Differentiation III

Differentiating functions of functions: the chain rule

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Answers to additional problems

15.1 To enable an opportunity to use the chain rule, we will ignore the general rule for differentiation of a sine, and say that $\omega t = u$ and $V = \sin u$.

For this example, the chain rule takes the form, $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}t}$

$$\text{If} \quad V = \sin u \quad \text{then} \quad \frac{\mathrm{d}V}{\mathrm{d}u} = \cos u$$

$$\text{If} \quad u = \omega\, t \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}t} = \omega$$

Inserting terms into the chain-rule expression yields,

$$\mathrm{d}V/\mathrm{d}t = \cos u \times \omega$$

and substituting for u yields,

$$\mathrm{d}V/\mathrm{d}t = \omega \cos \omega t$$

15.2 The jumble of constants before the sin term is a constant, so we will call it 'A'. We rewrite the expression, saying $I = A\,(\sin\theta)^2$. Therefore, $I = Au^2$ and $u = \sin\theta$

For this example, the chain rule takes the form, $\frac{\mathrm{d}I}{\mathrm{d}\theta} = \frac{\mathrm{d}I}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}\theta}$

$$\text{If} \quad I = Au^2 \quad \text{then} \quad \frac{\mathrm{d}I}{\mathrm{d}u} = 2Au$$

$$\text{If} \quad u = \sin\theta \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}\theta} = \cos\theta$$

Inserting terms into the chain-rule expression yields,

$$\frac{\mathrm{d}I}{\mathrm{d}\theta} = 2A\,u \times \cos\theta$$

and substituting for u yields,

$$\frac{\mathrm{d}I}{\mathrm{d}\theta} = 2A \sin\theta \times \cos\theta$$

Finally, we substitute for A,

$$\frac{\mathrm{d}I}{\mathrm{d}\theta} = I_o \frac{2\pi\alpha^2}{\varepsilon_r^2 \lambda^4 r^2} \sin\theta \cos\theta$$

15.3 The term within the root can be u, so the expression becomes, $\mathcal{E} = u^{1/2}$ where $u = \frac{1}{2}\mathcal{E}_0^2 + \frac{1}{2}\mathcal{E}_0^2 \cos(2\omega t)$. (Notice how the bracket has been multiplied out.)

For this example, the chain rule takes the form, $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}t}$

If $\mathcal{E} = u^{1/2}$ then $\dfrac{d\mathcal{E}}{dt} = \dfrac{1}{2}u^{-1/2} = \dfrac{1}{2u^{1/2}}$

If $u = \dfrac{1}{2}\mathcal{E}_0^2 + \dfrac{1}{2}\mathcal{E}_0^2\cos(2\omega t)$ then $\dfrac{du}{dt} = -2\omega \times \dfrac{1}{2}\mathcal{E}_0^2 + \dfrac{1}{2}\mathcal{E}_0^2\sin(2\omega t) = -\omega\,\mathcal{E}_0^2\sin(2\omega t)$

Inserting terms into the chain-rule expression yields,

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2u^{1/2}} \times -\omega\,\mathcal{E}_0^2\sin(2\omega t)$$

Re-inserting for u yields,

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2\left[\frac{1}{2}\mathcal{E}_0^2 + \frac{1}{2}\mathcal{E}_0^2\cos(2\omega t)\right]} \times -\omega\,\mathcal{E}_0^2\sin(2\omega t)$$

Tidying creates a neater expression,

$$\frac{d\mathcal{E}}{dt} = \frac{\omega\,\mathcal{E}_0^2\sin(2\omega t)}{2\left[\frac{1}{2}\mathcal{E}_0^2(1+\cos(2\omega t))\right]^{1/2}}$$

- This worked example assumes the differentiation was performed with θ expressed in radians.

15.4 We first rewrite the expression slightly, saying, $d = 3(\cos\theta)^2 - 1$. We now say $\cos\theta = u$, so $d = 3u^2 - 1$.

For this example, the chain rule takes the form, $\dfrac{dd}{d\theta} = \dfrac{dd}{du} \times \dfrac{du}{d\theta}$

If $d = 3u^2 - 1$ then $\dfrac{dd}{du} = 6u$

If $u = \cos\theta$ then $\dfrac{du}{d\theta} = -\sin\theta$

Inserting terms into the chain-rule expression yields,

$$\frac{dd}{du} = 6u \times -\sin\theta$$

and re-inserting for u yields,

$$\frac{dd}{du} = -6\,(\cos\theta) \times \sin\theta$$

A little tidying up will make the expression look neater,

$$\frac{dd}{du} = -6\cos\theta\sin\theta$$

- We cannot say $\mathrm{d}d = \mathrm{d}^2$ here because the upright d is an operator and the italic d is a variable.
- This worked example assumes the differentiation was performed with θ expressed in radians.

15.5 We firstly rearrange the equation to make d the subject, $d = \dfrac{n\lambda}{2\sin\theta}$.

We then rewrite the expression slightly, saying, $\sin\theta = u$, so $d = \frac{1}{2}n\lambda u^{-1}$.

For this example, the chain rule takes the form, $\dfrac{dd}{d\theta} = \dfrac{dd}{du} \times \dfrac{du}{d\theta}$

If $d = \frac{1}{2}n\lambda u^{-1}$ then $\dfrac{dd}{du} = -\frac{1}{2}n\lambda u^{-2} = -\dfrac{n\lambda}{2u^2}$

If $u = \sin\theta$ then $\dfrac{du}{d\theta} = \cos\theta$

Inserting terms into the chain-rule expression yields,

$$\mathrm{d}d/\mathrm{d}\theta = -\frac{n\lambda}{2u^2} \times \cos\theta = -\frac{n\lambda\cos\theta}{2u^2}$$

and substituting for u yields,

$$\mathrm{d}d/\mathrm{d}\theta = -\frac{n\lambda\cos\theta}{2(\sin\theta)^2} = -\frac{n\lambda\cos\theta}{2\sin^2\theta}$$

- We cannot say $\mathrm{d}d = \mathrm{d}^2$ here, because the upright d is an operator and the italic d is a variable.
- This worked example assumes the differentiation was performed with θ expressed in radians.

15.6 We want to find the derivative $\mathrm{d}P/\mathrm{d}n$. We start by simplifying the expression slightly. We say $P = k\exp(-An^2)$, where k is the first bracket above and $A = 1/(2N)$.

We next identify the functions. $u = -An^2$ and $P = k\exp u$.

For this example, the chain rule takes the form, $\frac{\mathrm{d}P}{\mathrm{d}n} = \frac{\mathrm{d}P}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}n}$

$$\text{If} \quad P = k\exp u \quad \text{then} \quad \frac{\mathrm{d}P}{\mathrm{d}u} = k\exp u$$

$$\text{If} \quad u = -An^2 \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}n} = -2An$$

Inserting terms into the chain-rule expression yields,

$$\frac{\mathrm{d}P}{\mathrm{d}n} - 2kAn\exp(-An^2)$$

Finally, we back-substitute for A and k,

$$\frac{\mathrm{d}P}{\mathrm{d}n} = -2\frac{n}{2N}\left(\frac{2}{\pi N}\right)^{1/2}\exp\left(-\frac{n^2}{2N}\right)$$

Tidying and cancelling yields our final result,

$$\frac{\mathrm{d}P}{\mathrm{d}n} = -n\left(\frac{2}{\pi N^3}\right)^{1/2}\exp\left(-\frac{n^2}{2N}\right)$$

15.7 We start by saying, $\psi = A\exp(u)$, where $u = -\alpha r^2$ and $A = \left(\frac{2\alpha}{\pi}\right)^{3/4}$.

For this example, the chain rule takes the form, $\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{\mathrm{d}\psi}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}r}$

$$\text{If} \quad \psi = A\exp(u) \quad \text{then} \quad \frac{\mathrm{d}\psi}{\mathrm{d}u} = A\exp(u)$$

$$\text{If} \quad u = -\alpha r^2 \quad \text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}r} = -2\alpha r$$

Inserting terms into the chain-rule expression yields,

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = A\exp(u) \times (-2\alpha r)$$

Tidying the expression yields,

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = -2\alpha r A\exp(u)$$

And substituting for u yields,

$$\frac{d\psi}{dr} = -2\alpha r A \exp(-\alpha r^2)$$

Finally, we substitute for A,

$$\frac{d\psi}{dr} = -2\alpha r\left(\frac{2\alpha}{\pi}\right)^{3/4} \exp(-\alpha r^2)$$

15.8 We start by saying, $\psi = K\cos u$ where $u = \dfrac{x\sqrt{2mE}}{h}$.

In this example, the chain rule takes the form, $\dfrac{d\psi}{dE} = \dfrac{d\psi}{du} \times \dfrac{du}{dE}$

$$\text{If} \quad \psi = K\cos u \quad \text{then} \quad \frac{d\psi}{du} = -K\sin u$$

$$\text{if} \quad u = \frac{x\sqrt{2mE}}{h} = \left(\frac{x\sqrt{2m}}{h}\right)E^{1/2} \quad \text{then} \quad \frac{du}{dE} = \frac{1}{2}\times\left(\frac{x\sqrt{2m}}{h}\right)E^{-1/2} = \frac{x}{h}\left[\frac{m}{2E}\right]$$

Inserting terms into the chain-rule expression yields,

$$\frac{d\psi}{dE} = -K\sin(u) \times \frac{x}{h}\sqrt{\frac{m}{2E}}$$

Tidying the expression slightly yields,

This worked example assumes the differentiation was performed with θ expressed in **radians**.

$$\frac{d\psi}{dE} = -\frac{Kx}{h}\sqrt{\frac{m}{2E}}\sin(u)$$

Finally, substituting for u yields,

$$\frac{d\psi}{dE} = -\frac{Kx}{h}\sqrt{\frac{m}{2E}}\sin\left(\frac{x\sqrt{2mE}}{h}\right)$$

15.9 Let the term within the square bracket be u, so $U = D_e u^2$

For this example, the chain rule takes the form, $\dfrac{dU}{dr} = \dfrac{dU}{du} \times \dfrac{du}{dr}$

$$\text{If} \quad U = D_e u^2 \quad \text{then} \quad \frac{dU}{du} = 2D_e u$$

$$\text{If} \quad u = 1 - \exp(-\beta r) \quad \text{then} \quad \frac{du}{dr} = \beta\exp(-\beta r)$$

(There is no minus sign in front of the first β because the two minus signs have cancelled.) Inserting terms into the chain-rule expression yields,

$$\frac{dU}{dr} = 2D_e u \times \beta\exp(-\beta r)$$

And substituting for u gives,

$$\frac{dU}{dr} = 2D_e(1 - \exp(-\beta r)) \times \beta\exp(-\beta r)$$

Further tidying makes the expression look a little neater,

$$\frac{dU}{dr} = 2\beta D_e \exp(-\beta r)\,[1 - \exp(-\beta r)]$$

15.10 Let the term within the root be u, and let the $\{(4a^2/(n^2(h^2+k^2+l^2))\}$ term within the root be A. In which case, $\lambda = \sqrt{u}$ where $u = A\sin^2\theta$.

For this example, the chain rule takes the form, $\dfrac{d\lambda}{d\theta} = \dfrac{d\lambda}{du} \times \dfrac{du}{d\theta}$

The derivative of $\sin^2\theta$ requires an additional chain-rule cycle. We find the necessary working within Additional Problem 15.2.

$$\text{If} \quad \lambda = \sqrt{u} \qquad \text{then} \quad \frac{d\lambda}{du} = \tfrac{1}{2}u^{-1/2} \quad \text{so} \quad \frac{1}{2\sqrt{u}}$$

$$\text{If} \quad u = A\sin^2\theta \quad \text{then} \quad \frac{du}{d\theta} = 2A\sin\theta\cos\theta$$

Inserting terms into the chain-rule expression,

$$\frac{d\lambda}{d\theta} = \frac{1}{2\sqrt{u}} \times 2A\sin\theta\cos\theta = \frac{A\sin\theta\cos\theta}{\sqrt{u}}$$

And substituting for u yields, $\dfrac{A\cos\theta\sin\theta}{\sqrt{A\sin^2\theta}} = \dfrac{A\cos\theta\sin\theta}{\sin\theta \times \sqrt{A}} = \dfrac{A\cos\theta}{\sqrt{A}} = \sqrt{A}\cos\theta$

Back-substituting for A yields, $\sqrt{\dfrac{4a^2}{n^2(h^2+k^2+l^2)}}\cos\theta$

This problem could be performed more easily if we had noticed that it simplifies to, $\lambda = \sqrt{A}\sin\theta$. We then obtain the solution directly as, $\dfrac{d\lambda}{d\theta} = \sqrt{A}\cos\theta$, which can be solved using eqn. (14.6).

- This worked example assumes that we performed the differentiation with θ in radians.