THE CHEMIST'S TOOLKIT 30 Integration by the method of partial fractions

To solve an integral of the form

$$I=\int \frac{1}{(a-x)(b-x)}\mathrm{d}x \tag{30.1}$$

where a and b are constants with $a \neq b$, use the **method of partial fractions** in which a fraction that is the product of terms (as in the denominator of this integrand) is written as a sum of fractions. To implement this procedure write the integrand as

$$\frac{1}{(a-x)(b-x)}=\frac{1}{b-a}\left(\frac{1}{a-x}-\frac{1}{b-x}\right)$$

Then integrate each term on the right. It follows that

$$I=\frac{1}{b-a}\left(\overbrace{\int\frac{\mathrm{d}x}{a-x}}^{\text{Integral A.2}}-\overbrace{\int\frac{\mathrm{d}x}{b-x}}^{\text{Integral A.2}}\right)$$

$$=\frac{1}{b-a}\left(\ln\frac{1}{a-x}-\ln\frac{1}{b-x}\right)+\text{constant} \tag{30.2}$$

Further information

Although the condition $a \neq b$ has been specified, the result is also valid for $a = b$ provided the equality is interpreted as the limit $b \to a$. Thus, write $b = a + \delta$, with $\delta \to 0$; then, by using $\ln(1 + z) = z + \tfrac{1}{2}z^2 + \ldots = z + O(z^2)$,

$$\lim_{\delta\to 0}\frac{1}{b-a}\left(\ln\frac{1}{a-x}-\ln\frac{1}{b-x}\right)$$

$$=\lim_{\delta\to 0}\frac{1}{a+\delta-a}\left(\ln\frac{1}{a-x}-\ln\frac{1}{a+\delta-x}\right)$$

$$=\lim_{\delta\to 0}\frac{1}{\delta}\ln\frac{a+\delta-x}{a-x}=\lim_{\delta\to 0}\frac{1}{\delta}\ln\left(1+\frac{\delta}{a-x}\right)$$

$$=\lim_{\delta\to 0}\frac{1}{\delta}\left(\frac{\delta}{a-x}+O\left(\delta^2\right)\right)=\frac{1}{a-x}$$

That is, in this limit

$$\int\frac{1}{(a-x)^2}\mathrm{d}x=\frac{1}{a-x}+\text{constant}$$