Supplementary Section 6S.12

Rules of Passage

Formulas of **F** may sometimes be written equivalently with quantifiers having narrow scope (often buried deep within a formula) or wide scope (out in front). When translating sentences of natural language into **F**, it is useful to keep the scope of our quantifiers as narrow as possible, introducing them only when needed. When constructing derivations in formal logic, it is often useful to have a wide scope for our quantifiers, so that we can instantiate quickly and easily. Moving quantifiers through a formula, switching between narrow and wide scopes, can alter the meaning of the formula, so we have to be sure to do it correctly. This section presents some rules for moving quantifiers through a formula.

These so-called rules of passage are unnecessary in our systems of deduction for **M** and **F**, but they are interesting and can be useful. The rules appear in Whitehead and Russell's *Principia Mathematica*. The name is due to Jacques Herbrand, who made important contributions to proof theory and Hilbert's program to explain or justify our knowledge of infinity before dying while climbing in the Alps at age twenty-three.

QUANTIFIERS: NARROW AND WIDE SCOPE

In some cases, we can move quantifiers through a formula without much worry. For example, if all quantifiers are universal, we can pull them in or out at will, as long as we are careful not to accidentally bind any variables. 6S.12.1 can be written as any of 6S.12.2–6S.12.4.

6S.12.1 Everyone loves everyone
6S.12.2 (∀x)[Px ⊃ (∀y)(Py ⊃ Lxy)]
6S.12.3 (∀x)(∀y)[(Px • Py) ⊃ Lxy]
6S.12.4 (∀y)(∀x)[(Px • Py) ⊃ Lxy]

Technically, 6S.12.4 is 'everyone is loved by everyone'. But all three statements are logically equivalent. Similarly, 6S.12.5 can be written as any of 6S.12.6–6S.12.8.

6S.12.5 Someone loves someone.
6S.12.6 (∃x)[Px • (∃y)(Py • Lxy)]
6S.12.7 (∃x)(∃y)[(Px • Py) • Lxy]
6S.12.8 (∃y)(∃x)[(Px • Py) • Lxy]

6S.12.8 is 'someone is loved by someone'. But, again, 6S.12.6–6S.12.8 are all logically equivalent.

In contrast, when we mix universal quantifiers with existential quantifiers, changing the scope of the quantifiers is not so easy. Slight alterations, like reversing the order of the quantifiers, can change the meaning of a proposition. None of 6S.12.9–6S.12.12 are equivalent.

6S.12.9	Everyone loves someone.	$(\forall x)(\exists y)[Px \supset (Py \bullet Lxy)]$
6S.12.10	Everyone is loved by someone.	$(\forall x)(\exists y)[Px \supset (Py \bullet Lyx)]$
6S.12.11	Someone loves everyone.	$(\exists x)(\forall y)[Px \bullet (Py \supset Lxy)]$
6S.12.12	Someone is loved by everyone.	$(\exists x)(\forall y)[Px \bullet (Py \supset Lyx)]$

Note that the first word in each translation above corresponds to the leading quantifier. Also, note that the operators that directly follow the 'Px' and the 'Py' are determined by the quantifier binding that variable. This tendency is clearer if we take the quantifiers inside, as in 6S.12.9′–6S.12.12′.

6S.12.9′	$(\forall x)[Px \supset (\exists y)(Py \bullet Lxy)]$
6S.12.10′	$(\forall x)[Px \supset (\exists y)(Py \bullet Lyx)]$
6S.12.11′	$(\exists x)[Px \bullet (\forall y)(Py \supset Lxy)]$
6S.12.12′	$(\exists x)[Px \bullet (\forall y)(Py \supset Lyx)]$

While all of these shifts of quantifiers are acceptable, we cannot put quantifiers just anywhere in a formula without changing its meaning. For example, 6S.12.13 and 6S.12.14 are *not* equivalent, as a possible interpretation of each shows.

6S.12.13	$(\forall x)[(\exists y)Lxy \supset Hx]$	All lovers are happy.
6S.12.14	$(\forall x)(\exists y)(Lxy \supset Hx)$	Everything has something such that loving it will make it (the lover) happy.

From 6S.12.13 to 6S.12.14, we have moved the existential quantifier out front, and merely brought the 'Hx' into the scope of '$(\exists y)$', which does not bind it. But 6S.12.13 does not commit to the existence of something that, by being loved, makes something happy while 6S.12.14 does. Consider the universe in which there are things that can never be happy, in other words, for which nothing could make them happy. 6S.12.13 could still be true, but 6S.12.14 would be false.

We need a set of rules to determine which moves of quantifiers are acceptable. Also motivating the need for such rules, there are metalogical proofs that require that every statement of **F** is equivalent to a statement with all quantifiers having wide scope. Such a form is called prenex normal form (PNF), and was used by Skolem for his proof procedure in 1922. In order to transform formulas to PNF, we can use what are sometimes called rules of passage. The rules of passage are rules of equivalence, justified by the equivalence of statements of the paired forms and applicable to whole lines or to parts of lines.

RULES OF PASSAGE

For all variables α and all formulas Γ and Δ:

RP1:	$(\exists\alpha)(\Gamma \vee \Delta)$	$\rightleftarrows$	$(\exists\alpha)\Gamma \vee (\exists\alpha)\Delta$
RP2:	$(\forall\alpha)(\Gamma \bullet \Delta)$	$\rightleftarrows$	$(\forall\alpha)\Gamma \bullet (\forall\alpha)\Delta$

For all variables α, all formulas Γ containing α, and all formulas Δ not containing α:

RP3:	$(\exists\alpha)(\Delta \bullet \Gamma\alpha)$	$\rightleftarrows$	$\Delta \bullet (\exists\alpha)\Gamma\alpha$
RP4:	$(\forall\alpha)(\Delta \bullet \Gamma\alpha)$	$\rightleftarrows$	$\Delta \bullet (\forall\alpha)\Gamma\alpha$
RP5:	$(\exists\alpha)(\Delta \vee \Gamma\alpha)$	$\rightleftarrows$	$\Delta \vee (\exists\alpha)\Gamma\alpha$
RP6:	$(\forall\alpha)(\Delta \vee \Gamma\alpha)$	$\rightleftarrows$	$\Delta \vee (\forall\alpha)\Gamma\alpha$
RP7:	$(\exists\alpha)(\Delta \supset \Gamma\alpha)$	$\rightleftarrows$	$\Delta \supset (\exists\alpha)\Gamma\alpha$
RP8:	$(\forall\alpha)(\Delta \supset \Gamma\alpha)$	$\rightleftarrows$	$\Delta \supset (\forall\alpha)\Gamma\alpha$
RP9:	$(\exists\alpha)(\Gamma\alpha \supset \Delta)$	$\rightleftarrows$	$(\forall\alpha)\Gamma\alpha \supset \Delta$
RP10:	$(\forall\alpha)(\Gamma\alpha \supset \Delta)$	$\rightleftarrows$	$(\exists\alpha)\Gamma\alpha \supset \Delta$

TRANSFORMATIONS WITH THE RULES OF PASSAGE

Let's look at a few examples of transformations using the rules of passage. 6S.12.15 and 6S.12.16 are equivalent by RP4.

6S.12.15 $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)]$
6S.12.16 $(\exists x)(\forall y)[Px \bullet (Qy \supset Rxy)]$

To see that these formulas are actually equivalent, we can derive 6S.12.15 from 6S.12.16 and 6S.12.16 from 6S.12.15, as I do in an appendix to this section.

When moving quantifiers using the rules of passage, be careful not to accidentally bind any variables or to accidentally remove variables from binding. Sometimes, you have to change quantifier variables, as in the transformation from 6S.12.17 to 6S.12.18, using RP2.

6S.12.17 $(\forall x)(Ax \supset Bx) \cdot (\forall y)(Dy \supset Ey)$
6S.12.18 $(\forall x)[(Ax \supset Bx) \cdot (Dx \supset Ex)]$

6S.12.19 and 6S.12.20 are equivalent by RP8.

6S.12.19 $(\forall x)(\forall y)[Px \supset (Qy \supset Rxy)]$
6S.12.20 $(\forall x)[Px \supset (\forall y)(Qy \supset Rxy)]$

6S.12.14, above, is equivalent to 6S.12.19 by RP9.

6S.12.14 $(\forall x)(\exists y)(Lxy \supset Hx)$
6S.12.21 $(\forall x)[(\forall y)Lxy \supset Hx]$

The transformation between 6S.12.14 and 6S.12.21 might strike one as strange. It might even make one call RP9 into question. But notice that we can make the same transformation without RP9.

6S.12.14	1.	$(\forall x)(\exists y)(Lxy \supset Hx)$	
	2.	$(\forall x)(\exists y)(\sim Lxy \vee Hx)$	1, Impl
	3.	$(\forall x)(\exists y)(Hx \vee \sim Lxy)$	2, Com
	4.	$(\forall x)[Hx \vee (\exists y)\sim Lxy]$	3, RP5
	5.	$(\forall x)[(\exists y)\sim Lxy \vee Hx]$	4, Com
	6.	$(\forall x)[\sim(\forall y)Lxy \vee Hx]$	5, QE
6S.12.21	7.	$(\forall x)[(\forall y)Lxy \supset Hx]$	6, Impl

6S.12.13, above, is equivalent by RP10 to 6S.12.22. Both formulas are good translations of 'If anyone loves someone, then s/he is happy'.

6S.12.13 $(\forall x)[(\exists y)Lxy \supset Hx]$
6S.12.22 $(\forall x)(\forall y)(Lxy \supset Hx)$

6S.12.23 and 6S.12.24 are equivalent, also by RP10.

6S.12.23 $(\forall x)[Px \supset (\exists y)Qy]$
6S.12.24 $(\exists x)Px \supset (\exists y)Qy$

EXERCISES 6S.12a

Using the rules of passage, transform each formula with a quantifier having narrow scope into one with quantifiers of wider scope.

1. $(\exists x)(Ax \bullet \sim Bx) \vee (\exists x)(Cx \vee Dx)$
2. $(\forall x)Fx \supset (\exists x)Gx$
3. $(\exists x)[Hx \bullet (\exists y)(Iy \bullet Jxy)]$
4. $(\forall x)Px \supset Ra$
5. $(\exists x)[Kx \bullet (\forall y)(Ly \supset Mxy)]$
6. $(\exists x)Jx \supset (\exists y)Ky$
7. $(\exists x)(Px \bullet Qx) \supset (Pa \bullet Ra)$
8. $(\exists x)[Nx \vee (\exists y)(Oy \bullet Pxy)]$
9. $(\forall x)(Ex \supset Fx) \bullet (\forall x)(\sim Fx \equiv Gx)$
10. $(\forall x)[Qx \vee (\forall y)(Ry \supset Sxy)]$
11. $(\forall x)[(\forall y)Dxy \supset Ex]$
12. $(\forall x)[Tx \supset (\exists y)(Uy \bullet Vxy)]$
13. $(\forall x)[Ax \supset (\forall y)(By \supset Cxy)]$
14. $(\forall x)[(\exists y)Rxy \supset (Px \bullet Qx)]$

EXERCISES 6S.12b

Using the rules of passage, transform each formula with a quantifier having wide scope into one with quantifiers of narrower scope.

1. $(\forall x)[Rx \bullet (Tx \vee Sx)]$
2. $(\exists x)(\exists y)[(Kx \equiv Lx) \bullet (My \bullet Nxy)]$
3. $(\exists x)(\forall y)[(Dx \bullet Ex) \vee (Fy \supset Gxy)]$
4. $(\forall x)(\exists y)[Hx \supset (Iy \bullet Jxy)]$
5. $(\exists x)[(Ox \vee Qx) \supset (\exists y)Py]$
6. $(\forall x)(\forall y)[Axy \supset (Bx \vee Cx)]$

7. $(\forall x)(\forall y)[(Ax \vee {\sim}Cx) \supset (By \supset Dxy)]$
8. $(\exists x)(\forall y)[(Fx \bullet Gx) \bullet (Hy \supset Exy)]$
9. $(\forall x)(\exists y)[Ixy \supset (Jx \bullet Kx)]$
10. $(\exists x)[(Lx \equiv Mx) \vee Nx]$
11. $(\forall x)[(Px \bullet {\sim}Qx) \supset (\exists y)Oy]$
12. $(\forall x)(\exists y)[Sx \supset (Ty \bullet Rxy)]$

PROVING THE EQUIVALENCE OF RP10

We will not prove the equivalence of all of the rules of passage. Most of them are quite intuitive. RP9 and RP10 are the two oddballs. Let's take a moment to prove RP10 semantically.

RP10 $(\forall\alpha)(\Gamma\alpha \supset \Delta)$ $\rightleftarrows$ $(\exists\alpha)\Gamma\alpha \supset \Delta$

Consider first what happens when Δ is true, and then when Δ is false. (As an example, in 6S.12.23, Δ is '$(\exists y)Qy$'.)

If Δ is true, then both formulas will turn out to be true.

The consequent of the formula on the right is just Δ.
So, if Δ is true, the whole formula on the right is true.
$\Gamma\alpha \supset \Delta$ is true for every instance of α, since the consequent is true.
So, the universal generalization of each such formula (the formula on the left) is true.

If Δ is false, then the truth value of each formula will depend on the antecedents of the conditionals.

To show that the truth values of each formula will be the same, we will show that the formula on the right is true in every case that the formula on the left is true and that the formula on the left is true in every case that the formula on the right is.
If the formula on the left turns out to be true when Δ is false, it must be because $\Gamma\alpha$ is false, for every α.
But then, $(\exists\alpha)\Gamma\alpha$ is false, and so the formula on the right turns out to be true.
If the formula on the right turns out to be true, then it must be because $(\exists\alpha)\Gamma\alpha$ is false.
And so, there will be no value of α that makes $\Gamma\alpha$ true, and so the formula on the right will also turn out to be (vacuously) true.

Since the formulas on the left and right of RP10 have the same truth values in all cases, whether Δ is true or false, they are logically equivalent.

QED

A similar argument may be made for RP9, which I leave to you.

PRENEX NORMAL FORM

It is not the case that any given formula has a unique prenex form. For example, consider 6S.12.25 (which comes from Quine), and a natural, narrow-scope regimentation into **F**, at 6S.12.26.

6S.12.25 If there is a philosopher whom all philosophers contradict, then there is a philosopher who contradicts him or herself.

6S.12.26 (∃x)[Fx • (∀y)(Fy ⊃ Gyx)] ⊃ (∃x)(Fx • Gxx)

In order to put this sentence into prenex form, we have first to change the latter 'x's to 'z's, as in 6S.12.27, so that when we stack the quantifiers in front, we won't get accidental binding.

6S.12.27 (∃x)[Fx • (∀y)(Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)

In the first set of transformations to prenex form, I will work with the 'z', then the 'y' then the 'x' and then the 'y' again.

6S.12.27	(∃x)[Fx • (∀y)(Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)	
	(∃z){(∃x)[Fx • (∀y)(Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP7
	(∃z){(∃x)(∀y)[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP4
	(∃z)(∀x){(∀y)[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP10
6S.12.28	(∃z)(∀x)(∃y){[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP9

In the second set, from, 6S.12.27 to 6S.12.29, I will work with the 'x', then the 'y', then the 'z'.

6S.12.27	(∃x)[Fx • (∀y)(Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)	
	(∀x){[Fx • (∀y)(Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)}	by RP10
	(∀x){(∀y)[Fx • (Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)}	by RP4
	(∀x)(∃y){[Fx • (Fy ⊃ Gyx)] ⊃ (∃z)(Fz • Gzz)}	by RP9
6S.12.29	(∀x)(∃y)(∃z){[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	by RP7

6S.12.28 and 6S.12.29 are equivalent to 6S.12.27. 6S.12.28 and 6S.12.29 are both in prenex form. But they differ in form from each other.

EXERCISE 6S.12c

Use the rules of passage in different orders to find two other prenex forms equivalent to 6S.12.27.

RULES OF PASSAGE IN TRANSLATIONS

RP10 allows us to translate 6S.12.30 as 6S.12.31 or as 6S.12.32; the latter two are thus logically equivalent.

6S.12.30 If anything was damaged, then everyone gets upset.
6S.12.31 $(\exists x)Dx \supset (\forall x)(Px \supset Ux)$
6S.12.32 $(\forall x)[Dx \supset (\forall y)(Py \supset Uy)]$

Using the rules of passage, we can transform any statement of predicate logic into prenex normal form, with all the quantifiers out front. 6S.12.33 uses only monadic predicates; still the rules of passage apply.

6S.12.33 If there are any wildebeest, then if all lions are hungry, they will be eaten.
6S.12.34 $(\forall x)\{Wx \supset [(\forall y)(Ly \supset Hy) \supset Ex]\}$
6S.12.35 $(\forall x)\{Wx \supset (\exists y)[(Ly \supset Hy) \supset Ex]\}$ by RP9
6S.12.36 $(\forall x)(\exists y)\{Wx \supset [(Ly \supset Hy) \supset Ex]\}$ by RP7

6S.12.34 is the most natural translation of 6S.12.33. It would be unlikely that any one would translate 6S.12.33 as either 6S.12.35 or 6S.12.36. But our rules of inference allow us only to remove quantifiers from whole lines (i.e., when they are the main operators). So for the purposes of derivations, it may be useful to have the quantifiers out front.

RULES OF PASSAGE IN DERIVATIONS

The rules of passage are rules of equivalence. You use them to transform any formula or subformula in a derivation. In some cases, you can shorten derivations by using them. The proof of 5.5.9 took twelve steps, but with the rules of passage, we can do it in nine steps, as I do at 6S.12.37.

5.5.9

	1. $(\exists x)Mx$	
	2. $(\forall x)(\forall y)[(Mx \bullet My) \supset x=y]$	/ $(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$

6S.12.37

	1. $(\exists x)Mx$	
	2. $(\forall x)(\forall y)[(Mx \bullet My) \supset x=y]$	/ $(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$
	3. $(\forall x)(\forall y)[Mx \supset (My \supset x=y)]$	2, Exp
	4. $(\forall x)[Mx \supset (\forall y)(My \supset x=y)]$	3, RP8
	5. Ma	1, EI
	6. $Ma \supset (\forall y)(My \supset a=y)]$	4, UI
	7. $(\forall y)(My \supset a=y)]$	6, 5, MP
	8. $Ma \bullet (\forall y)(My \supset a=y)]$	5, 7, Conj
	9. $(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$	8, EG
QED		

Summary

Our systems **M** and **F** are complete without the rules of passage. Still, the rules are useful in metalogic, to put propositions into prenex normal form with the quantifiers having wide scope. And they can be helpful in maneuvering between translation and derivation. Translation tends to be most natural when quantifiers have narrow scope, but derivations can be easier when the quantifiers are all out in front.

For Further Research and Writing

1. Return to the derivations in sections 5.3, 5.5, and 5.7, using the rules of passage. Compare the results. Are they effective in shortening your derivations?
2. Another, perhaps more challenging task is to prove the equivalence of the rules of passages. You can do this in at least two ways. First, you can take a formula of the form on one side of a rule of passage and transform it, using our standard set of rules, into a formula of the form on the other side. Then, you do a derivation in the opposite order. (See the appendix to this section for an example.) If each formula entails the other, they are equivalent. Another option is to follow the semantic form I used above to prove RP10.
3. The rules of passage do not include transformations for the biconditional. Determine the relations among the schemas B1–B4.

B1	$(\exists x)(\alpha \equiv \mathcal{F}x)$	B2	$\alpha \equiv (\exists x)\mathcal{F}x$
B3	$(\forall x)(\alpha \equiv \mathcal{F}x)$	B4	$\alpha \equiv (\forall x)\mathcal{F}x$

4. Explore the uses of prenex normal form in the Quine and Mendelson texts suggested below.

Suggested Readings

Herbrand, Jacques. *Logical Writings*. Edited, with an introduction, by Warren Goldfarb. Translated by Jean van Heijenoort. Cambridge, MA: Harvard University Press, 1971. See "On the Fundamental Problem of Mathematical Logic (1931)," Rule 3 (p. 225), for an early formulation and use of the rules of passage.

Mendelson, Elliott. *Introduction to Mathematical Logic*, 4th ed. Boca Raton, FL: Chapman & Hall/CRC, 1997. See pp. 106–109 for the discussion of prenex normal form and its use in metalogic.

Quine, W. V. *Methods of Logic*, 4th ed. Cambridge, MA: Harvard University Press, 1982. See section 23.

SOLUTIONS TO EXERCISES 6S.12a

Alternative solutions to some of these transformations are possible.

1. $(\exists x)[(Ax \bullet {\sim}Bx) \vee (Cx \vee Dx)]$	RP1
2. $(\exists x)[Fx \supset (\exists y)Gy]$	RP9
or $(\exists y)[(\forall x)Fx \supset Gy]$	RP7
3. $(\exists x)(\exists y)[Hx \bullet (Iy \bullet Jxy)]$	RP3
4. $(\exists x)(Px \supset Ra)$	RP9
5. $(\exists x)(\forall y)[Kx \bullet (Ly \supset Mxy)]$	RP4
6. $(\forall x)[Jx \supset (\exists y)Ky]$	RP10
or $(\exists y)[(\exists x)Jx \supset Ky]$	RP7
7. $(\forall x)[(Px \bullet Qx) \supset (Ra \bullet Pa)]$	RP10
8. $(\exists x)(\exists y)[Nx \vee (Oy \bullet Pxy)]$	RP5

9. (∀x)[(Ex ⊃ Fx) • (~Fx ≡ Gx)]	RP2
10. (∀x)(∀y)[Qx ∨ (Ry ⊃ Sxy)]	RP6
11. (∀x)(∃y)(Dxy ⊃ Ex)	RP9
12. (∀x)(∃y)[Tx ⊃ (Uy • Vxy)]	RP7
13. (∀x)(∀y)[Ax ⊃ (By ⊃ Cxy)]	RP8
14. (∀x)(∀y)[Rxy ⊃ (Px • Qx)]	RP10

SOLUTIONS TO EXERCISES 6S.12b

Alternative solutions to some of these transformations are possible.

1. (∀x)Rx • (∀x)(Tx ∨ Sx)	RP2
2. (∃x)[(Kx ≡ Lx) • (∃y)(My • Nxy)]	RP5
3. (∃x)[(Dx • Ex) ∨ (∀y)(Fy ⊃ Gxy)]	RP6
4. (∀x)[Hx ⊃ (∃y)(Iy • Jxy)]	RP7
5. (∀x)(Ox ∨ Qx) ⊃ (∃y)Py	RP9
6. (∀x)[(∃y)Axy ⊃ (Bx ∨ Cx)]	RP10
7. (∀x)[(Ax ∨ ~Cx) ⊃ (∀y)(By ⊃ Dxy)]	RP8
8. (∃x)[(Fx • Gx) • (∀y)(Hy ⊃ Exy)]	RP4
9. (∀x)[(∀y)Ixy ⊃ (Jx • Kx)]	RP9
10. (∃x)(Lx ≡ Mx) ∨ (∃x)Nx	RP1
11. (∃x)(Px • ~Qx) ⊃ (∃y)Oy	RP10
12. (∀x)[Sx ⊃ (∃y)(Ty • Rxy)]	RP7

SOLUTIONS TO EXERCISES 6S.12c

(∃z)(∃x)(∀y){[Fx • (Fy ⊃ Gyx)] ⊃(Fz • Gzz)}	(by RP4 and RP7)
(∀x)(∃z)(∃y){[Fx • (Fy ⊃ Gyx)] ⊃ (Fz • Gzz)}	(by RP4, RP10, RP7, and RP9)

APPENDIX TO 6S.12

Deriving 6S.12.16 from 6S.12.15

1. (∃x)[Px • (∀y)(Qy ⊃ Rxy)]	
2. Pa • (∀y)(Qy ⊃ Ray)	1, EI
3. Pa	2, Simp
4. (∀y)(Qy ⊃ Ray) • Pa	2, Com
5. (∀y)(Qy ⊃ Ray)	4, Simp
6. Qy ⊃ Ray	5, UI
7. Pa • (Qy ⊃ Ray)	3, 6, Conj
8. (∀y)[Pa • (Qy ⊃ Ray)]	7, UG
9. (∃x)(∀y)[Px • (Qy ⊃ Rxy)]	8, EG

QED

Deriving 6S.12.15 from 6S.12.16

1. $(\exists x)(\forall y)[Px \bullet (Qy \supset Rxy)]$	
2. $(\forall y)[Pa \bullet (Qy \supset Ray)]$	1, EI
3. $Pa \bullet (Qy \supset Ray)$	2, UI
4. $(Qy \supset Ray) \bullet Pa$	3, Com
5. $Qy \supset Ray$	4, Simp
6. $(\forall y)(Qy \supset Ray)$	5, UG
7. Pa	3, Simp
8. $Pa \bullet (\forall y)(Qy \supset Ray)$	7, 6, Conj
9. $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)$	8, EG

QED