d^2y}{dx^2} = 6\left\{2x^2 - x - \frac{2}{x^2} + \frac{1}{x^4}\right\}$

17.5 Using the chain rule yields, $\frac{dy}{dx} = 5(3x^2 + 1)(x^3 + x)^4$

The second differentiation step requires the product rule,

$$u = 5(3x^2 + 1) \quad \frac{du}{dx} = 5 \times 6x$$

$$v = (x^3 + x)^4 \quad \frac{dv}{dx} = 4(3x^2 + 1)(x^3 + x)^3$$

The differentiation of v here requires the use of the chain rule in Chapter 15.

Inserting terms into the product-rule expression (see Chapter 16),

$$\frac{dy}{dx} = 5\left\{(3x^2 + 1)\left[4(3x^2 + 1)(x^3 + x)^3\right] + (x^3 + x)^4[6x]\right\}$$

Factorizing yields, $\frac{dy}{dx} = 10(x^3 + x)^3\left[2(3x^2 + 1)(3x^2 + 1) + 3x^4 + 3x^2\right]$

then further tidying, $\frac{dy}{dx} = 10(x^3 + x)^3\left[21x^4 + 15x^2 + 2\right]$

17.6 A first differentiation requires the product rule,

$$u = \sin x \quad \frac{du}{dx} = \cos x$$

$$v = \cos x \quad \frac{du}{dx} = -\sin x$$

Inserting terms into the product-rule expression,

$$\frac{dy}{dx} = (\sin x)[-\sin x] + (\cos x)[\cos x], \quad \text{so} \quad \frac{dy}{dx} = -\sin^2 x + \cos^2 x$$

This worked example assumes that we performed the differentiation with x expressed in **radians**.

The second differentiation step requires the repeated use of the chain rule,

$$\frac{d^2y}{dx} = -(2\sin x\cos x) + (-2\sin x\cos x) \quad \text{so} \quad \frac{d^2y}{dx} = -4\sin x\cos x$$

17.7 $\frac{dy}{dx} = 6x^2 + 12x + 6$

Factorizing this expression yields, $6(x^2 + 2x + 1) = 6(x + 1)(x + 1)$ or $6(x + 1)^2$. The turning points therefore occur when $x = -1$. Back-substitution yields $y = -2$.

The second differential is $\frac{d^2y}{dx^2} = 12x + 12$.

When $x = -1$, $\frac{d^2y}{dx^2} = -12 + 12 = 0$.

To check this is an inflection point, we check that the value of the second differential at $x = -0.9$ and $x = -1.1$ have opposite signs.

When $x = -1.1$ $\frac{d^2y}{dx^2} = -13.2 + 12 = -1.2$ (a negative number)

When $x = -0.9$ $\frac{d^2y}{dx^2} = -10.8 + 12 = 1.2$ (a positive number)

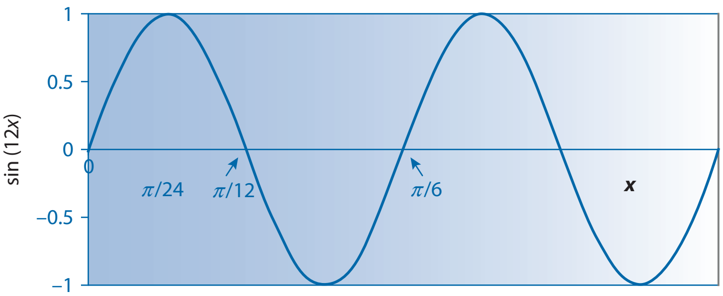
There is an inflection point at $(-1, -2)$

17.8 **1.** $\frac{dy}{dx} = 3x^2 + 2x$

At the turning points, $3x^2 + 2x = 0$.
Factorizing yields, $x(3x + 2) = 0$, so $x = 0$ or $x = -2/3$

2. $\frac{d^2y}{dx^2} = 6x + 2$
Inserting $x = 0$ yields a positive result so the curve has a minimum at $x = 0$.
Inserting $x = -2/3$ yields a negative result so the curve has a maximum at $x = -2/3$.
Therefore, the turning point at (0, 1) is a minimum and at $(-2/3, 1.15)$ is a maximum.

17.9



Before we start, it is worth noting that we expect many turning points. We are considering only the first in this question.

1. $\frac{dy}{dx} = 12\cos(12x)$
At the turning point, $\frac{dy}{dx} = 0$. Therefore $12\cos(12x) = 0$.The first possible solution of this expression is $12x = \pi/2$, $x = \pi/24$. Therefore, $x = 0.131$ radians $\approx 7.5°$.

2. $\frac{d^2y}{dx^2} = -12^2\sin(12x)$ or $-144\sin(12x)$
When $x = \pi/24$ radians, $\frac{d^2y}{dx^2} = -12^2\sin 12x$ is negative, implying a local maximum.

- This worked example assumes that we performed the differentiation with x expressed in radians.

17.10 **1.** $\frac{dy}{dx} = \frac{1}{x} - 2x$

At the turning point, $\frac{dy}{dx} = 0$. Therefore, $0 = \frac{1}{x} - 2x$. Slight rearranging gives, $1/x = 2x$ and hence, $1 = 2x^2$. More rearranging gives, $0.5 = x^2$ so $x = 0.707$ and $y = \ln(3 \times 0.707) - 0.5^2 = 0.252$.

2. $\frac{d^2y}{dx^2} = -\frac{1}{x^2} - 2$. Inserting values, at the turning point, $\frac{d^2y}{dx^2} = -\frac{1}{(0.707)^2} - 2$

The second derivative is negative when $x = 0.707$, which tells us the turning point is a maximum.

The turning point (0.707, 0.252) is a maximum.