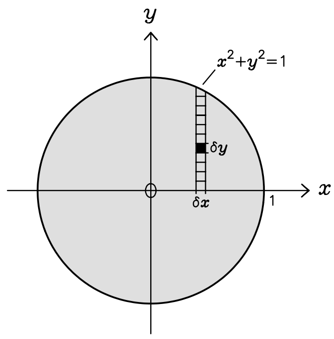
12 Multiple integrals

(1) Evaluate the double integral $\iint x^2\left(1-x^2-y^2\right) \mathrm{d}x\,\mathrm{d}y$ over an origin-centred circle of radius 1 in (a) Cartesian, and (b) polar, coordinates.



$$\text{(a)}\quad \iint\limits_{\text{circle}} x^2\left(1-x^2-y^2\right) \mathrm{d}x\,\mathrm{d}y \;=\; 4\int_{x=0}^{x=1} x^2\,\mathrm{d}x \int_{y=0}^{y=\sqrt{1-x^2}} \left(1-x^2-y^2\right) \mathrm{d}y$$

$$= 4\int_0^1 x^2\left[y - x^2 y - \tfrac{1}{3}y^3\right]_{y=0}^{y=\sqrt{1-x^2}} \mathrm{d}x$$

$$= 4\int_0^1 x^2\sqrt{1-x^2}\left[1-x^2-\tfrac{1}{3}\left(1-x^2\right)\right] \mathrm{d}x$$

$$= \tfrac{8}{3}\int_0^1 x^2\left(1-x^2\right)^{3/2} \mathrm{d}x$$

Putting $x=\sin\theta$, $\displaystyle\int_0^1 x^2\left(1-x^2\right)^{3/2} \mathrm{d}x = \int_0^{\pi/2} \sin^2\theta\,\cos^4\theta\,\mathrm{d}\theta$

and

$$\sin^2\theta\,\cos^4\theta = \left(\sin\theta\,\cos\theta\right)^2\cos^2\theta$$

$$= \left(\tfrac{1}{2}\sin 2\theta\right)^2 \times \tfrac{1}{2}\left(1+\cos 2\theta\right)$$

$$= \tfrac{1}{8}\left(1-\cos 4\theta\right) \times \tfrac{1}{2}\left(1+\cos 2\theta\right)$$

$$= \tfrac{1}{16}\left[1+\cos 2\theta - \cos 4\theta - \tfrac{1}{2}\left(\cos 2\theta + \cos 6\theta\right)\right]$$

$$= \tfrac{1}{32}\left(2+\cos 2\theta - 2\cos 4\theta - \cos 6\theta\right)$$

$$\therefore\quad \iint\limits_{\text{circle}} x^2\left(1-x^2-y^2\right) \mathrm{d}x\,\mathrm{d}y = \tfrac{1}{12}\int_0^{\pi/2}\left(2+\cos 2\theta - 2\cos 4\theta - \cos 6\theta\right) \mathrm{d}\theta$$

$$\therefore \quad \iint\limits_{\text{circle}} x^2\left(1-x^2-y^2\right)\mathrm{d}x\,\mathrm{d}y \;=\; \tfrac{1}{12}\left[2\theta+\tfrac{1}{2}\sin 2\theta-\tfrac{1}{2}\sin 4\theta-\tfrac{1}{6}\sin 6\theta\right]_0^{\pi/2}$$

$$= \underline{\tfrac{\pi}{12}}$$

We replaced the integral over the whole circle by four times that for the positive quadrant. This was permissible because of the symmetry of the integrand, $x^2\left(1-x^2-y^2\right)$, which has the same value for a given magnitude of x and y independent of their signs.

(b) In polar coordinates, r and θ, the area element $\mathrm{d}x\,\mathrm{d}y$ takes the form $r\,\mathrm{d}r\,\mathrm{d}\theta$. Therefore, putting $x=r\cos\theta$ and $y=r\sin\theta$, we have

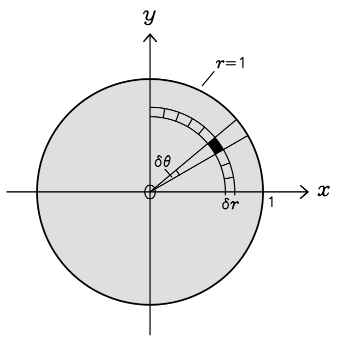
$$\iint\limits_{\text{circle}} x^2\left(1-x^2-y^2\right)\mathrm{d}x\,\mathrm{d}y \;=\; \iint\limits_{\text{circle}} r^2\cos^2\theta\left(1-r^2\right)r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_{r=0}^{r=1}\left(r^3-r^5\right)\mathrm{d}r\int_{\theta=0}^{\theta=\pi/2}\cos^2\theta\;\mathrm{d}\theta$$

$$= 4\left[\tfrac{1}{4}r^4-\tfrac{1}{6}r^6\right]_0^1\;\tfrac{1}{2}\int_0^{\pi/2}\left(1+\cos 2\theta\right)\mathrm{d}\theta$$

$$= \left(\tfrac{1}{2}-\tfrac{1}{3}\right)\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{(3-2)}{6}\times\frac{\pi}{2} \;=\; \underline{\frac{\pi}{12}}$$



Reassuringly, we obtain the same value of the double integral using both Cartesian and polar coordinates. The effort required to do the calculation is far less for (b) than (a), however, and illustrates the point that problems are best formulated in a coordinate system which matches the symmetry of the situation being considered. In this case we were integrating over a circular region, and so polar coordinates represent the most natural choice; if it had been a rectangular or triangular (with a right-angle) domain, then Cartesians would have been better.