Supplementary Section 6S.13

Second-Order Logic and Set Theory

In chapters 4 and 5, we explored first-order predicate logic. In first-order logic, the quantifiers range over objects, the values of the variables, which appear only as singular terms. Extending our uses of variables to the predicate places, as I do in this section, creates second-order logic. Logics of even higher orders are also possible.

We will both look at the syntax of higher-order logics, mainly focusing on second-order logic and its expressive power, and examine a philosophical question that arises from the introduction of variables in predicate positions. One salient question about higher-order logics is whether we should consider them to be logic, rather than mathematics. Debate over this question tends to focus on whether second-order logic is set theory. Set theory is important in mathematics because of its role as a foundation for mathematics. So to contextualize the question of whether second-order logic is logical or mathematical, we will delve a little into set theory.

SECOND-ORDER LOGIC: SYNTAX

Let's start with an inference that might be taken as naturally logical, at 6S.13.1.

6S.13.1 There are red apples.
There are red fire trucks.
So, some apples and some fire trucks have something in common.

A natural way to express the inference at 6S.13.1 is to quantify over the predicates themselves, treating the predicates as if they are variables, as I do in line 8 of 6S.13.2.

6S.13.2

1. (∃x)(Rx • Ax)	
2. (∃x)(Rx • Fx)	
3. Ra • Aa	1, EI
4. Rb • Ab	3, EI
5. Ra	3, Simp
6. Rb	4, Simp
7. Ra • Rb	5, 6, Conj
8. (∃X)(Xa • Xb)	7, by existential generalization over predicates

The generalization at line 8 is not possible in first-order logic. To accommodate it, I'll introduce a second-order language, which I'll call **S**.

In our first-order language, singular terms are divided between constants and variables. In our second-order language, S, we apply that same distinction to the predicates. I now reserve 'V', 'W', 'X', 'Y', and 'Z' as predicate variables, keeping the others as predicate constants. The quantifiers, as in the last line of 6S.13.2, use the corresponding predicate variables, the last five capital letters. So the vocabulary of S, is the same as that of **FF**, with just the addition of a distinction between predicate constants and predicate variables.

Vocabulary of S

Capital letters

A . . . U, used as predicate constants
V, W, X, Y, and Z, used as predicate variables

Lower-case letters

a, b, c, d, e, i, j, k . . . u are used as constants.
f, g, and h are used as functors.
v, w, x, y, z are used as singular variables.

Five propositional operators: $\sim$, $\bullet$, $\vee$, $\supset$, $\equiv$
Quantifiers: $\exists$, $\forall$
Punctuation: (,), [,], {, }

The formation rules for second-order quantifiers in S are exactly parallel to the rules for forming first-order quantifiers, with the same restriction to avoid overlapping quantifiers. A first-order quantifier that uses an 'x' may overlap with a second-order quantifier that uses an 'X'.

Formation rules for wffs of S

1. An *n*-place predicate constant or predicate variable followed by *n* singular terms (constants, variables, or functor terms) is a wff.
2. For any singular variable β, if $\mathcal{F}$ is a wff that does not contain either '$(\exists\beta)$' or '$(\forall\beta)$', then '$(\exists\beta)\mathcal{F}$' and '$(\forall\beta)\mathcal{F}$' are wffs.
3. For any predicate variable β, if $\mathcal{F}$ is a wff that does not contain either '$(\exists\beta)$' or '$(\forall\beta)$', then '$(\exists\beta)\mathcal{F}$' and '$(\forall\beta)\mathcal{F}$' are wffs.
4. If α is a wff, so is $\sim\alpha$.
5. If α and β are wffs, then so are:

   $(\alpha \bullet \beta)$
   $(\alpha \vee \beta)$
   $(\alpha \supset \beta)$
   $(\alpha \equiv \beta)$

6. These are the only ways to make wffs.

In **F** and **FF**, I did not allow zero-place predicates. For **S**, it will be convenient to permit zero-place predicate constants and variables, thus allowing for unanalyzed propositions that look and act just like the sentential variables of **PL**. For example, the law of the excluded middle, which we saw as a metalinguistic schematic sentence at 1.6.5, is neatly regimented in second-order logic, as at 6S.13.3.

6S.13.3 $(\forall X)(X \vee {\sim}X)$

Normally, though, we will expect predicates (both predicate constants and predicate variables) to be followed by one or more singular terms.

Second-order logic is only one of the higher-order logics. All logics beyond first-order logic are called higher-order. To create third-order logic, we introduce attributes of attributes. For example, one might say that some properties are virtuous; virtue may be seen as a property of properties. We won't pursue higher-order logics much here, but it is interesting to note that one can productively construct logics of very high order. A logic of infinite order is called type theory. Whitehead and Russell, in their foundational work, *Principia Mathematica*, use type theory to avoid the paradoxes of naive set theory that Russell found in Frege's work.

SECOND-ORDER LOGIC: TRANSLATION

Let's proceed to see the expressive power of second-order logic. We can regiment some general claims about properties.

6S.13.4 Everything has some relation to itself.
$(\forall x)(\exists V)Vxx$

6S.13.5 No two distinct things have all properties in common.
$(\forall x)(\forall y)[x \neq y \supset (\exists X)(Xx \cdot {\sim}Xy)]$

6S.13.6 Identical objects share all properties.
$(\forall x)(\forall y)[x = y \supset (\forall Y)(Yx \equiv Yy)]$

6S.13.6 is Leibniz's law. We saw Leibniz's law and its converse, the identity of indiscernibles, at 5.4.8 and 5.4.9, written as schematic sentences in the metalanguage. In a second-order language, we can write them as simple object-level sentences. The identity of indiscernibles is 6S.13.7.

6S.13.7 $(\forall x)(\forall y)[(\forall Z)(Zx \equiv Zy) \supset x = y]$

At 6S.13.8 and 6S.13.9, sentences about properties become a little more complex.

6S.13.8 All people have some property in common.
$(\forall x)(\forall y)[(Px \cdot Py) \supset (\exists Y)(Yx \cdot Yy)]$

6S.13.9 No two people have every property in common.
$(\forall x)(\forall y)[(Px \cdot Py \cdot x \neq y) \supset (\exists Z)(Zx \cdot {\sim}Zy)]$

Second-order logic allows us to regiment analogies, like 6S.13.10.

6S.13.10 Cat is to meow as dog is to bark.
$(\exists X)(Xcm \cdot Xdb)$

Note the odd use of constants at 6S.13.10; we're taking 'cat', 'meow', 'dog', and 'bark' to be names of particular things. We can think of them as names of collections or abstract singular terms.

Second-order logic allows us to regiment three important characteristics of relations: reflexivity, symmetry, and transitivity.

A relation is **reflexive** if every object bears that relation to itself.

Being the same size as something is a reflexive relation. So is being equidistant from a given point. Being a sibling is not reflexive, I think, because people aren't their own siblings. But it is a symmetric relation.

A relation is **symmetric** if whenever one thing bears that relation to another, the reverse is also true.

While being a sibling is symmetric, being older than is asymmetric, which means that if a relation holds in one direction (if I am older than you, for example) then it follows that it does not hold in the other direction (it follows that you are not older than me).

Lastly, transitivity is exemplified by hypothetical syllogism. If a relation is transitive, then if *x* bears the relation to *y* and *y* bears the relation to *z*, then *x* also bears the relation to *z*. Being older than, or larger than, or earlier than are all transitive relations.

We can express that any particular relation is reflexive, symmetric, or transitive without any use of second-order quantification, as 6S.13.11–6S.13.13 do, for some relation R.

6S.13.11	Reflexivity	$(\forall x)Rxx$
6S.13.12	Symmetry	$(\forall x)(\forall y)(Rxy \equiv Ryx)$
6S.13.13	Transitivity	$(\forall x)(\forall y)(\forall z)[(Rxy \cdot Ryz) \supset Rxz]$

Second-order logic allows us to do more with these characteristics. We can quantify over them and make assertions concerning these properties, as at 6S.13.14.

6S.13.14 Some relations are transitive.
$(\exists X)(\forall x)(\forall y)(\forall z)[(Xxy \cdot Xyz) \supset Xxz]$

In mathematics, many relations, like 'greater than', are antisymmetric, which we can also represent. Be careful to distinguish asymmetry from antisymmetry. 6S.13.15 says that relation S is asymmetric. 6S.13.16 says that relation T is antisymmetric.

6S.13.15	Asymmetry	$(\forall x)(\forall y)(Sxy \equiv {\sim}Syx)$
6S.13.16	Antisymmetry	$(\forall x)(\forall y)[(Txy \cdot Tyx) \supset x{=}y)]$

Putting them together, we can express that each property holds of some relations, at 6S.13.17.

6S.13.17 Some relations are symmetric, while some are antisymmetric.
$(\exists X)(\forall x)(\forall y)(Xxy \equiv Xyx) \cdot (\exists X)(\forall x)(\forall y)(Xxy \equiv {\sim}Xyx)$

These properties of relations are especially important because we can use them to characterize identity. We call any relation which is reflexive, symmetric, and transitive an equivalence relation. Identity is the most basic equivalence relation. The power of second-order logic entails that we need not reserve a special identity predicate, as we did in chapter 5. Instead, we can just introduce it as shorthand for the second-order claim on the right at 6S.13.18.

6S.13.18 $x=y \rightleftarrows (\forall X)(Xx \equiv Xy)$

Lastly on the expressive power of second-order logic, some philosophers of mathematics are especially interested in second-order logic because they see it as the appropriate logic for mathematics. Frege's logic, which developed from *Begriffsschrift* to *Grundgesetze*, was a higher-order language; he took numbers as properties of properties. Contemporary philosophers, Stewart Shapiro perhaps most notably, have defended the logic vigorously, especially for its ability to provide categorical theories; first-order mathematical theories allow for non-standard models.

One of the advantages of second-order logic for mathematics is that it allows us to generalize what first-order logic allows us to say only schematically. Recall the induction schema in Peano arithmetic, which we saw as the fifth axiom at 5.6.16. Since it is a schema, the theory is not finitely axiomatizable: infinitely many instances of the schema are all axioms. Second-order logic allows us to replace the induction schema with single axiom, 6S.13.19, which uses 'a' to stand for zero, 'Nx' for 'x is a number', and 'f(x)' for the successor function. The leading quantifier ranges over any mathematical property using the predicate variable 'X'.

6S.13.19 $(\forall X)\{\{Na \cdot Xa \cdot (\forall x)[(Nx \cdot Xx) \supset Xf(x)]\} \supset (\forall x)(Nx \supset Xx)\}$

EXERCISES 6S.13

Translate each of the following sentences into S. Exercises 16–20 are adapted from Spinoza's *Ethics*.

1. Liza has some attributes, but she lacks some attributes.
2. Cristóbal and Dante share no properties.
3. Reva has at least two different properties.

4. Everyone shares some property with Tudor.
5. Everyone shares some property with some monkeys.
6. Some chemists share some property with Einstein.
7. Gillian shares some attributes with a famous scientist.
8. All psychologists and biologists have some property in common.
9. Alec shares some of his mother's properties. (f(x): the mother of x)
10. Ron has all of his father's properties. (g(x): the father of x)
11. Some attributes are properties of nothing.
12. Some relations are transitive.
13. Something lacks all symmetric relations.
14. Some relations are both reflexive and symmetric.
15. Something lacks all transitive relations.
16. Two substances, whose attributes are different, have no properties in common (1p2). Note: The parenthetical citations are standard for Spinoza's work; for example, '1p2' refers to the second proposition in part 1.
17. Two or more distinct things are distinguished by the difference of their attributes (1p4).
18. There cannot exist in the universe two or more substances having some attribute (1p5).
19. Two things with no common properties cannot be the cause of one another (1p3). (f(x): the cause of x)
20. Two things with no common properties cannot be understood through each other (1a5). (Uxy: x is understood through y)

SET THEORY AND THE FOUNDATIONS OF ARITHMETIC

Calling S a logical theory is controversial. Many philosophers have argued that no higher-order logics are really logic. Perhaps most influentially, Quine calls second-order logic "set theory in sheep's clothing" (*Philosophy of Logic*, p. 66). Some philosophers, like Quine, take first-order logic with identity as a canonical language, the privileged language used for expressing one's most sincere beliefs and commitments. Many philosophers see the step from first-order logic to second-order logic as breaching a barrier.

The line between logical and nonlogical claims is not always clear or obvious. Most people who think about these things take identity to be a logical relation. Most take set theory to be mathematical. But the difference between first-order logic and some versions of set theory is mainly just the inclusion of one symbol, $\in$, used for set inclusion, and a few basic principles which govern that relation. These principles, the axioms of set theory, are very powerful. But they are neatly presentable in a compact form.

There are a wide variety of formulations of basic set theory. Some of these formulations differ in their consequences. There is dispute among mathematicians over which set theory is correct. There is dispute over whether there is a correct set theory. And, beyond basic set theory, there are lots of controversial extensions. These topics are for another place. Our interest in set theory is mainly just to consider the question of whether higher-order logics are logical or mathematical.

For the purposes of our discussion, then, we can consider one simple set of axioms of set theory, which is standardly called ZF.

Zermelo-Fraenkel Set Theory[1]

Substitutivity:	$(\forall x)(\forall y)(\forall z)[y=z \supset (y\in x \equiv z\in x)]$
Pairing:	$(\forall x)(\forall y)(\exists z)(\forall w)[w\in z \equiv (w=x \vee w=y)]$
Null Set:	$(\exists x)(\forall y) \sim y\in x$
Sum Set:	$(\forall x)(\exists y)(\forall z)[z\in y \equiv (\exists w)(z\in w \cdot w\in x)]$
Power Set:	$(\forall x)(\exists y)(\forall z)[z\in y \equiv (\forall w)(w\in z \supset w\in x)]$
Selection:	$(\forall x)(\exists y)(\forall z)[z\in y \equiv (z\in x \cdot \mathcal{F}z)]$
	for any formula $\mathcal{F}$ not containing y as a free variable
Infinity:	$(\exists x)(a\in x \cdot (\forall y)(y\in x \supset Sy\in x)$

Note that in addition to $\in$, the axiom of infinity uses ‘a’ for the empty set, whose existence is guaranteed by the null set axiom, and ‘S’ for the function, ‘$y \cup \{y\}$’, the definitions for the components of which are standard. ‘S’ is a successor function, essential to mathematics. In arithmetic, the successor function is used to generate the natural numbers. In ZF, we use it to generate an infinite set of sets.

Almost all of what we consider to be mathematics is derivable, with just the addition of further definitions, from the axioms of set theory. Let's take a moment to sketch how the powerful tools of the real numbers can be constructed out of set theory. The discussion in this section will get a little bit technical, but only the general form of the sketch is most important.

First, we can define the natural numbers, **N**, within set theory using any of various standard constructions, like those of Zermelo or Von Neumann. (Remember, ‘a’ stands for the empty set.)

[1]Many mathematicians adopt a further axiom, the axiom of choice, yielding a theory known as ZFC. Choice says that given any set of sets, there is a set that contains precisely one member of each of the subsets of the original set.

Zermelo	**Von Neumann**
0 = a	0 = a
1 = {a}	1 = {a}
2 = {{a}}	2 = {a, {a}}
3 = {{{a}}}	3 = {a, {a}, {a, {a}}}
. . .	. . .

The Zermelo sets and the von Neumann sets are different, but either can do the work of translating arithmetic into set theory.

Using the Peano axioms (see 5.6.16) and the notion of an ordered pair, which is easily definable within set theory, we can define standard arithmetic operations like addition and multiplication. We can define the integers, **Z**, in terms of the natural numbers by using subtraction. Since −3 is 5 − 8, we can define −3 as the ordered pair <5, 8>. But −3 could also be defined as <17, 20>. To avoid ambiguity, we take the negative numbers to be equivalence classes of such ordered pairs. The equivalence class for subtraction is defined using addition: <a, b> ~ <c, d> iff a + d = b + c, where <a, b> ~ <c, d> indicates that <a, b> is in the same equivalence class as <c, d>. So, we can define **Z** = . . . −3, −2, −1, 0, 1, 2, 3 . . . in terms of **N**, addition, and the notion of an ordered pair.

The rationals, **Q**, can be defined in terms of the integers, **Z**, by using ordered pairs of integers. The function 'a/b' can be defined as the equivalence class of ordered pairs <a,b>, where '<a,b> ~ <c,d> iff ad=bc' is the relevant identity clause. By adopting the definitions of **Z** and **Q**, we have translated the theory of rational numbers into the theory of natural numbers, with the background assumptions of set theory and logic. Anything we want to say about the rationals, we can say in a slightly more complicated fashion about the natural numbers.

The real numbers, **R**, are differentiated from the rationals by their continuity. There are a variety of ways to define continuity set-theoretically, and a variety of ways to define the reals in terms of the rationals. Dedekind's definition, from 1872, relies on the concept of a cut, which has become known as a Dedekind cut. The real numbers are identified with separations of the rationals, **Q**, into two sets, Q_1 and Q_2, such that every member of Q_1 is less than or equal to the real number and every member of Q_2 is greater. So even though √2 is not rational, it divides the rationals into two such sets; we know for any rational whether it is greater or less than √2. Not all cuts are produced by rational numbers. So we can distinguish the continuity of the reals from the discontinuity of the rationals on the basis of these cuts. Real numbers are thus defined in terms of sets of rationals, the set of rationals below the cut. These sets have no largest member, since for any rational less than √2, for example, we can find another one larger. But they do have an upper bound in the reals (i.e., the real number being defined).

By adding our definition of the real numbers in terms of sets of rational numbers to our definitions of the rationals in terms of the natural numbers, we have shown how to define the reals in terms of the natural numbers. Such definitions do two things. First, they make it clear that analysis (including calculus, differential equations, theories of real and complex numbers, analytic functions, and measurement theory) is accessible to finite (or at least denumerable) methods. Second, they make it plausible that we can reduce the problem of justifying our knowledge of arithmetic to the problem of justifying our knowledge of the natural numbers. Arithmetic is, in some sense, reducible to set theory.

SECOND-ORDER LOGIC AND SET THEORY

The question we are pursuing is whether second-order logic is logic or mathematics. We are taking set theory to be mathematics and **M**, **F**, and **FF** to be logic. When we interpret first-order languages, we specify a domain for the variables to range over. Sometimes we use restricted domains. If we want to interpret number theory, for example, we restrict our domain to the integers. If we want to interpret a biological theory, we might restrict our domain to species. For our most general reasoning, we take an unrestricted domain: the universe, everything there is. Consider 6S.13.20.

6S.13.20 There are blue hats.
$(\exists x)(Bx \cdot Hx)$

On standard semantics, for 6S.13.20 to be true, there must exist a thing that will serve as the value of the variable 'x', and that has both the property of being a hat and being blue. As Quine says, to be is to be the value of a variable. Our most sincere commitments arise from examining the domain of quantification for our best theory of everything.

Now, consider a sentence of second-order logic, 6S.13.21.

6S.13.21 Some properties are shared by two people.
$(\exists X)(\exists x)(\exists y)(Px \cdot Py \cdot x \neq y \cdot Xx \cdot Xy)$

For 6S.13.21 to be true, there must exist two people, and there must exist a property. The value of the variable 'X' is not an ordinary object, but a property of an object. By quantifying over properties, we take properties as kinds of objects; we need some thing to serve as the value of the variable.

The commitments of second-order logic to properties, in addition to the objects which have those properties, are thus apparently profligate and definitely controversial. The first-order sentence about blue hats referred only to an object with properties. The second-order sentence reifies properties. Is there really blueness, in addition to blue things? What are properties like blueness?

One classic (and classical) way to understand properties comes from Plato's work. We could take the objects that serve as the values of predicate variables to be Platonic forms, or eternal ideas. Such an interpretation would be extremely contentious.

The least controversial way to understand properties is to take them to be sets of the objects that have those properties. We call this conception of properties extensional. On an extensional interpretation, 'blueness' refers to the collection of all blue things; the taller-than relation is just the set of ordered pairs of objects whose first element is taller than its second element. Thus, second-order logic seems at least to commit us to the existence of sets. Remember, ordered pairs are just kinds of sets. Second-order logic, in its least-controversial interpretation, seems to be some form of set theory.

If we believe that there are mathematical objects, we can include sets in our ontology, allowing them in our first-order domain of discourse by taking them to be values of first-order variables. We need not include them under the guise of second-order logic, sneaking them in through the interpretations of second-order variables. Quine's complaints about second-order logic, that it is set theory in sheep's clothing, are based in part on this sneakiness.

In favor of second-order logic, it is difficult to see how one could regiment in first-order logic sentences like many we have already seen in this section. The possibility of deriving the properties of identity from the second-order axioms, rather than introducing a special predicate with special inferential properties, is especially tempting. Perhaps the most famous example motivating quantification beyond the basic first-order logic we have seen is called the Geach-Kaplan sentence, 6S.13.22.

6S.13.22 Some critics admire only one another.

6S.13.22 is difficult to regiment into first-order logic, indeed provably so, if one uses the predicates that appear in the sentence.

Quine favors using schematic predicate letters in lieu of predicate variables. With schematic letters, we regiment the law of the excluded middle, for example, as 6S.13.23, rather than 6S.13.3, above, with the understanding that any wff of **F** can be substituted for 'α'.

6S.13.23 $\alpha \vee {\sim}\alpha$

Schematic letters are metalinguistic variables. Those who favor schematic letters to second-order logic are admitting that we cannot formulate claims like 6S.13.23 in our canonical object language. We must, instead, ascend to a metalanguage, using metalinguistic variables.

So, while it seems that second-order logic is some form of set theory, precisely what form of set theory it is depends on the semantics for the specific language of second-order logic we adopt. And there may be an option between first- and second-order logics, called plural quantification. Plural quantification regiments the natural-language locution, "There are some *P*s," and can handle the Geach-Kaplan sentence more naturally than standard first-order logic.

Given that second-order logic is some type of set theory, a natural question to ask is whether that is a problem for the language. Does its value in expressing some natural claims and their inferences outweigh its controversial ontological commitments? What exactly differentiates logic and mathematics? Is there a firm line between the disciplines? What is the purpose of logic? Is there one right logic? These are interesting and complex questions. If you've completed the work in *Introduction to Formal Logic with Philosophical Applications*, you are ready to begin to answer them.

Summary

Most of the formal work in *Introduction to Formal Logic with Philosophical Applications* covers themes that philosophers sometimes call baby logic, an introduction to a vast and burgeoning field that raises many important questions about human reasoning, ontology, epistemology, and other philosophical topics. To answer questions about which logic is the right logic, and what that would mean for our views of ourselves and our intellectual capacities, one should explore the wide variety of logics available beyond baby logic.

TELL ME MORE

- What paradoxes did Russell find in Frege's work? See 7.7: Logicism.

For Further Research and Writing

1. The debate over second-order logic is one aspect of a larger question of determining a canonical language, and whether there even is such a best language. Is there a correct logic? What is the purpose of logic? Could there be a different logic for natural language and for mathematics? What differentiates logic and mathematics? There are lots of good potential paper topics here. Quine, Hacking, and Shapiro are excellent sources.
2. Some philosophers, notably George Boolos, have explored a formal tool that seems to lie between first-order and second-order logic, called plural quantification. Is plural quantification more like first-order logic or more like second-order logic? Is it appropriate for formal versions of natural languages? Is it useful in mathematics? The entry by Linnebo in the suggested readings is an excellent place to start.
3. What problems arise for translating the Geach-Kaplan sentence into first-order logic? How does plural quantification help? See Linnebo, "Plural Quantification," and Boolos, "To Be Is to Be a Value of a Variable" in Boolos's *Logic, Logic, and Logic*. Quine takes the Geach-Kaplan sentence to be an argument for set theory; see *Methods of Logic* section 46.

Suggested Readings

Boolos, George. *Logic, Logic, and Logic*. Cambridge, MA: Harvard University Press, 1998. This collection of Boolos's papers contain two especially important papers on second-order logic, "On Second-Order Logic," and "To Be Is to Be a Value of a Variable."

Enderton, Herbert B. "Second-Order and Higher-Order Logic." In *The Stanford Encyclopedia of Philosophy*. http://plato.stanford.edu/archives/fall2015/entries/logic-higher-order.

Accessed January 29, 2016. An excellent, efficient, mathematical overview of second-order logic.

Hacking, Ian. "What Is Logic?" In *A Philosophical Companion to First-Order Logic*, edited by R. I. G. Hughes, 225–258. Indianapolis: Hackett, 1993. Hacking evaluates the debate over the status of second-order logic in section 13.

Linnebo, Øystein. "Plural Quantification." In *The Stanford Encyclopedia of Philosophy*. http://plato.stanford.edu/archives/fall2014/entries/plural-quant. Accessed February 7, 2016.

Nolt, John. *Logics*. Belmont, CA: Wadsworth, 1997. Chapter 14 includes semantics and rules for semantic tableaux for higher-order logics.

Quine, W. V. *Philosophy of Logic*, 2nd ed. Cambridge, MA: Harvard University Press, 1986. Quine's arguments against second-order logic are mainly found in chapter 5.

Quine, W. V. *Method of Logic*, 4th ed. Cambridge, MA: Harvard University Press, 1982. For Quine's view of schemata, see especially sections 27, 28, and 46; the last contains a lucid discussion of set theory and second-order logic.

Read, Stephen. *Thinking About Logic*. Oxford: Oxford University Press, 1995. The section of chapter 2 on compactness (which first-order logic is and second-order logic is not) can be illuminating.

Sainsbury, Mark. *Logical Forms: An Introduction to Philosophical Logic*, 2nd ed. Oxford, UK: Blackwell, 2001. Chapter 4, section 19, contains interesting and accessible observations on second-order logic.

Shapiro, Stewart. 1991. *Foundations Without Foundationalism: A Case for Second-Order Logic*. Oxford: Oxford University Press. Shapiro makes a compelling case against Quine and for second-order logic.

Simpson, Stephen, 1999. *Subsystems of Second Order Arithmetic*. Berlin: Springer. For those who wish to brave the mathematics.

Väänänen, Jouko. "Second-Order Logic and Foundations of Mathematics." *Bulletin of Symbolic Logic* 7 (2001): 504–520. A brief and fruitful discussion of how second-order logic may serve as a foundation for mathematics.

SOLUTIONS TO EXERCISES 6S.13

1. $(\exists X)Xl \bullet (\exists X) {\sim}Xl$
2. $(\forall X)(Xc \equiv {\sim}Xd)$
3. $(\exists X)(\exists Y)[Xr \bullet Yr \bullet (\exists x) {\sim}(Xx \equiv Yx)]$
4. $(\forall x)[Px \supset (\exists X)(Xt \bullet Xx)]$
5. $(\forall x)[Px \supset (\exists X)(\exists y)(My \bullet Xx \bullet Xy)]$
6. $(\exists x)[Cx \bullet (\exists X)(Xx \bullet Xe)]$
7. $(\exists x)[(Fx \bullet Sx) \bullet (\exists X)(Xg \bullet Xx)]$
8. $(\forall x)\{Px \supset (\forall y)[By \supset (\exists X)(Xx \bullet Xy)]\}$
9. $(\exists X)(Xa \bullet Xf(a))$
10. $(\forall X)(Xg(r) \supset Xr)$
11. $(\exists X)(\forall x) {\sim}Xx$
12. $(\exists X)\{(\forall x)(\forall y)(\forall z)[(Xxy \bullet Xyz) \supset Xxz]\}$

13. $(\exists z)(\forall X)[(\forall x)(\forall y)(Xxy \equiv Xyx) \supset (\forall w)(\sim Xzw \bullet \sim Xwz)]$
14. $(\exists X)[(\forall x)Xxx \bullet (\forall x)(\forall y)(Xxy \equiv Xyx)]$
15. $(\exists w)(\forall X)\{(\forall x)(\forall y)(\forall z)[(Xxy \bullet Xyz) \supset Xxz] \supset (\forall v)(\sim Xwv \bullet \sim Xvw)\}$
16. $(\forall x)(\forall y)\{[Sx \bullet Sy \bullet \sim(\forall X)(Xx \equiv Xy)] \supset \sim(\exists X)(Xx \bullet Xy)\}$
17. $(\forall x)(\forall y)[x \neq y \supset (\exists X)(Xx \equiv \sim Xy)]$
18. $(\forall x)(\forall y)\{[Sx \bullet Ux \bullet Sy \bullet Uy \bullet (\exists X)(Xx \bullet Xy)] \supset x=y\}$
19. $(\forall x)(\forall y)[\sim(\exists X)(Xx \bullet Xy) \supset (\sim x=f(y) \bullet \sim y=f(x))]$
20. $(\forall x)(\forall y)[\sim(\exists X)(Xx \bullet Xy) \supset (\sim Uxy \bullet \sim Uyx)]$